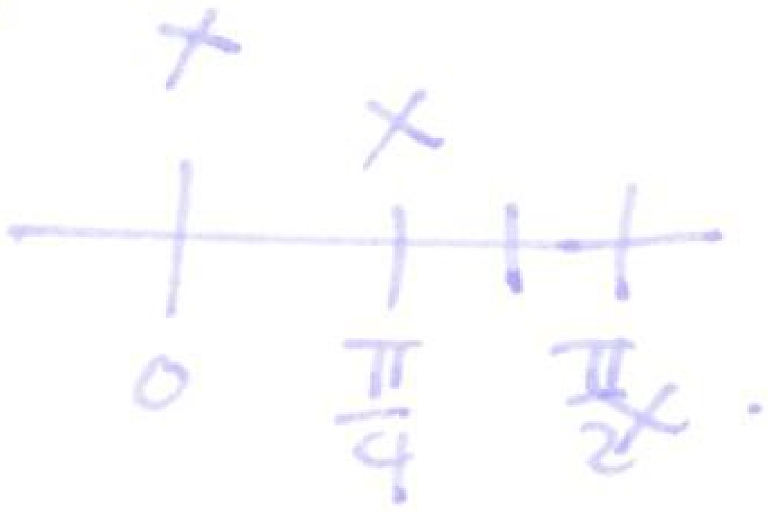
$f(0) = +\infty.$

$f(\frac{\pi}{2}) = \frac{2}{\pi} - \sin(\frac{\pi}{2}) = \frac{2}{\pi} - 1 \simeq -0.3634$

now try halfway point: $\frac{\pi}{4}$

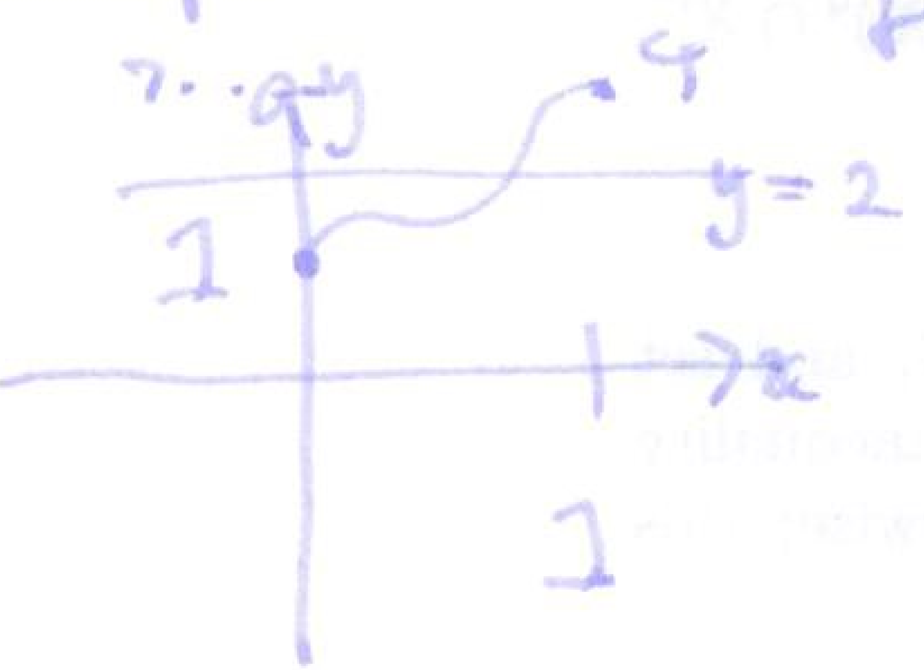


$f(\frac{\pi}{4}) = \frac{4}{\pi} - \sin(\frac{\pi}{4}) \simeq 0.566$ +ve.

so there is a solution in $[\frac{\pi}{4}, \frac{\pi}{2}]$ now try $\frac{3\pi}{8}$ ...

and so on.

