check $\frac{d}{dx}\left(\sinh^{-1}x\right)$ $\quad f(x) = \sinh x \quad f'(x) = \cosh x$

$g(x) = \sinh^{-1}(x).$

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cosh(\sinh^{-1}(x))}$$

$\cosh t, \quad t = \sinh^{-1}(x)$

$\sinh t = x.$ use $\cosh^2 t - \sinh^2 t = 1.$

$\cosh^2 t = 1 + \sinh^2 t = 1 + x^2.$

$\cosh t = \sqrt{1+x^2}.$

so $\frac{d}{dx}\left(\sinh^{-1}(x)\right) = \frac{1}{\sqrt{1+x^2}}$ $\square$.

§3.11 Related rates

Example falling ladders.

$x(t)$ distance of foot of ladder from wall.
$h(t)$ height of top of ladder.

$\frac{dx}{dt} = 3\text{ft/s}.$ Q: what is $\frac{dh}{dt}$?

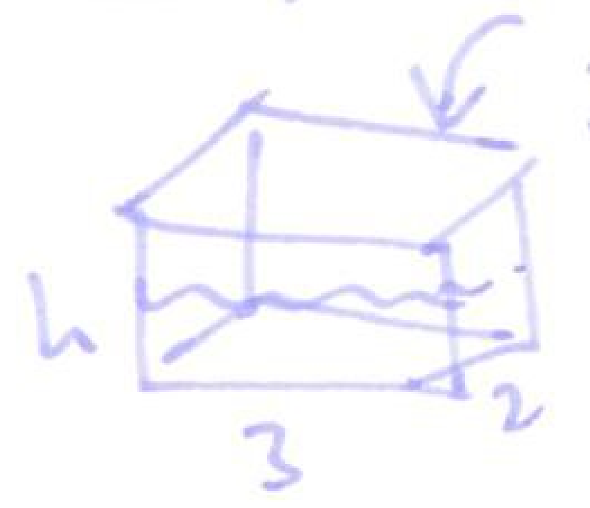
$$x^2 + h^2 = 16^2$$

differentiate wrt t: $2x\underbrace{\frac{dx}{dt}}_{=3} + 2h\frac{dh}{dt} = 0.$

$$\frac{dh}{dt} = -3\frac{x}{h}\ \text{ft/s}.$$

$t=0$: $x = 5$, $h \approx 15.2$ $\quad \frac{dh}{dt} \simeq -1$ $\qquad t=1$ $\quad x = 8$, $h \simeq 13.9$ $\quad \frac{dh}{dt} \simeq -1.7$ ft/s.

Example filling a rectangular tank.



how fast is the water rising?

$V =$ volume of water $\quad V(t)$

$\frac{dV}{dt} =$ rate of change of water with time

$h =$ height of water

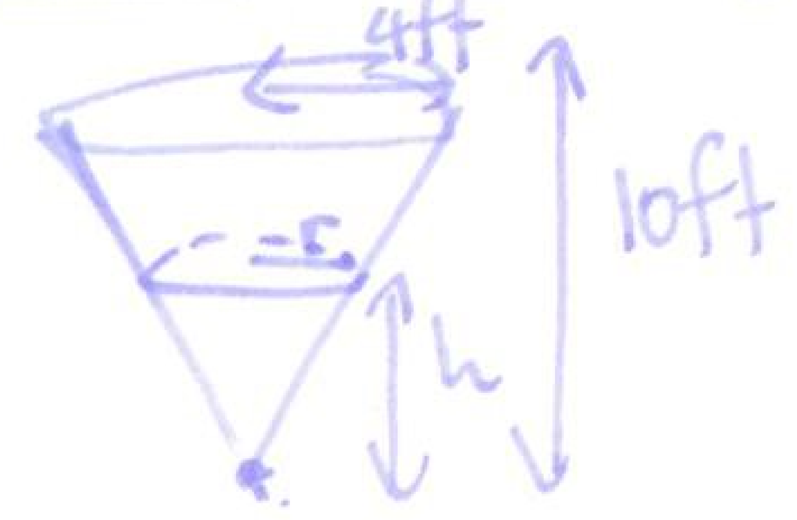
$\frac{dh}{dt} =$ rate of change of height of water.

