Rational functions $f(x) = \dfrac{p(x)}{q(x)}$.

$\deg p > \deg q \qquad \lim_{x \to \pm\infty} f(x) = \pm\infty.$

$\deg p < \deg q \qquad \lim_{x \to \pm\infty} f(x) = 0.$

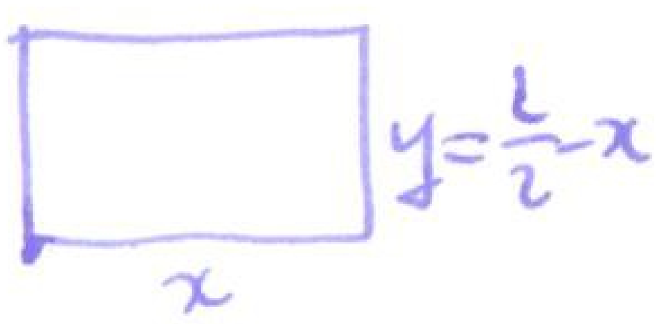
$\deg p = \deg q \qquad \lim_{x \to \pm\infty} f(x) = \dfrac{p_n}{q_n}$

$p(x) = p_n x^n + p_{n-1} x^{n-1} + \dots + p_0$

$q(x) = q_n x^n + q_{n-1} x^{n-1} + \dots + q_0.$

§4.6 Optimization

Example A piece of wire of length L is bent into a rectangle. What's the largest area rectangle we can make?



$2x + 2y = L$ $\qquad$ area $= x\left(\frac{L}{2} - x\right) = \frac{L}{2}x - x^2$

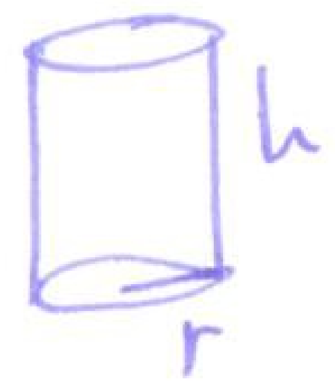
$y = \frac{L}{2} - x$

find max: $A'(x) = \frac{L}{2} - 2x$

$A'(x) = 0 \qquad \frac{L}{2} - 2x = 0 \qquad x = \frac{L}{4}.$

[This is a closed interval optimization problem - if we didn't know what the function looked like then we should also check the endpoints $x = 0, \frac{L}{2}$].

Example What shape of cylindrical tin can minimizes ~~volume~~ ~~the total surface~~ surface area, if you want total volume to be $1\ ft^3$.



$V = \pi r^2 h \qquad A = 2\pi r h + 2\pi r^2$

$1 = \pi r^2 h$

$h = \frac{1}{\pi r^2}$ $\qquad A(r) = 2\pi r\left(\frac{1}{\pi r^2}\right) + 2\pi r^2.$

