# §4.7 L'Hôpital's rule

**Thm** Suppose $f(x)$ and $g(x)$ are differentiable and $f(a)=g(a)=0$; and $g'(x)\neq 0$ for $x$ near $a$, then (or $f(a)=g(a)=\pm\infty$)

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$ (provided this limit exists)

**Examples** Warning this is totally different from the quotient rule!

① $$\lim_{x\to 1}\frac{x^3-1}{x-1} = \lim_{x\to 1}\frac{2x^2}{1} = 2$$

$\left.\begin{array}{l} f(1)=0 \\ g(1)=0 \end{array}\right\}$ indeterminate.

② $$\lim_{x\to \frac{\pi}{2}}\frac{\cos^2 x}{\sin x-1} = \lim_{x\to\frac{\pi}{2}}\frac{2\cos x.-\sin x}{\cos x} = \lim_{x\to\frac{\pi}{2}} -2\sin x = -2.$$

③ $$\lim_{x\to 0+} x\ln x = \lim_{x\to 0+}\frac{\ln x}{1/x} = \lim_{x\to 0+}\frac{1/x}{-x^{-2}} = \lim_{x\to 0+}(-x) = 0$$

④ $$\lim_{x\to 0}\frac{e^x-x-1}{\cos x-1} = \lim_{x\to 0}\frac{e^x-1}{-\sin x} = \lim_{x\to 0}\frac{e^x}{-\cos x} = -e$$

Warning doesn't work for $\lim_{x\to 0}\frac{x^2+1}{x+1}$ as not indeterminate form.

(5) $$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = \lim_{x\to 0}\frac{x-\sin x}{x\sin x} = \lim_{x\to 0}\frac{1+\cos x}{\sin x + x\cos x}$$

$$= \lim_{x\to 0}\frac{-\sin x}{\cos x+\cos x + x.-\sin x} = 0.$$

(6) $$\lim_{x\to 0+} x^x = \lim_{x\to 0+} e^{x\ln(x)}$$ $e^x$ cts, so suffices

to find $$\lim_{x\to 0+} x\ln x = \lim_{x\to 0+}\frac{\ln(x)}{1/x} = \lim_{x\to 0+}\frac{1/x}{-1/x^2} = 0$$

so $$\lim_{x\to 0+} e^{x\ln x} = e^0 = 1.$$

## Comparing growth of functions

Q: which grows faster, $(\ln x)^2$ or $\sqrt{x}$ ?

$$\lim_{x\to\infty}\frac{\sqrt{x}}{(\ln(x))^2} = \lim_{x\to\infty}\frac{\frac{1}{2}x^{-1/2}}{2\ln(x)/x} = \lim_{x\to\infty}\frac{x^{1/2}}{4\ln(x)}.$$

$$= \lim_{x\to\infty}\frac{\frac{1}{2}x^{-1/2}}{4/x} = \lim_{x\to\infty}\frac{1}{8}\sqrt{x} \to \infty \text{ as } x\to\infty.$$

Thm: $e^x$ grows faster than $x^n$ for every $n$.

Proof $$\lim_{x\to\infty}\frac{e^x}{x^n} = \lim_{x\to\infty}\frac{e^x}{nx^{n-1}} = \cdots = \lim_{x\to\infty}\frac{e^x}{n!}\to\infty.$$ □

# §4.9 Antiderivatives

**Defn** $F(x)$ is an <u>antiderivative</u> of $f(x)$ if $F'(x) = f(x)$.

**Example** $f(x) = x^2$, then $F(x) = \frac{1}{3}x^3$ is an antiderivative as $\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$. <u>Note</u>: $\frac{1}{3}x^3 + 4$ also works.

General antiderivative:

**Thm** Let $F(x)$ be an antiderivative for $f(x)$. Then any other antiderivative for $f(x)$ is of the form $F(x) + C$ for some constant $C$.