$V = 3 \times 2 \times h$ i.e. $V = 6h$

so $\frac{dV}{dt} = 6\frac{dh}{dt}$

$3 = \frac{dV}{dt} = 6\frac{dh}{dt}$ so $\frac{dh}{dt} = \frac{1}{2}$ ft/min.

Example filling a conical tank



water in at $10 ft^3/min$ i.e. $\frac{dV}{dt} = 10 ft^3/min$

volume of cone: $V = \frac{1}{3}\pi h r^2$

need: r in terms of h $\quad \frac{r}{h} = \frac{4}{10}$ (similar triangles).

$r = \frac{4h}{10} = \frac{2h}{5}$

$V = \frac{1}{3}\pi h\left(\frac{2h}{5}\right)^2 = \frac{4\pi h^3}{3.25}$

$\frac{dV}{dt} = \frac{4\pi}{75} \cdot 3h^2\frac{dh}{dt} \quad \frac{4\pi}{25}h^2\frac{dh}{dt}$

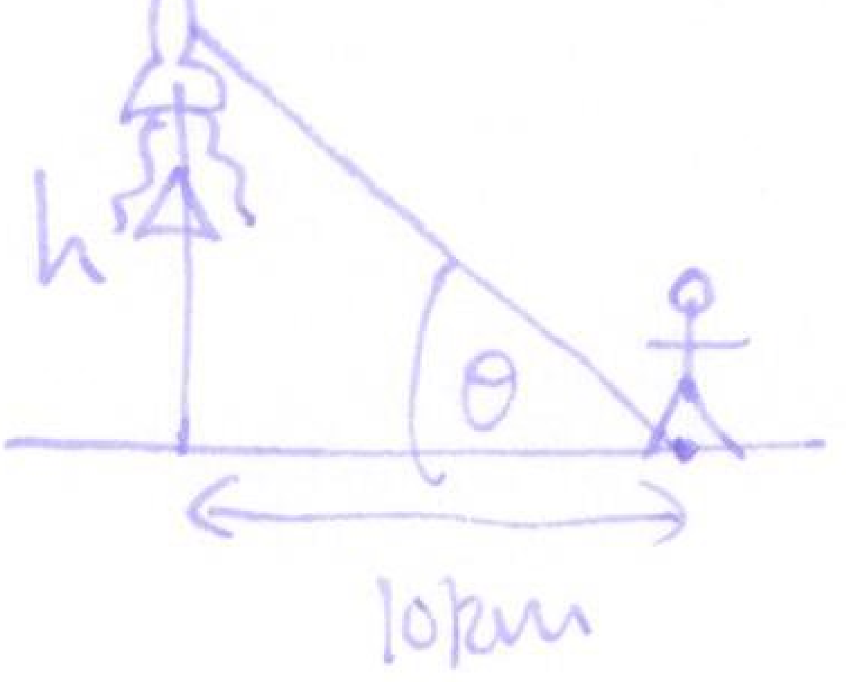
so e.g. when $h=5$ water rising at rate $\frac{10}{4\pi}$ ft/min things to try: ① give things names: (height h, Volume V etc.)

② write down relations between the things and differentiate

③ plug in numbers (if necessary)

Example

Tracking rockets:

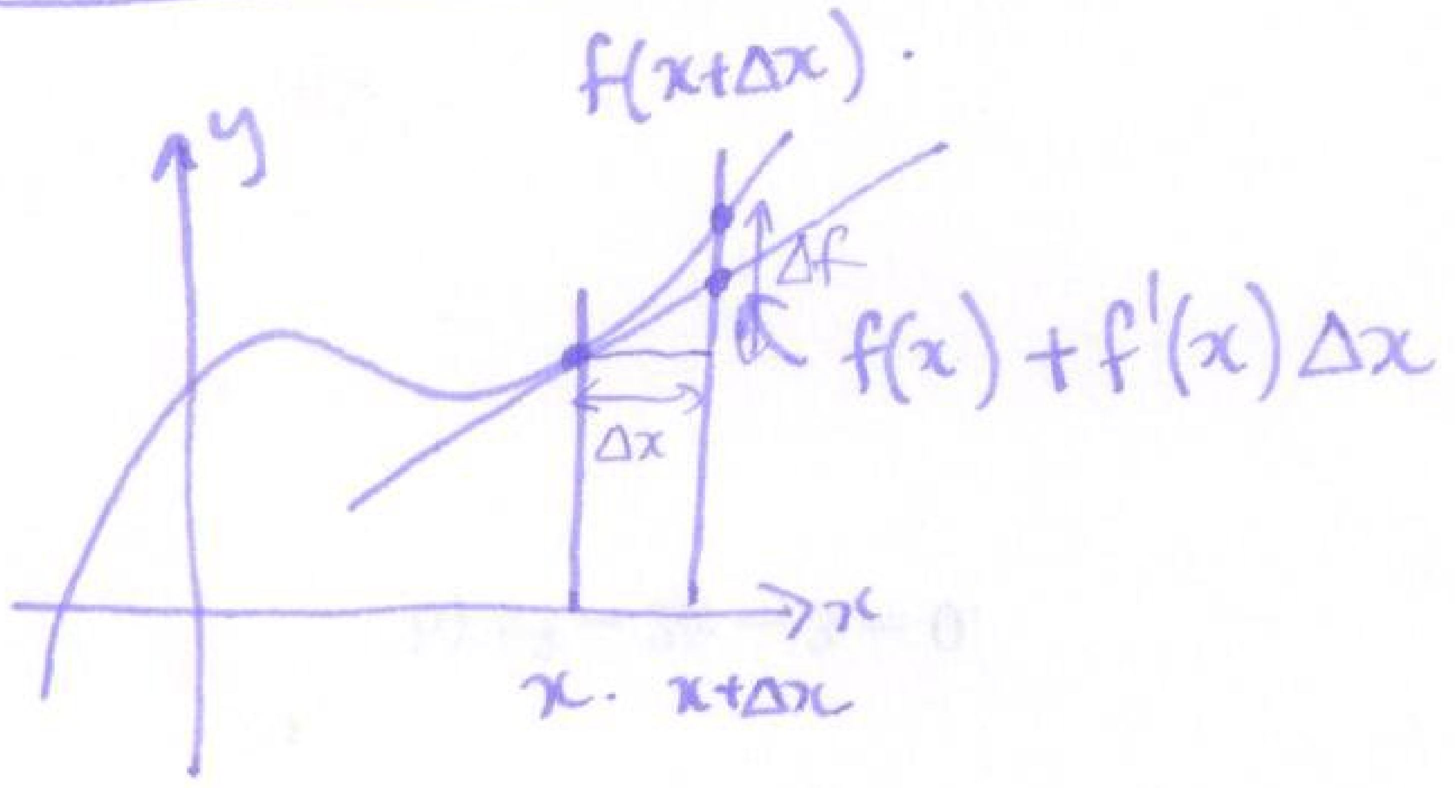


if angle is $\frac{\pi}{3}$ and rate of change of angle is $\frac{1}{2}$ radians/min how fast is the rocket going?

$$\frac{h}{10} = \tan\theta \qquad \frac{1}{10}\frac{dh}{dt} = \sec^2\theta \frac{d\theta}{dt}$$

$$\frac{dh}{dt} = 10\sec^2\left(\frac{\pi}{3}\right)\cdot\frac{1}{2} \approx 20\text{km/min}.$$

§4.1 Linear approximations



If f is differentiable at x, and Δx is small then

$$\boxed{\Delta f \approx f'(x)\Delta x.}$$

Example what is $\sqrt{103}$?

$f(x) = \sqrt{x} = x^{1/2}$ $\qquad f(100) = \sqrt{100} = 10$

$f'(x)$, $\frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ $\qquad f'(100) = \frac{1}{20}$ $\qquad \Delta f \approx f'(x)\Delta x.$

