Proof (of chain rule)

$$[f(g(x))]' = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

answer should be $f'(g(x))g'(x)$.

so write this as:

$$\lim_{h\to 0} \underbrace{\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}}\ \underbrace{\frac{g(x+h)-g(x)}{h}}_{\text{this limit} = g'(x)}$$

set $k = g(x+h)-g(x)$ then as $\underline{g \text{ cts}}$ $h\to 0 \Rightarrow k \to 0$.

then $\displaystyle \lim_{k\to 0} \frac{f(g(x)+k) - f(g(x))}{k} = \lim f'(g(x))$ $\square$.

More examples

$$\frac{d}{dx}\left(g(x)^n\right) = n\,(g(x))^{n-1}\cdot g'(x)$$

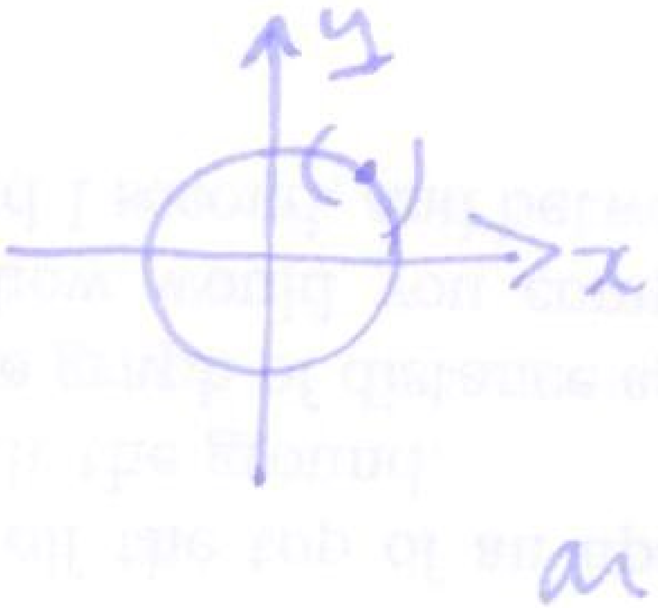
$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}\cdot g'(x)$$

$$\frac{d}{dx}\left(f(kx+b)\right) = kf'(kx+b)$$

§3.8 Implicit differentiation

explicit function: $f(x) = x^2+x+1$

implicit "function": $x^2+y^2=1$



"local" function near a point (x,y) on the curve we can think of y as a function of x $(x, y(x))$.

differentiate: $\displaystyle \frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1)$

$$2x + \frac{d}{dx}\left(y(x)^2\right) = 0$$

$$2x + 2y(x)\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = -\frac{x}{y}$$

$\uparrow$ still implicit!

<u>Example</u> find tangent line to circle $x^2+y^2=1$. at

point $\left(\frac{3}{5}, \frac{4}{5}\right)$ $\quad \frac{dy}{dx} = -\frac{x}{y} = \frac{-3/5}{4/5} = -3/4$

$$y - \frac{4}{5} = -\frac{3}{4}\left(x - \frac{3}{5}\right).$$

<u>Example</u> find tangent line to $y^4 + xy = x^3 - x + 2$ at $(1,1)$

implicit differentiation: $\quad 4yy' + y + xy' = 3x^2 - 1$

set $x=1$, $y=1$ $\quad 4y' + 1 + y' = 3 - 1$

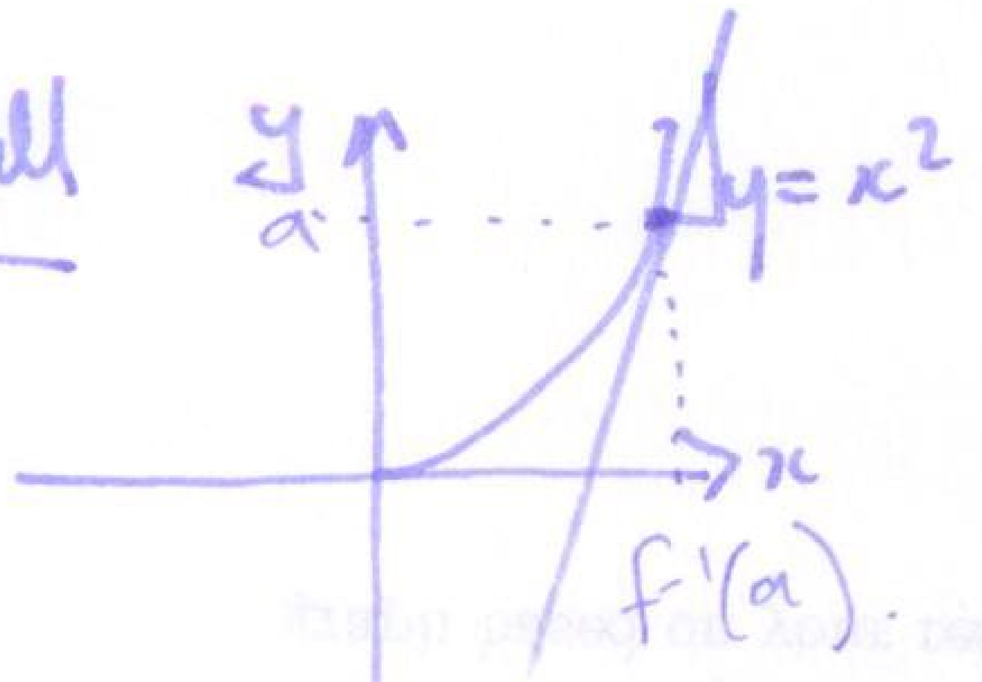
$5y' = 1 \qquad y' = 5 \qquad y - 1 = 5(x-1)$.