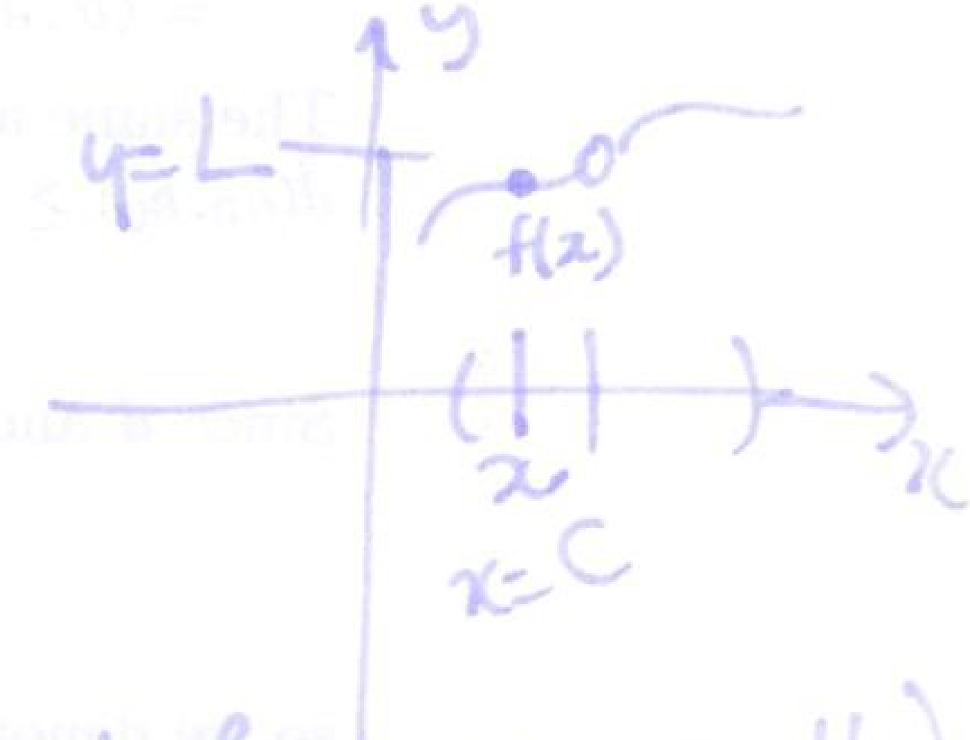
Example show that $\sqrt{c} + \sqrt{c+1} = 2$ for some number c.



c	$f(c) =$ $f(c)$
0	1
1	$1 + \sqrt{2} \simeq 2.4$
2	$\sqrt{2} + \sqrt{3} \simeq$
3	

§2.8 Formal definition of a limit

recall: $\lim_{x \to c} f(x) = L$ if $|f(x) - L|$ becomes arbitrarily small as x gets close to c.



make this precise: give names to things!

$|x - c|$ horizontal gap "small" $< \delta$ (think of as some small number)

