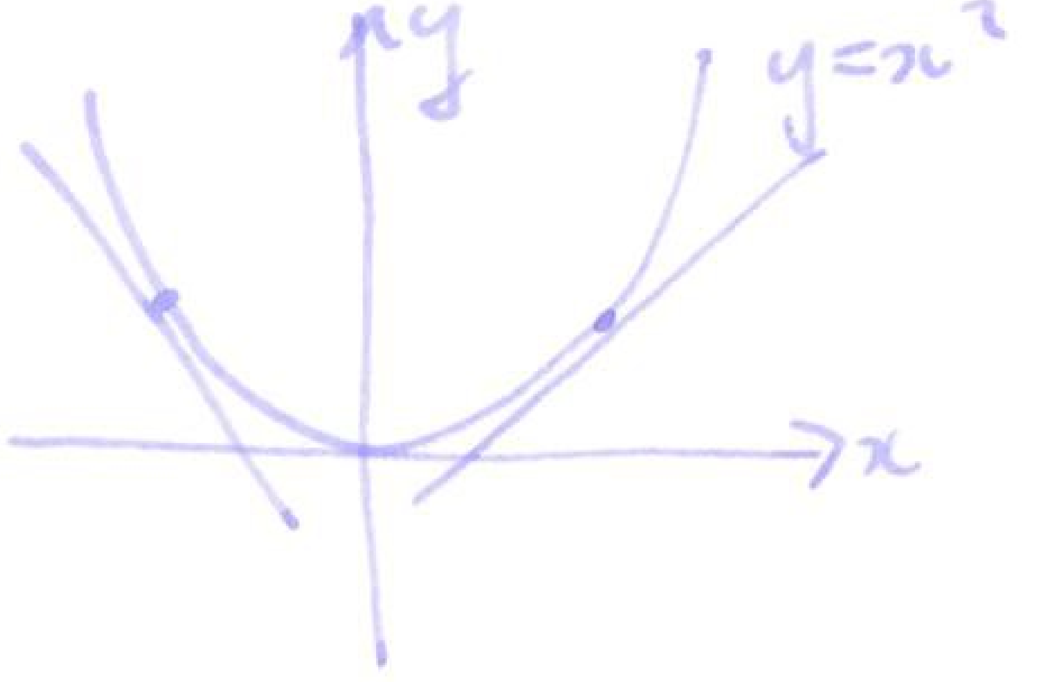
Observation if $f(x)=b$ is a constant function, then $f'(a)=0$ for all $x=a$.

§3.2 Derivative as a function



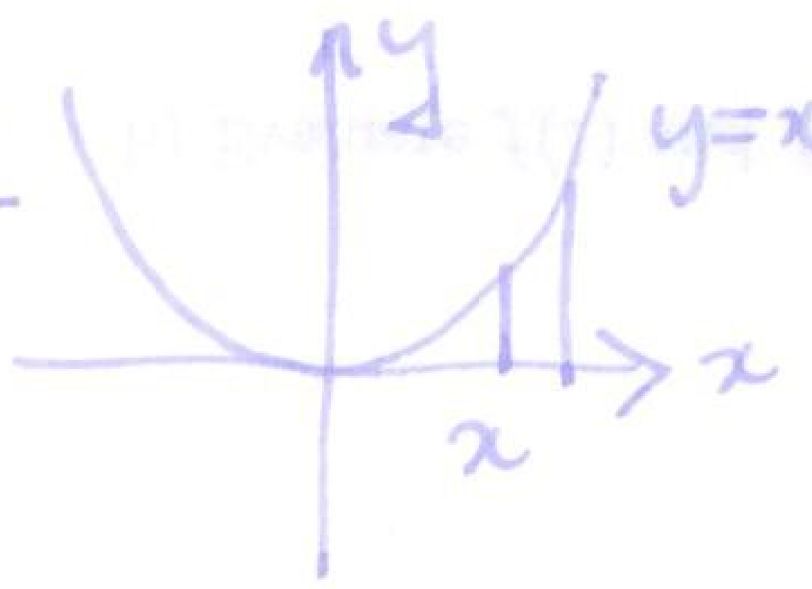
$$f:\mathbb{R}\to\mathbb{R}$$
$$x\mapsto x^2$$

at each point x, there is a slope of the tangent line, so we can define a function $x \longmapsto$ slope of tangent line ~~to~~ thru $(x,f(x))$

notation call this $f'(x)$

say the derivative of f.