$$= \frac{2}{r} + 2\pi r^2$$

$$A'(r) = -\frac{2}{r^2} + 4\pi r$$

$A'(r) = 0$

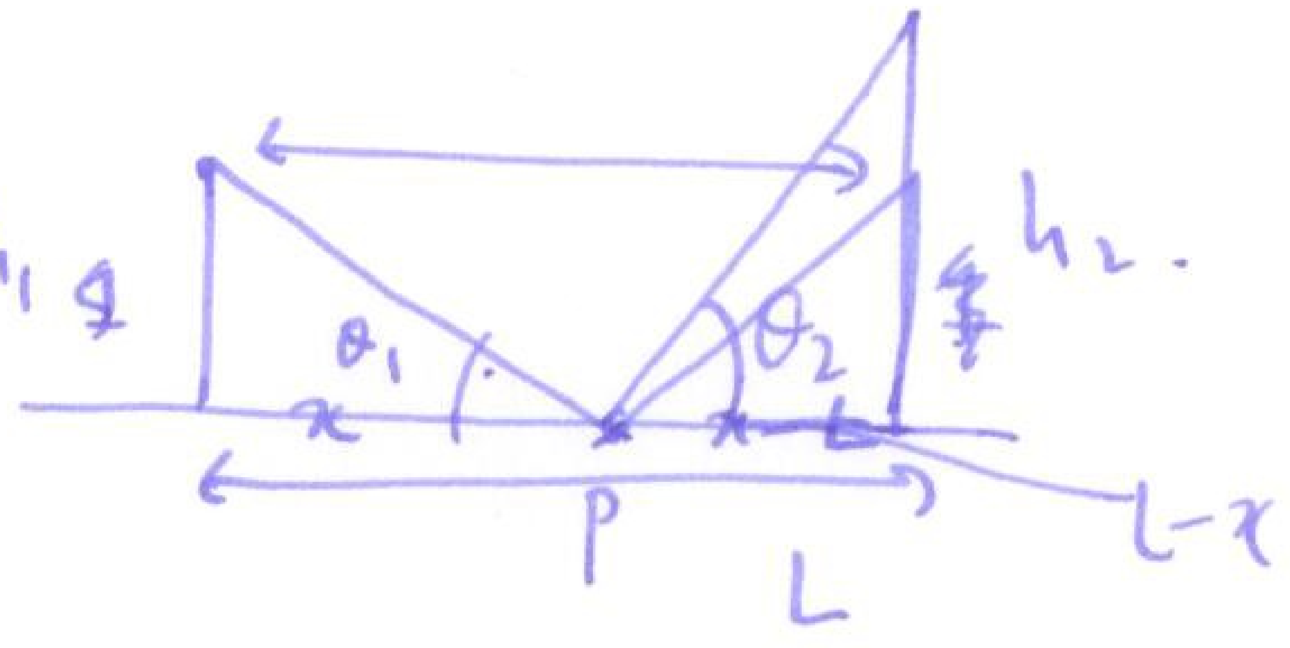
$$-\frac{2}{r^2} + 4\pi r = 0$$

$$4\pi r = \frac{2}{r^2}.$$

$$r^3 = \frac{1}{2\pi} \qquad r \sim \sqrt[3]{\frac{1}{2\pi}}.$$

Example

Minimize distance



show distance is minimized when $\theta_1 = \theta_2$.
(angle of reflection = angle of incidence)

$$f(x) \quad \text{distance} = \sqrt{h_1^2 + x^2} + \sqrt{h_2^2 + (\cancel{x-L})^2} \quad L-x$$

$$f'(x) = \frac{x}{\sqrt{h_1^2 + x^2}} - \frac{L-x}{\sqrt{h_2^2 + (L-x)^2}} = 0.$$

i.e.

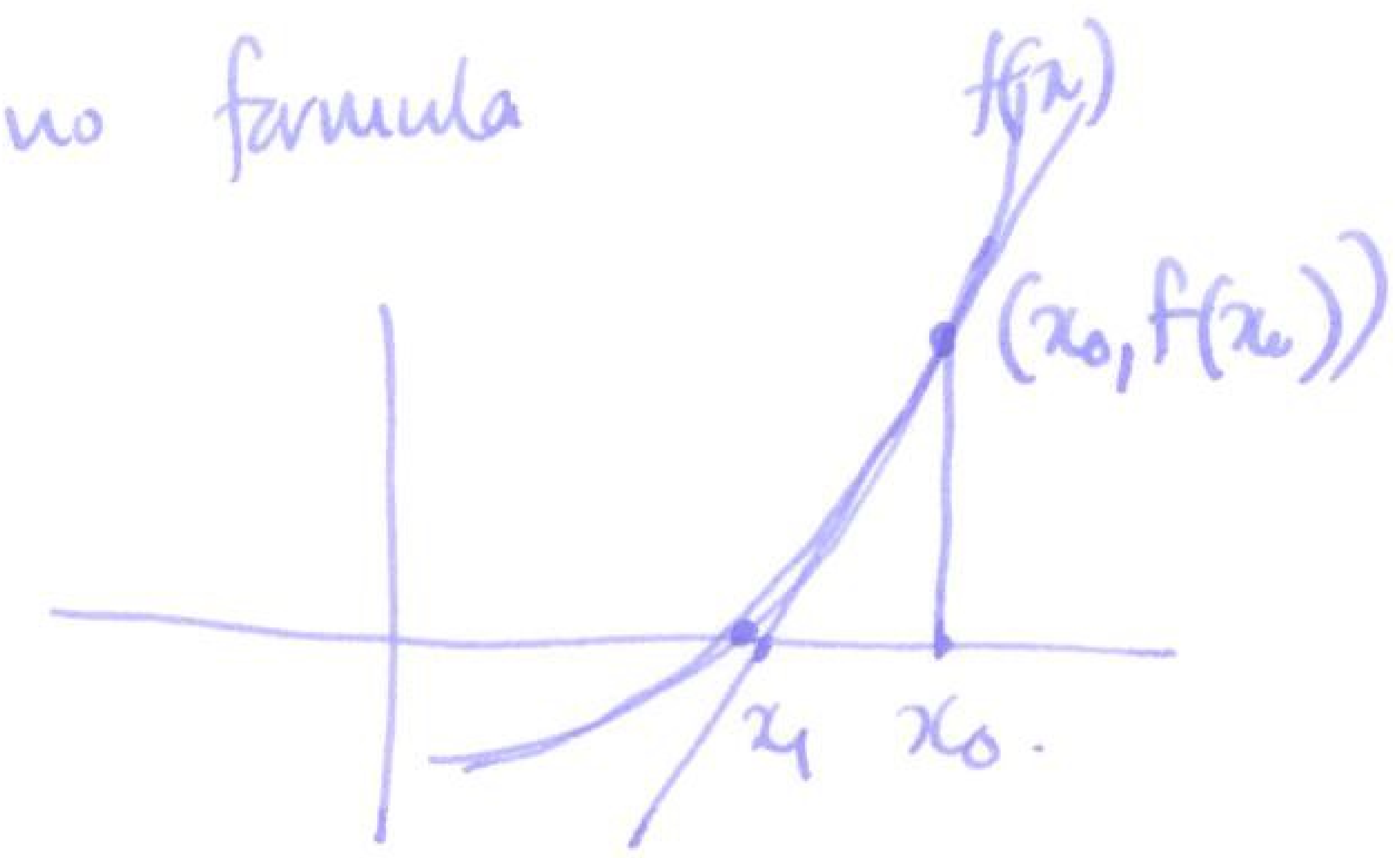
$$\underbrace{\frac{x}{\sqrt{h_1^2 + x^2}}}_{\cos\theta_1} = \underbrace{\frac{L-x}{\sqrt{h_2^2 + (L-x)^2}}}_{\cos\theta_2}. \quad \Rightarrow \theta_1 = \theta_2 \qquad 0 \leq \theta_i \leq \frac{\pi}{2}$$