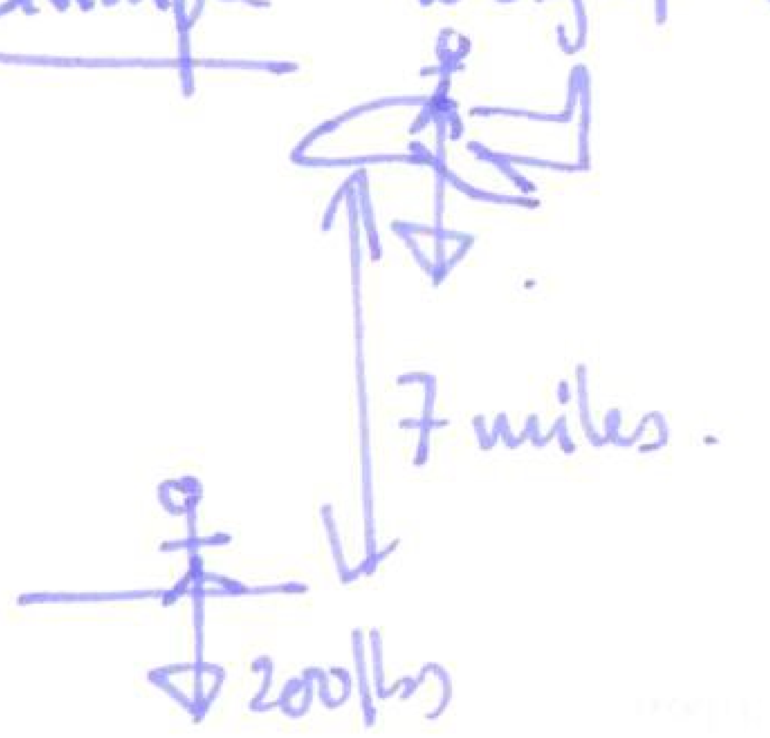
$\frac{1}{20}\cdot 3.$

so $\sqrt{103} \approx 10 + \frac{3}{20}$

$$\underset{100+3.}{f(x+\Delta x)} \approx f(x) + \underbrace{\Delta f}_{\approx f'(x)\Delta x}.$$

$f'(100)\,3.$

<u>Example</u> weight loss.



$$W(x) = \frac{wR^2}{x^2} = wR^2x^{-2}$$

$R = 3960$ miles, radius of earth

x = distance from center of earth.

w = weight at ground ($R = 3960$).

$$W'(x) = -\frac{2wR^2}{x^3}.$$

$$W(3960+7) \approx W(3960) + W'(3960)\Delta x$$

$$\approx 200 + -\frac{2 \cdot 200 \cdot 3960^2}{3960^3} \cdot 7 \approx 0.7 \text{ lbs.}$$

<u>Example</u> pizza size: you make an 18" pizza, but if your diameter is accurate to ± 0.4 in, how much size pizza could you gain or lose?

$A = \pi r^2 \qquad 2r = D \qquad A = \pi\left(\frac{D}{2}\right)^2 = \frac{1}{4}\pi D^2$