$|f(x) - L|$ vertical gap "small" $< \epsilon$

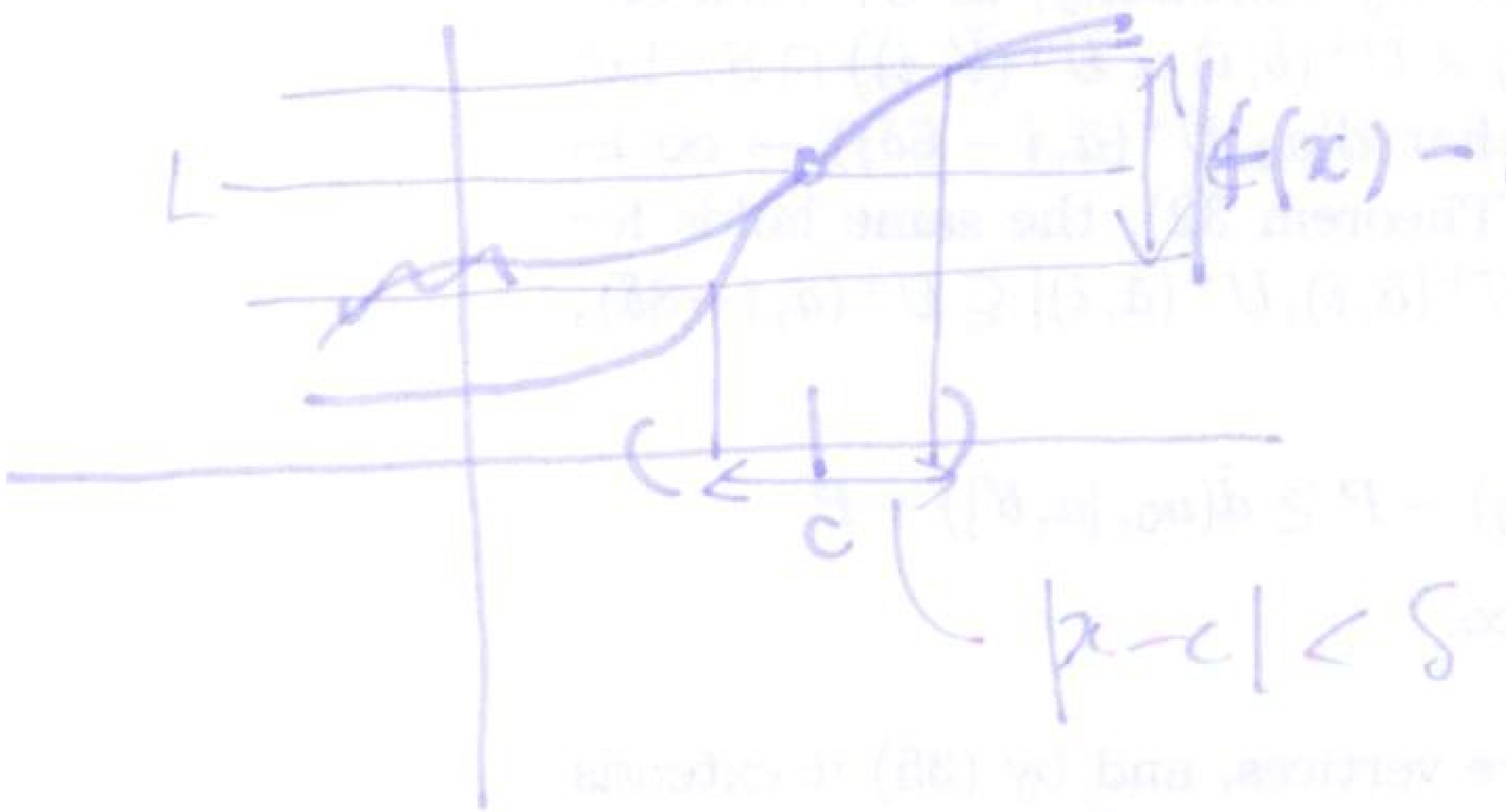
Def^n Suppose $f(x)$ is defined in an open neighbourhood containing c (but not necessarily at c). Then

$$\lim_{x \to c} f(x) = L$$

if for all $\epsilon > 0$, there exists a $\delta > 0$ (δ depends on ϵ, $f(x)$)

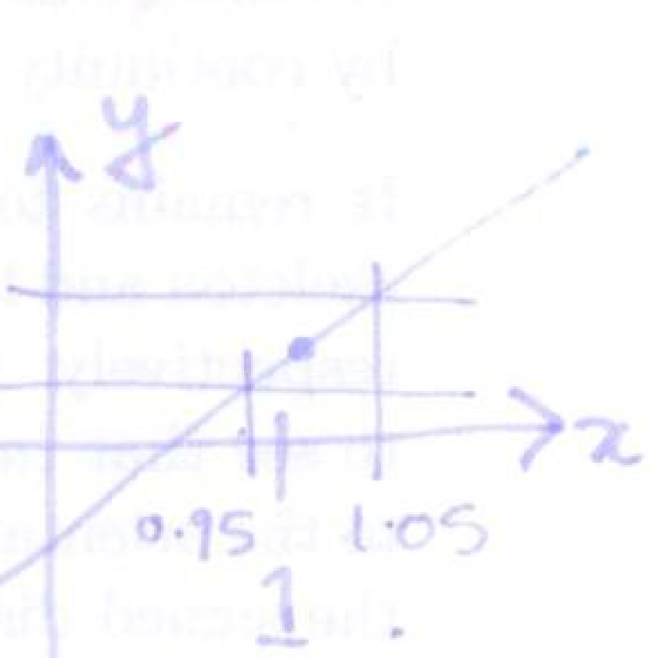
such that $|f(x) - L| < \epsilon$ for all $|x - c| < \delta$

vertical gap — horizontal gap.