§4.8 Newton's method

Q: find roots of $x^5 - x - 1$ (one real root)

problem: no formula

algorithm:



guess initial answer x_0

use linear approximation to find better approximation x_1

equation of tangent line: $y - y_0 = m(x - x_0)$

$$y - f(x_0) = f'(x_0)(x - x_0)$$

hits $y = 0$ at:

$$0 - f(x) = f'(x_0)(x - x_1).$$

$$\frac{-f(x_0)}{f'(x_0)} = x - x_0.$$

check

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

$x^5 - x - 1$ $\qquad x_0 = 1$

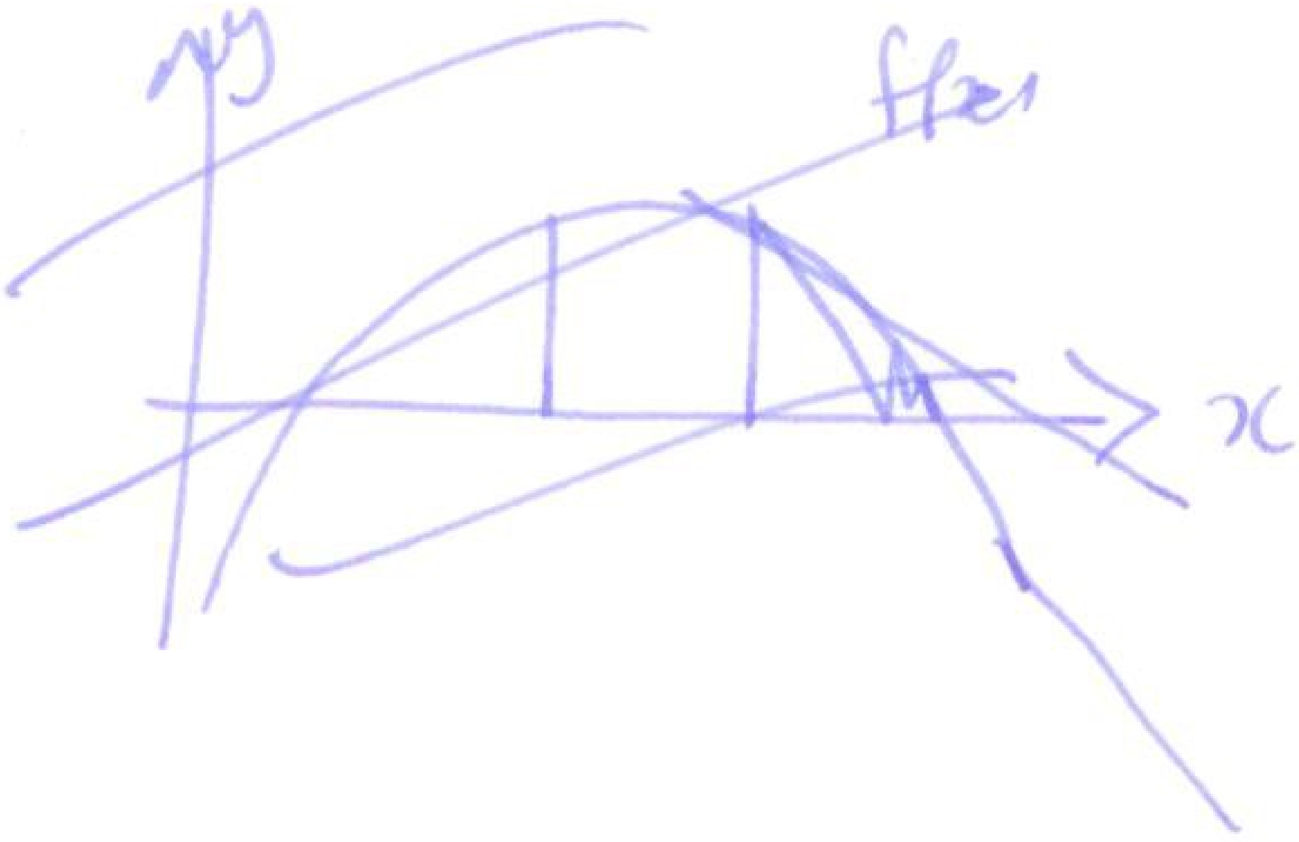
$f'(x) = 5x^4 - 1$ $\qquad x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 1 - \dfrac{-1}{4}$