§3.9 Derivatives of inverse functions

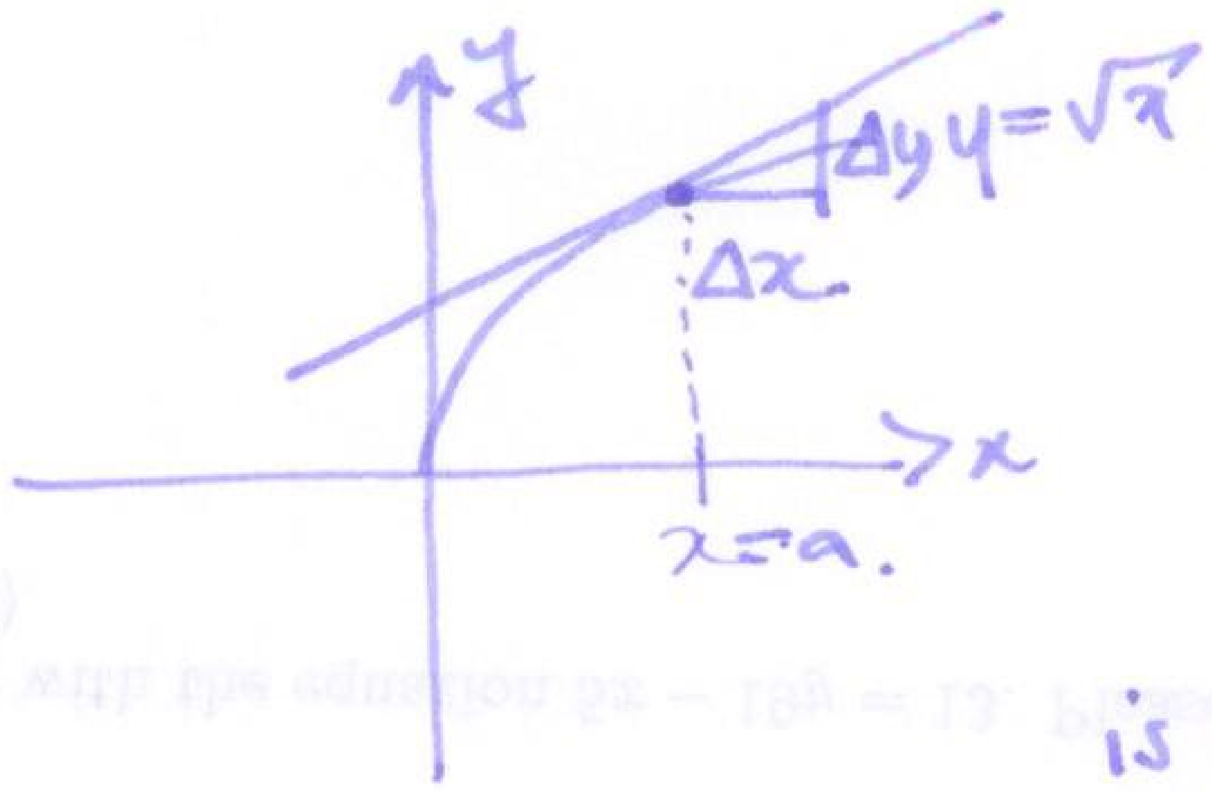
<u>Thm</u> $f(x)$ differentiable and one-to-one with inverse $g(x) = f^{-1}(x)$

then $g'(x) = \dfrac{1}{f'(g(x))}$ as long as $f'(g(x)) \neq 0$

<u>recall</u>



inverse (reflect in $x=y$)



$\frac{\Delta y}{\Delta x}$ here is ~~inverse of~~ 1/slope of f.

