Example Continuous but not differentiable.

$f(x) = |x|$

cf. ✓ did earlier.

differentiable? $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{|0+h| - 0}{h}$

$= \lim_{h \to 0} \frac{|h|}{h}$ DNE.

Local picture: if $f(x)$ is differentiable at $x = c$, then if you look close enough, it looks close to a straight line.

§3.3 Product and quotient rules

new functions from old: $f(x)g(x)$ product, $f(x)/g(x)$ quotient.

Thm Product rule $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$.

$$\frac{d}{dx}(fg) = \frac{df}{dx} g + f \frac{dg}{dx}.$$

Warning: $(fg)' \neq f'g'$!!

Example ① $\frac{d}{dx} x^2 = \frac{d}{dx}(x) \cdot x + x \cdot \frac{d}{dx}(x)$

$= 1 \cdot x + x \cdot 1 = 2x$.

② $\frac{d}{dx}\left(3x^2(x^2+1)\right) = \frac{d}{dx}(3x^2) \cdot (x^2+1) + 3x^2 \frac{d}{dx}(x^2+1)$.

$= 6x(x^2+1) + 3x^2(2x)$

$= 6x^3 + 6x + 6x^3 = 12x^3 + 6x$.

③ $\frac{d}{dx}(x^2 e^x) = \frac{d}{dx}(x^2)\, e^x + x^2 \frac{d}{dx}(e^x)$.

$= 2xe^x + x^2 e^x = e^x(x^2+2x)$.

Proof (of product rule) (assume f, g both differentiable at x)

$$(fg)'(x) = \lim_{h \to 0} \cancel{fg}\, \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Trick: $$\lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}.$$

$$\lim_{h \to 0} f(x+h) \frac{g(x+h) - g(x)}{h} + g(x) \frac{f(x+h) - f(x)}{h}$$

sum: $$\lim_{h \to 0} f(x+h) \frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} g(x) \frac{f(x+h) - f(x)}{h}$$

~~f cts~~ product:

$$= \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} + g(x) \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

f cts :

$$= f(x)\,g'(x) + g(x)\,f'(x). \quad \square$$

Thm Quotient rule. Suppose f, g are differentiable, then f/g is differentiable for all x s.t. $g(x) \neq 0$. and

$$(f/g)' = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2} \qquad \square .$$

Example

$$\frac{d}{dx}\left(\frac{x}{x+1}\right) = \frac{(x+1)\cdot(x)' - (x+1)'\cdot x}{(x+1)^2}.$$

$$= \frac{(x+1)\cdot(1) - (1)\cdot x}{(x+1)^2} = \frac{x+1-x}{(x+1)^2}.$$

$$= \frac{1}{(x+1)^2}.$$

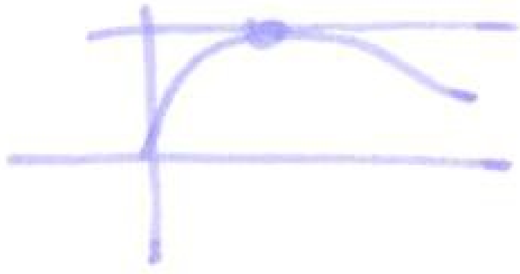
Example

$$\frac{d}{dt}\left(\frac{e^t}{e^t+1}\right) = \frac{(e^t+1)(e^t)' - (e^t)(e^t+1)'}{(e^t+1)^2}$$

$$= \frac{(e^t+1)e^t - (e^t)(e^t)}{(e^t+1)^2} = \frac{e^t}{(e^t+1)^2}.$$

Example battery power voltage V  resistance R

internal resistance r

Q: when does the battery give the maximal power?  find $\frac{dP}{dR} = 0$

power $P = \frac{V^2 R}{(r+R)^2}$ find $\frac{dP}{dR}$. assume V, r constant.

$$\frac{dP}{dR} = \frac{(r+R)^2 (V^2R)' - ((r+R)^2)'(V^2R)}{(r+R)^4} = \cancel{\frac{(r+R)V^2 - (1)(V^2R)}{(r+R)^2}}.$$

$$= \cancel{\frac{V^2(r+R-R)}{(r+R)^2}} = \cancel{\frac{V^2 r}{r+R}}$$

$$\frac{dP}{dR} = \frac{(r+R)^2 V^2 - (r^2+2rR+R^2)' V^2 R}{(r+R)^4} = \frac{V^2\left[r^2+2rR+R^2-(2r+2R)R\right]}{(r+R)^4}.$$

$$= \frac{V^2\left[r^2+2rR+R^2-2rR-2R^2\right]}{(r+R)^4} = V^2\frac{r^2-R^2}{(r+R)^4} = V^2\frac{r-R}{(r+R)^3}.$$

solve $\frac{V^2(r-R)}{(r+R)^3} = 0 \Rightarrow r-R=0 \Rightarrow r=R$ □.