given a vertical gap of size ϵ, we can find a horizontal gap of size δ s.t. for all $|x-c| < \delta$, the corresponding ~~value~~ vertical gaps are smaller than ϵ

Example $\lim_{x \to 1} 2x - 1 = 1$



given $\epsilon = \frac{1}{10}$ how big can δ be?

$$\left.\begin{array}{l} 2x - 1 = 1.1 \Rightarrow x = 1.05 \\ 2x - 1 = 0.9 \Rightarrow x = 0.95 \end{array}\right\}$$ so can choose $\delta = 0.05$.

given $\epsilon = \frac{1}{100}$ how big can δ be?

$$\left.\begin{array}{l} 2x-1 = 1.01 \Rightarrow x = 1.005 \\ 2x-1 = 0.99 \Rightarrow x = 0.995 \end{array}\right\}$$ so can chose $\delta = 0.005$.

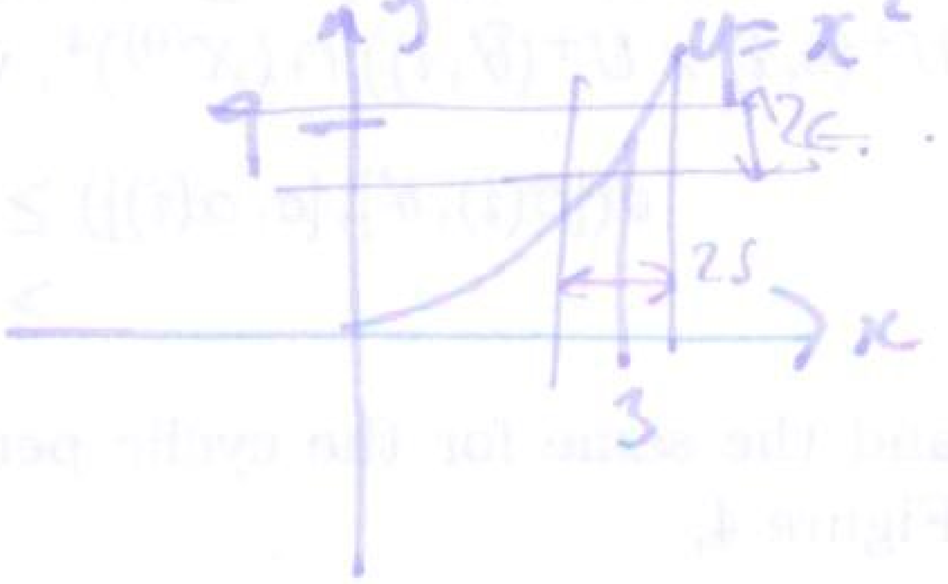
in general, ~~can~~ given ϵ, can chose $\delta = \epsilon/2$.

check: suppose $|x-1| < \epsilon/2$.

then $|f(x)-1| = |2x-1-1| = |2x-2|$

$= 2|x-1| < 2\,\epsilon/2 = \epsilon$ ✓.

<u>Example</u> show $\lim_{x \to 3} x^2 = 9$.



given ϵ, want to find δ s.t.

$|x-3| < \delta \Rightarrow |x^2-9| < \epsilon$.

~~note choose $\delta = \epsilon$ doesn't work.~~

use: $x^2-9 = (x-3)(x+3)$.