Example



slope at x:

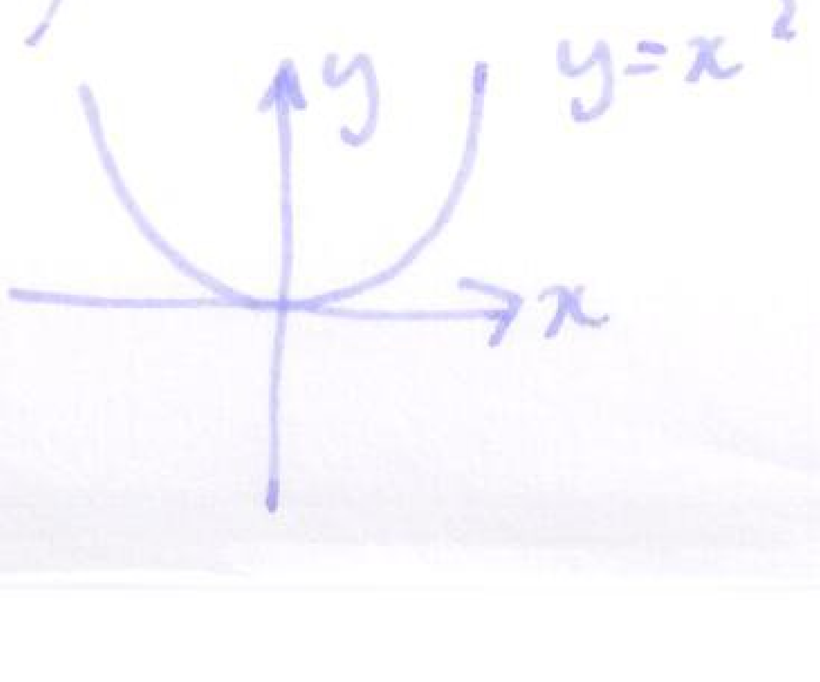
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