Example $f(x) = x^2$

$g(x) = f^{-1}(x) = \sqrt{x}$

$$g'(x) = \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2}.$$

$$g'(x) = \frac{1}{f'(g(x))}$$

$f(x) = x^2$

$f'(x) = 2x.$

$$= \frac{1}{2g(x)} = \frac{1}{2\sqrt{x}}.$$

Proof (use implicit differentiation)
(glassmic)

$y = f(x)$ $\qquad f^{-1}(y) = x$ } implicit differentiation/chain rule

$f^{-1}(y) = x$ $\qquad \overset{\shortparallel}{g(y)}$

$\overset{\shortparallel}{g(y)}$

$$g'(y)\frac{dy}{dx} = 1 \qquad \left(\frac{dy}{dx} = f'(x)\right).$$

$$\cancel{\frac{dy}{dx} = \frac{1}{g'(y)}}$$

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{f'(g(y))}.$$

$$\text{so } \quad g'(x) = \frac{1}{f'(g(x))} \quad \square.$$

Example **Thm** $\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$ $\qquad \frac{d}{dx}\left(\cos^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}}$

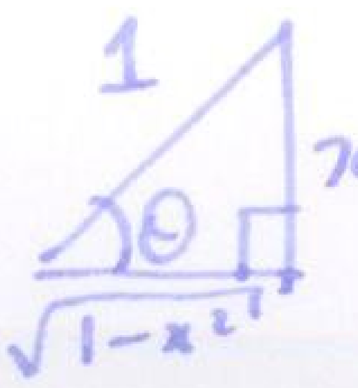
Proof use $g'(x) = \dfrac{1}{f'(g(x))}$

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\cos\left(\sin^{-1}(x)\right)} = \frac{1}{\sqrt{1-x^2}}$$

$f(x) = \sin(x)$

$g(x) = \sin^{-1}(x)$ $\qquad \cos\theta, \quad \theta = \sin^{-1}(x)$

$\sin\theta = x$



$$\frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{1}{-\sin(\cos^{-1}(x))} = \frac{-1}{\sqrt{1-x^2}}$$

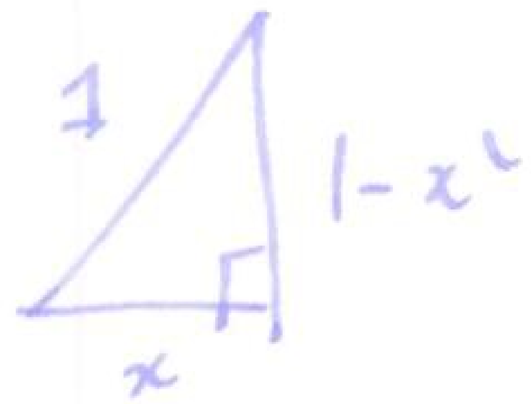
$f(x) = \cos(x)$

$g(x) = \cos^{-1}(x)$

$\sin\theta$, $\theta = \cos^{-1}(x)$

$\cos\theta = x$

$f'(x) = -\sin(x)$



Thm

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{x^2+1} \qquad \frac{d}{dt}\left(\cot^{-1}(x)\right) = -\frac{1}{x^2+1}$$

$$\frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dt}\left(\csc^{-1}(x)\right) = \frac{-1}{|x|\sqrt{x^2-1}}.$$

Proof $f(x) = \tan(x)$ $\quad f'(x) = \sec^2 x.$

$g(x) = \tan^{-1}(x)$

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\sec^2(\tan^{-1}(x))}$$

$\sec^2\theta \qquad \theta = \tan^{-1}x$

$\quad \| \qquad \tan\theta = x$

$1+\tan^2\theta = 1+x^2$

$$= \frac{1}{1+x^2} \quad \square.$$

Example $\frac{d}{dx}\left(\tan^{-1}(e^{2x})\right)$ 4x. $\qquad \frac{d}{dx}\left(\tan^{-1}(x)\right)' = \frac{1}{1+x^2}.$

+ chain rule

$$= \frac{1}{1+(e^{2x})^2}\cdot\left(e^{2x}\right)'$$

$$= \frac{1}{1+e^{4x}}\cdot e^{(2x)}\cdot(2x)' = \frac{2e^{2x}}{1+e^{4x}}.$$

§3.10 Derivatives of exponentials + logs

recall • $f(x)$, $g(x) = f^{-1}(x)$ then $g'(x) = \dfrac{1}{f'(g(x))}$

• if $f(x) = b^x$ then $f'(x) = m_b\, b^x$ • $\dfrac{d}{dx}(e^x) = e^x$.

Thm $f(x) = b^x$ then $f'(x) = \ln(b) \cdot b^x$

Proof $b^x = e^{\ln(b)x}$ use chain rule

$$\frac{d}{dx}(b^x) = e^{\ln(b)x} \cdot (\ln(b)x)' = \ln(b)\, e^{\ln(b)x} = \ln(b)\, \cancel{e}\, b^x.$$

Example $f(x) = \cancel{4^{3x}}\ 3^{4x}$

chain rule. $\ln(3)\, 3^{4x} \cdot (4x)' = 4\ln(3) \cdot 3^{4x}$.

Thm $\dfrac{d}{dx}(\ln(x)) = \dfrac{1}{x}$

Proof $f(x) = e^x$ $f'(x) = e^x$

$g(x) = \ln(x)$

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{e^{\ln(x)}} = \frac{1}{x} \quad \square.$$

Example $\dfrac{d}{dx}(x \ln x)$ (product rule)

$$x \frac{1}{x} + 1 \cdot \ln x = 1 + \ln(x).$$