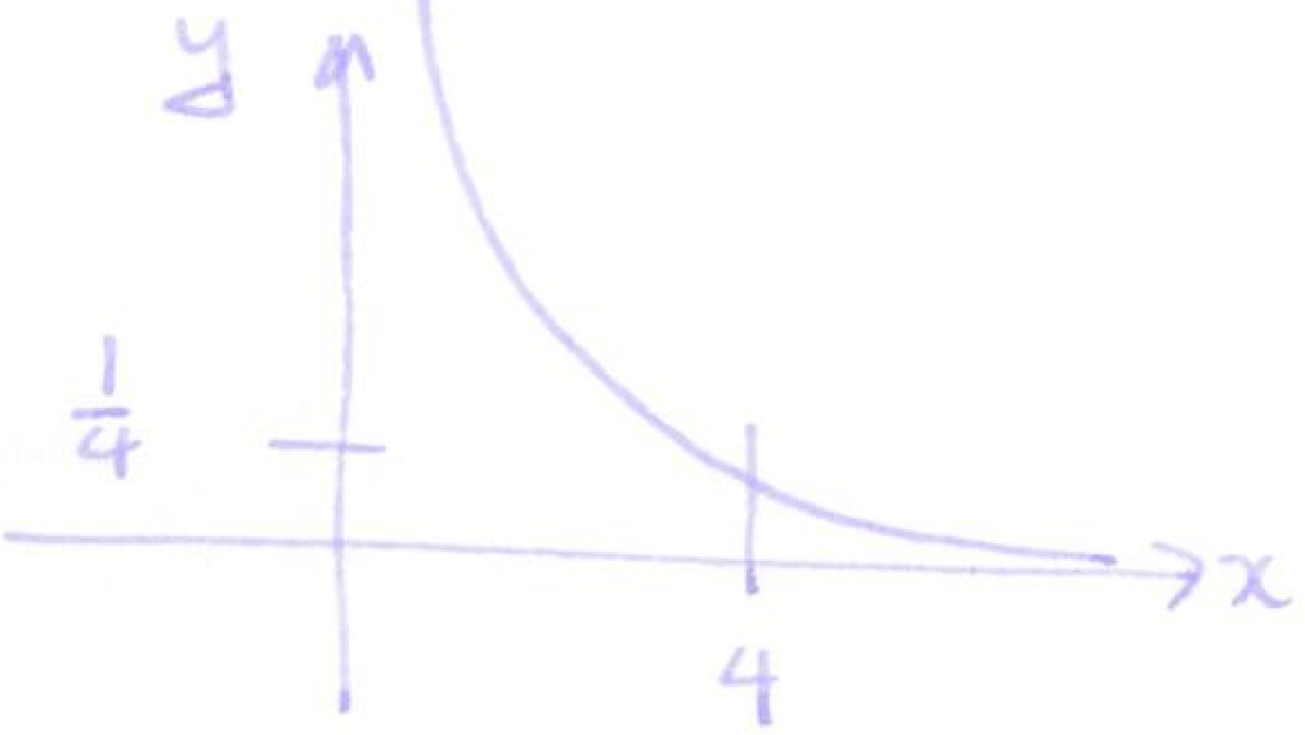
so $|x^2-9| \leq |x-3|\,|x+3|$.

~~so choose~~ assume $\delta \leq 1$ then $|x^2-9| \leq |x-3|.4$.