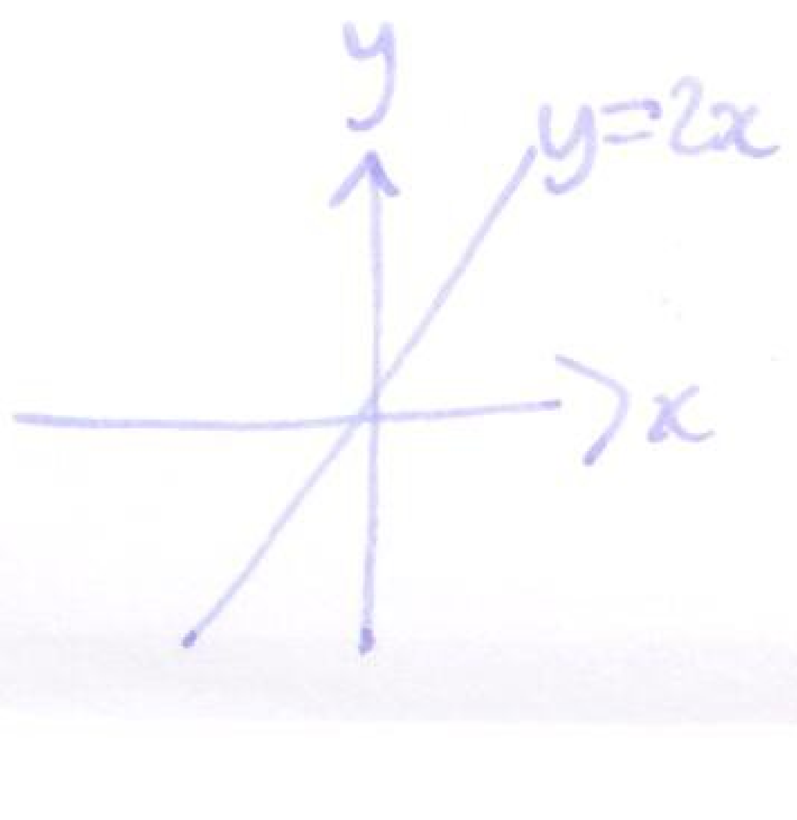
if $f(x)=x^2$

then $f'(x) = \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h}.$

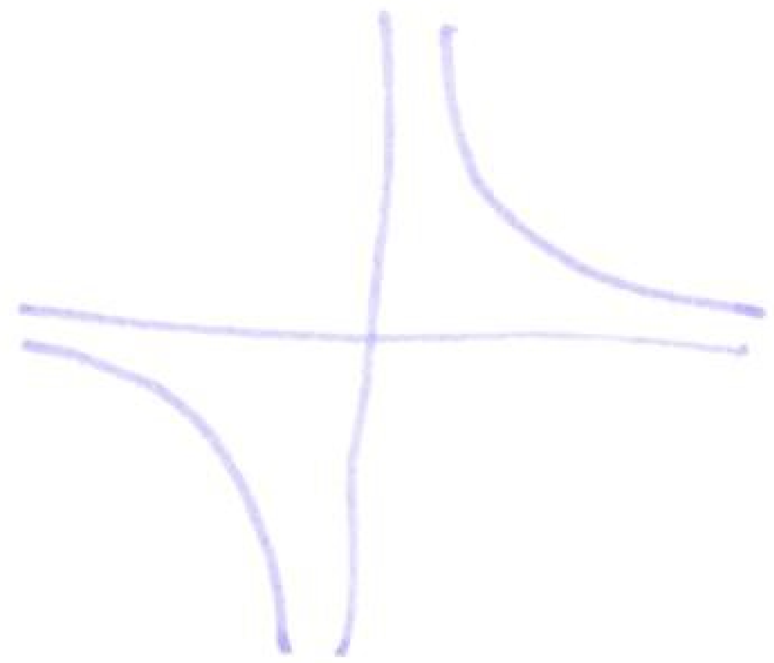
$$= \lim_{h\to 0} 2x+h = 2x.$$

$\therefore$ if $f(x)=x^2$ then $f'(x)=2x$.