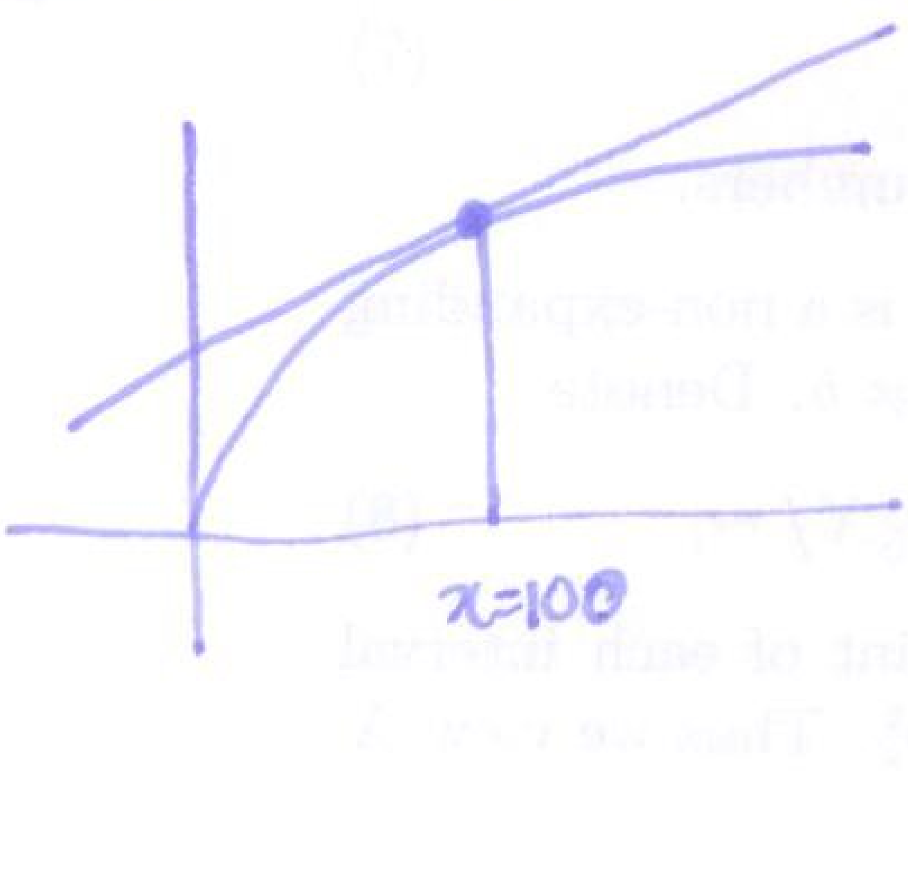
$$A'(D) = \frac{1}{2}\pi D$$

$$\Delta A \approx A'(18)\,\Delta D = \frac{1}{2}\pi \cdot 18 \cdot 0.4 \approx 11 \text{ in}^2.$$

is this good or bad? compare with actual area $\frac{\pi}{4} \cdot 18^2 \approx$

<u>Linear approximation</u> we can approximate $f(x)$ by its tangent line.

<u>Example</u>



$f(x) = \sqrt{x}$

tangent line at $x = 100$.

slope $f'(100) = \frac{1}{20}$ point $(100, 10)$.

$$y - 10 = \frac{1}{20}(x - 100)$$

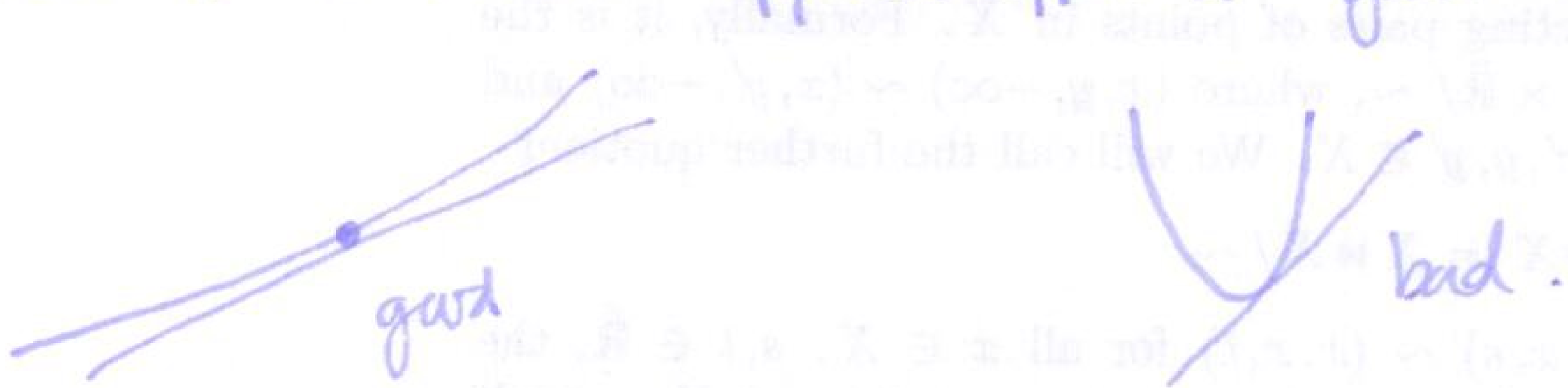
$$y = \frac{1}{20}x + 5.$$

	x	$f(x)$	linearization	error	percentage error
good approx	121	11	$\frac{121}{20}+5 = 11.05$	0.05	$\frac{0.05}{11} \times 100$
bad approx	400	20	$\frac{400}{20}+5 = 25$	5	$\frac{5}{20} \times 100$

$$\text{percentage error} = \left| \frac{\text{error}}{\text{actual value}} \right| \times 100$$

observation:

When is a linear approximation a good approximation?



$f''(x)$ small $\qquad$ $f''(x)$ big.

§4.2 Extreme values (maxima/minima)

Absolute max/min:

Def^{n}: Let $f(x)$ be defined on an interval I. We say $f(a)$ is the absolute max if $f(x) \le f(a)$ for all $x \in I$

absolute min $\quad f(x) \ge f(a)$

warning: not every function has absolute max/min.

Example