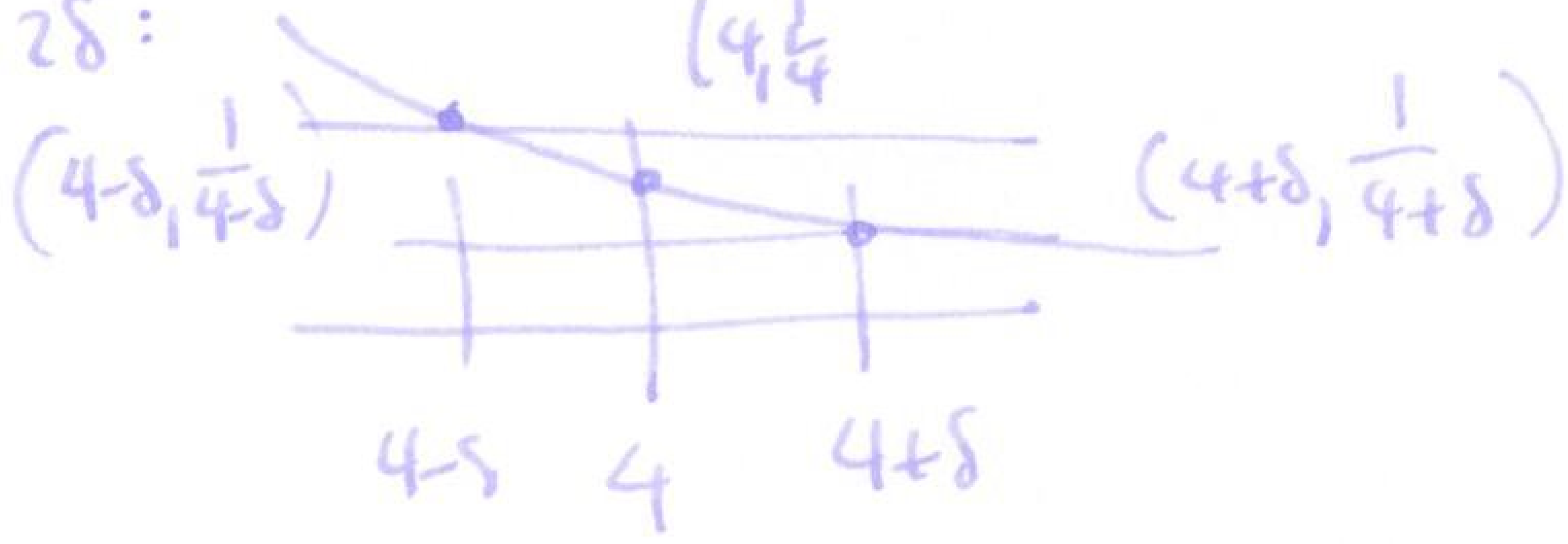
so choose $\delta = \frac{\epsilon}{4}$

then $|x^2-9| \leq \frac{\epsilon}{4}.4 = \epsilon$

Example show $\lim_{x\to 4} \frac{1}{x} = \frac{1}{4}$



close up: suppose horizontal gap is 2δ:



so vertical gap is max:

$$\frac{1}{4-\delta} - \frac{1}{4}, \quad \frac{1}{4} - \frac{1}{4+\delta}.$$

$$= \frac{\delta}{4(4-\delta)} \qquad = \frac{\delta}{4(4+\delta)}$$

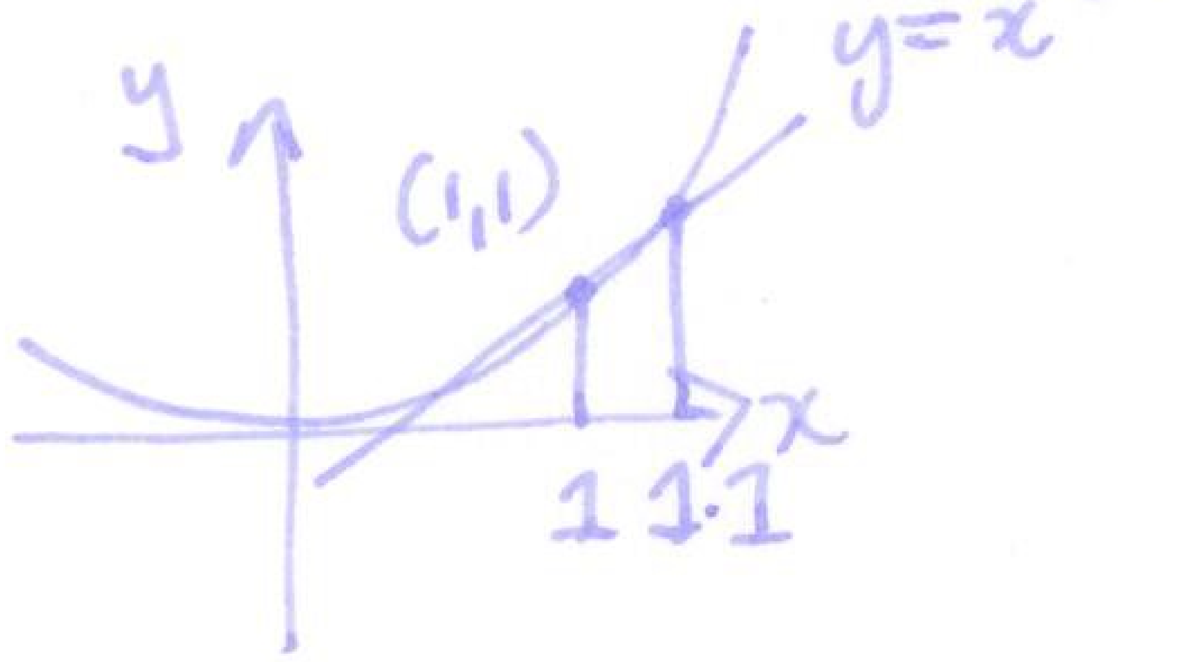
assume $|\delta| < 1$

then $\left| \frac{\delta}{4(4-\delta)} \right| \le \frac{1}{12}|\delta|$

so given ϵ, choose $\delta = \frac{1}{12}\epsilon$.

§3.1 Definition of derivative

recall



we can compute average rate of change of a function over same interval. $[x_1, x_2]$

$\cancel{f(x+a)\ \Delta f}$

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}.$$

e.g. $\dfrac{f(1.1)-f(1)}{1.1-1} = \dfrac{(1.1)^2-1^2}{0.1} \simeq 2.$

Q: how can we compute the slope of the tangent line at $(x, f(x))$?

idea: look at average rate of change on a small interval $[x, x+h]$ and take limit as $h \to 0$.

Defn

slope of tangent line at $x=a$ is $\displaystyle\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

notation: slope of tangent line at $x=a$ written $f'(a)$ or $\dfrac{df(a)}{dx}$

(~~Leibniz~~ Newton) (~~Newton~~ Leibniz)

if this limit exists, we say f is differentiable at $x=a$. (otherwise not differentiable)

note: $\displaystyle\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ equivalent to $\displaystyle\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$.

<u>Def</u>: the tangent line to $f(x)$ at the point $(a, f(a))$ is the line through $(a, f(a))$ with slope $f'(a)$

$$y - y_0 = m(x - x_0)$$

the equation for this line is $y - f(a) = f'(a)(x-a)$

<u>Example</u> Find tangent line to $y = x^2$ at $(1,1)$

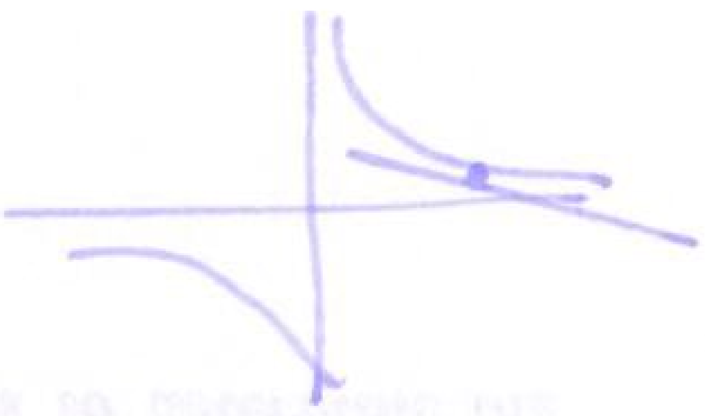
$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2 - 1}{h}$$

$$= \lim_{h\to 0} \frac{1 + 2h + h^2 - 1}{h} = \lim_{h\to 0} \frac{2h + h^2}{h} = \lim_{h\to 0} 2 + h = 2$$

so equation for tangent line is $y - 1 = 2(x-1)$.

<u>Example</u> Find slope of tangent line to $y = 1/x$ at $(4, 1/4)$

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{\frac{1}{4+h} - \frac{1}{4}}{h}$$



$$= \lim_{h\to 0} \frac{1}{h} \frac{4 - (4+h)}{(4+h)4} = \lim_{h\to 0} \frac{-h}{4h(4+h)} = \lim_{h\to 0} \frac{-1}{4(4+h)} = \frac{-1}{16}$$

<u>Example</u> straight line $y = mx + b$ find tangent line at $x = a$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{m(a+h) + b - (ma+b)}{h}.$$

$$= \lim_{h\to 0} \frac{ma + mh + b - ma - b}{h} = \lim_{h\to 0} \frac{mh}{h} = \lim_{h\to 0} m = m.$$