Example $f(x) = \frac{1}{x}$ find $f'(x)$

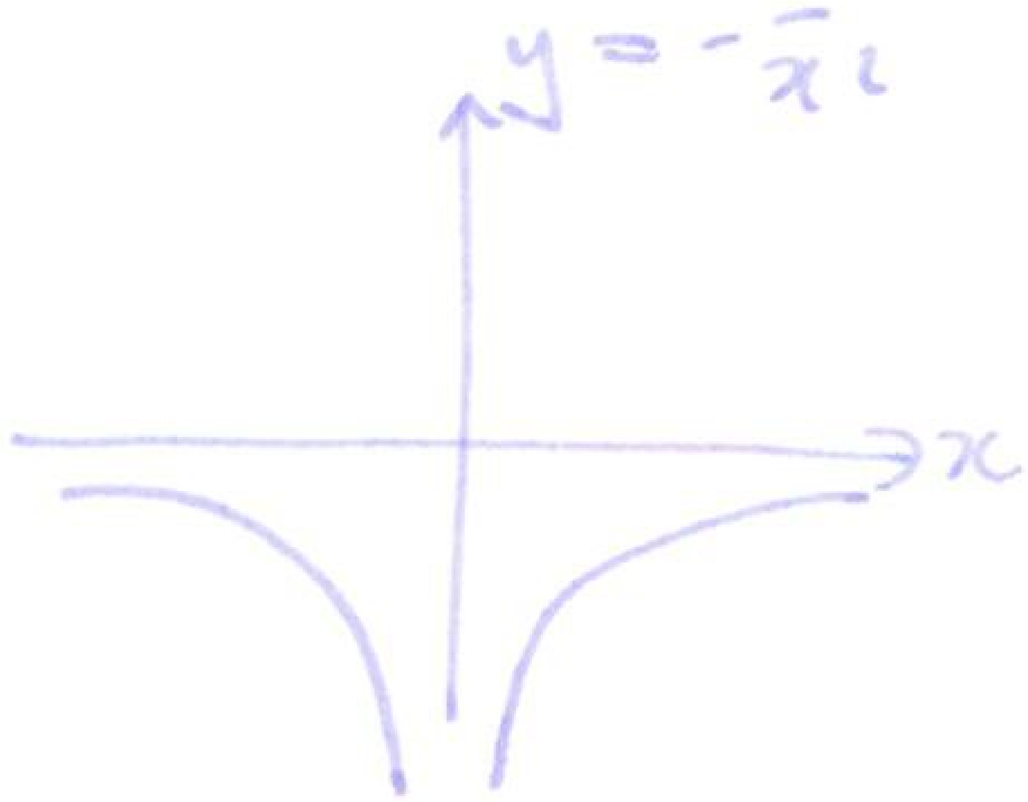


$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{x-(x+h)}{h(x+h)x}$$

$$= \lim_{h\to 0} \frac{-h}{h(x+h)x} = \lim_{h\to 0} \frac{-1}{x^2} = -\frac{1}{x^2}$$

so $f'(x) = -\frac{1}{x^2}$



Remarks

1. functions $f: \mathbb{R} \to \mathbb{R}$.

$$\{\text{functions}\} \xrightarrow{\text{derivative}} \{\text{functions}\}$$
$$f \longmapsto f' \text{ or } \frac{df}{dx}$$

2. "calculus" means rules for doing calculations — we won't have to compute explicit limits all the time.

Example $f(x) = x^3$ $\quad f'(x) = \lim_{x\to 0} \frac{(x+h)^3 - x^3}{h}$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2.$$

<u>Thm</u> Powers of x : $\frac{d}{dx}(x^n) = nx^{n-1}$

<u>Example</u> $\frac{d}{dx}(x^2) = 2x$ $\frac{d}{dx}(x^3) = 3x^2$ $\frac{d}{dx}(x^{-1}) = -x^{-2} = \frac{-1}{x^2}$.

$\frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{+1}{2\sqrt{x}}$.

<u>Proof</u> Let $f(x) = x^n$

then $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$

$(x+h)^n = x^n + nx^{n-1}h + \underbrace{\binom{n}{2}x^{n-2}h^2 + \dots + h^n}_{\text{all terms have at least } h^2 \dots}$ $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

so $f'(x) = \lim_{h\to 0} nx^{n-1}\cancel{\tfrac{h}{h}} + \underbrace{\binom{n}{2}x^{n-2}h + \dots + h^{n-1}}_{\text{all terms contain a power of } h}$

$= nx^{n-1}$ $\square$

<u>Warning</u> this rule works for polynomials, not exponentials.

$f(x) = x^2$ power of x

$f(x) = 2^x$ <u>not</u> a power of x.

other useful rules

Thm (Linearity) If f and g are differentiable, then $f+g$ is differentiable and $(f+g)' = f' + g'$

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

also if c is a constant: $(cf)' = cf'$

$$\frac{d}{dx}(cf) = c\frac{df}{dx}.$$

Proof

$$(f+g)' = \lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$

$$= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right)$$

f, g differentiable means $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists

and $g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$ exists.

so $= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x)$

constant multiple: $(cf)' = \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h} = c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= c f'(x)$. $\square$.

<u>Example</u> $f(x) = x^2 - 3x + 2$ find $f'(x)$.

$$\frac{df}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(-3x) + \frac{d}{dx}(2)$$