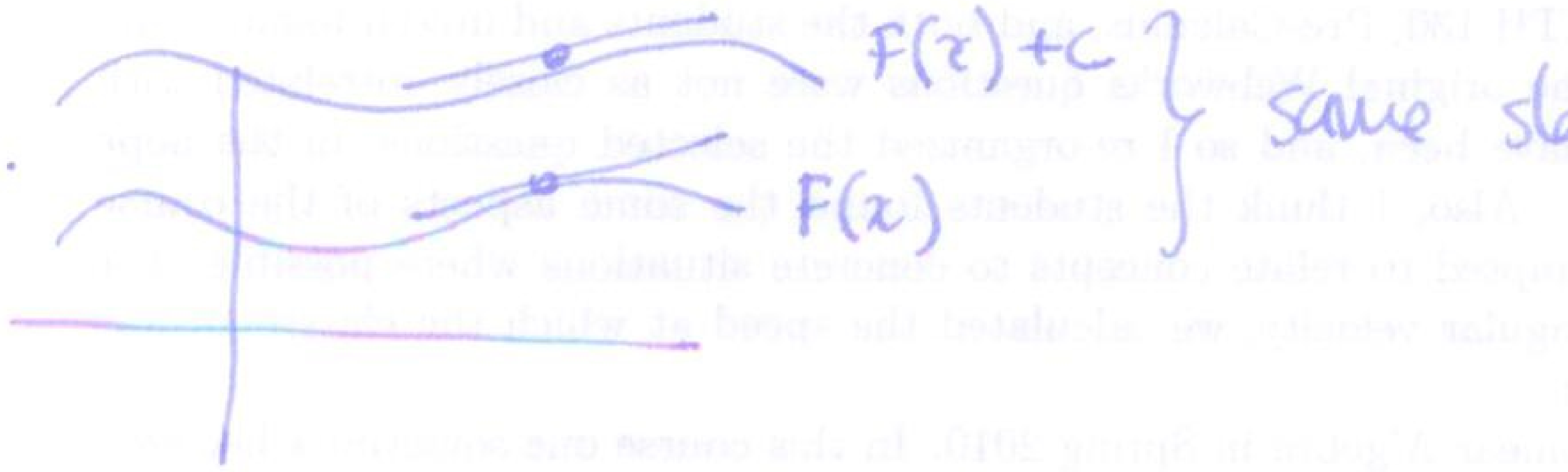
**Proof** suppose $F(x)$ and $G(x)$ are antiderivatives of $f(x)$,

so $F'(x) = f(x)$ and $G'(x) = f(x)$. Then consider

$G(x) - F(x)$ which has derivative $(G(x) - F(x))' = G'(x) - F'(x)$

$= f(x) - f(x) = 0$, so $G(x) - F(x) = \text{constant}$. $\square$

picture:




**Example** find the general antiderivative to $f(x) = \sin 4x$.

guess: $\frac{d}{dx}(\cos 4x) = -4\sin(4x)$

so try $\frac{d}{dx}\left(-\frac{1}{4}\cos 4x\right) = -\frac{1}{4}\cdot 4\sin 4x = \sin 4x$

so general antiderivative is $F(x) = -\frac{1}{4}\cos(4x) + C$.

**Notation** Indefinite integral

$\int f(x)\,dx = F(x) + C$ means $F(x)+C$ is the general antiderivative for $f(x)$.

**Thm** $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ for $n \neq -1$.

**Proof** $\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1} + C\right) = \frac{1}{n+1}(n+1)x^n = x^n$ ✓ $\square$.

**Thm** $\int \frac{1}{x}\,dx = \ln|x| + C$

**Proof** $\frac{d}{dx}\left(\ln(x) + C\right) = \frac{1}{x}$. ✓ $\square$ $x > 0$.

$x < 0$ $\quad \frac{d}{dx}\ln(-x) = \frac{1}{-x}\cdot -1 = \frac{1}{x}$ ✓ $\square$.

**Thm** sums and constant multiples:

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int c f(x)\,dx = c\int f(x)\,dx \quad \square.$$

WARNING: NO PRODUCT | QUOTIENT | CHAIN RULE!

**Useful integrals**

$\int \sin x\,dx = -\cos x + C$ $\qquad \int e^x\,dx = e^x + C$

$\int \cos x\,dx = \sin x + C$

**Examples** $\int x^2 + \frac{1}{x} + \sin x\,dx$.