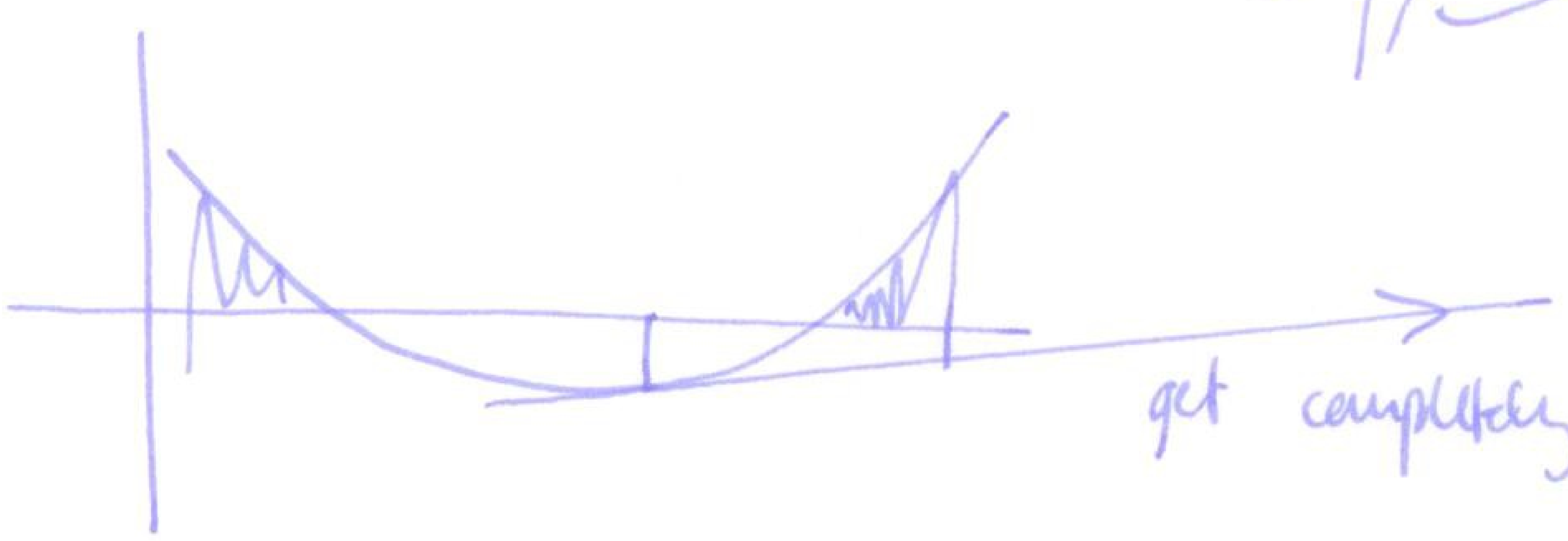
$= 1.25$.

$x_2 = 1.25 - \dfrac{f(1.25)}{f'(1.25)} = \ldots$ and so on.

picture

problems may be many roots:





get completely different root.

may not converge at all....