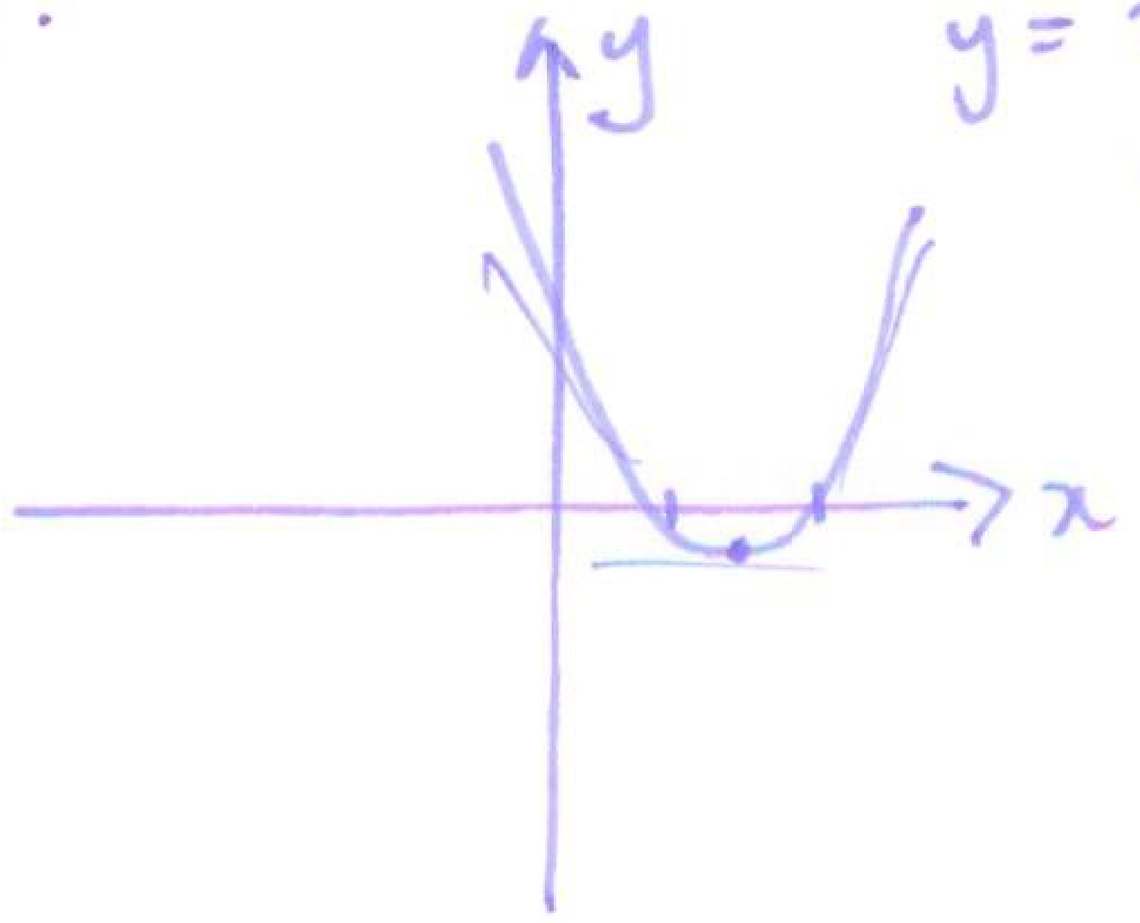
$$= \frac{d}{dx}(x^2) - 3\frac{d}{dx}(x) + \cancel{2}\frac{d}{dx}(2)$$

$$= 2x - 3 + 0 .$$

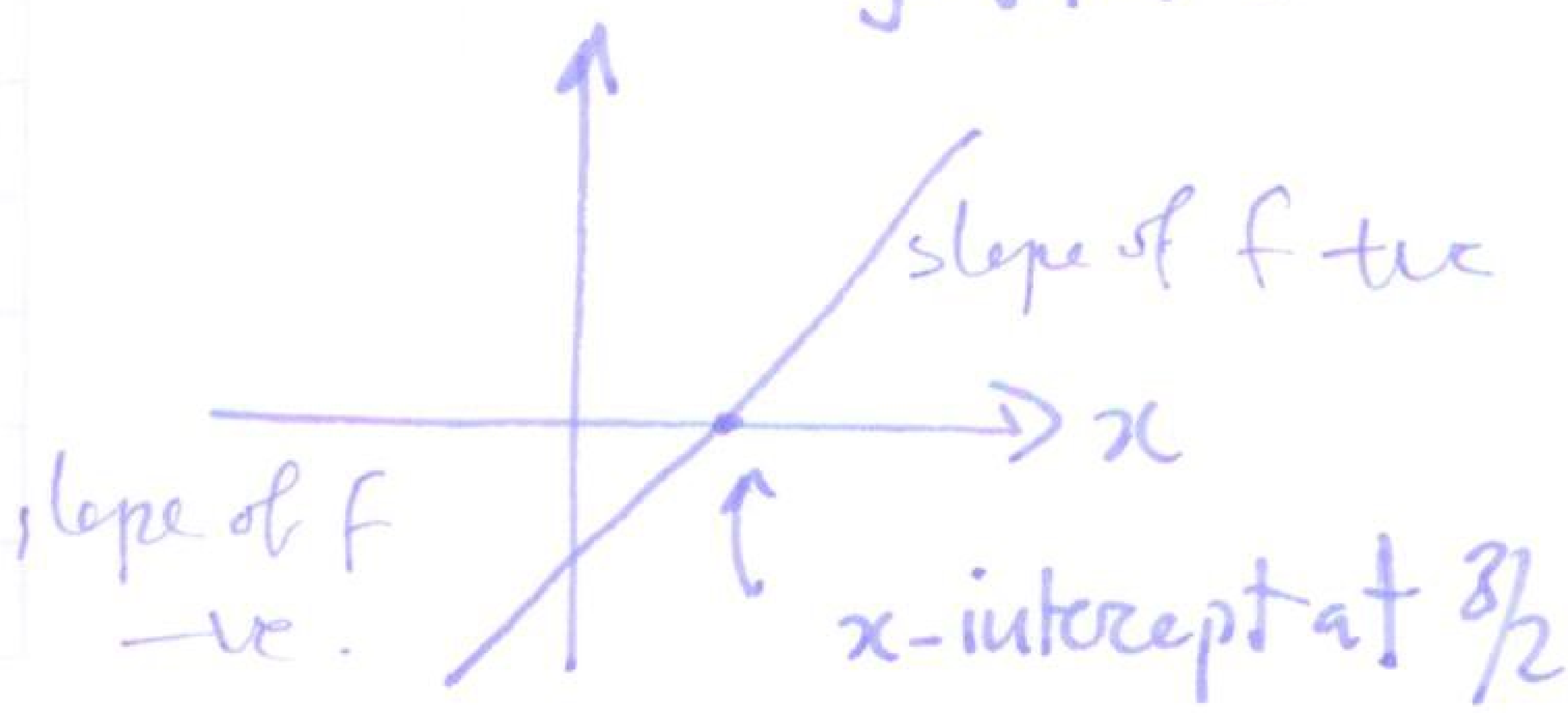
pictures:

$y = x^2 - 3x + 2 = (x-2)(x-1)$

$f(x)$



$y = f'(x)$.



<u>Derivative of e^x</u>

more generally, consider ~~g~~ $f(x) = b^x$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h}$$

$$= \lim_{h \to 0} b^x \left(\frac{b^h - 1}{h}\right) = b^x \underbrace{\lim_{h \to 0} \left(\frac{b^h - 1}{h}\right)}_{\text{assume this limit exists and call it } m_b}$$

exponential functions : derivative is proportional to function

$f(x) = b^x \quad f'(x) = m_b b^x.$

note slope at $x=0$ is $f'(0) = m_b b^0 = m_b$

recall e defined to be number such that slope of e^x at $x=0$ is 1. Therefore if $\boxed{f(x) = e^x \text{ then } f'(x) = e^x}$

$$\boxed{\frac{d}{dx} e^x = e^x}$$

Example differentiate. $f(x) = 7e^x + 4x^2$

$f'(x) = 7e^x + 8x$

Thm: Differentiable $\Rightarrow$ continuous.

If a function $f(x)$ is differentiable at $x = c$, then it is cts at $x = c$.

Proof f differentiable at $x=c$ means $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists. Want to show: $\lim_{x \to c} f(x) = f(c)$ (same as $\lim_{x \to c} f(x) - f(c) = 0$)

consider ~~f(x)~~ note $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

consider $f(x) - f(c) = (x-c)\left(\frac{f(x) - f(c)}{x-c}\right).$

$\lim_{x \to c} f(x) - f(c) = \lim_{x \to c} (x-c) \frac{(f(x) - f(c))}{x-c}.$ product limit.

$= \lim_{x \to c} (x-c) \lim_{x \to c} \frac{f(x) - f(c)}{x-c} = 0 \cdot f'(c) = 0$ ∎.