other indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty \cdot 0$, $\infty - \infty$.

Example $\lim_{x\to 3} \dfrac{x^2-4x+3}{x^2+x-12}$ $\qquad \dfrac{9-12+3}{9+3-12} = \dfrac{0}{0}$

indeterminate

factor: $\dfrac{(x-3)(x-1)}{(x-3)(x+4)} = \dfrac{x-1}{x+4} \quad (x \neq 3)$

$$\lim_{x\to 3} \frac{x-1}{x+4} = \frac{2}{7}.$$

Example $\lim_{x\to \frac{\pi}{2}} \dfrac{\tan x}{\cancel{\sin x}\ \sec x}$ $\qquad \tan(\pi/2) = \cancel{\infty}\ \dfrac{\sin(\pi/2)}{\cos(\pi/2)} = 1/0.$

$\cancel{\sin}\ \sec(\pi/2) = \dfrac{1}{\cos(\pi/2)} = 1/0.$

$\dfrac{\infty}{\infty}$ indeterminant form.

note $\dfrac{\tan x}{\sec x} = \dfrac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} = \sin x \quad (\text{if } \cos x \neq 0).$

so $\lim_{x\to \frac{\pi}{2}} \dfrac{\tan x}{\sec x} = \lim_{x\to\frac{\pi}{2}} \sin x = 1$

Example $\lim_{x\to 4} \dfrac{\sqrt{x}-2}{x-4} \quad \dfrac{\sqrt{x}+2}{\sqrt{x}+2}$ $\qquad \lim_{x\to 5} \dfrac{x-5}{\sqrt{x+4}-3} \quad \dfrac{\sqrt{x+4}+3}{\sqrt{x+4}+3} = 6$

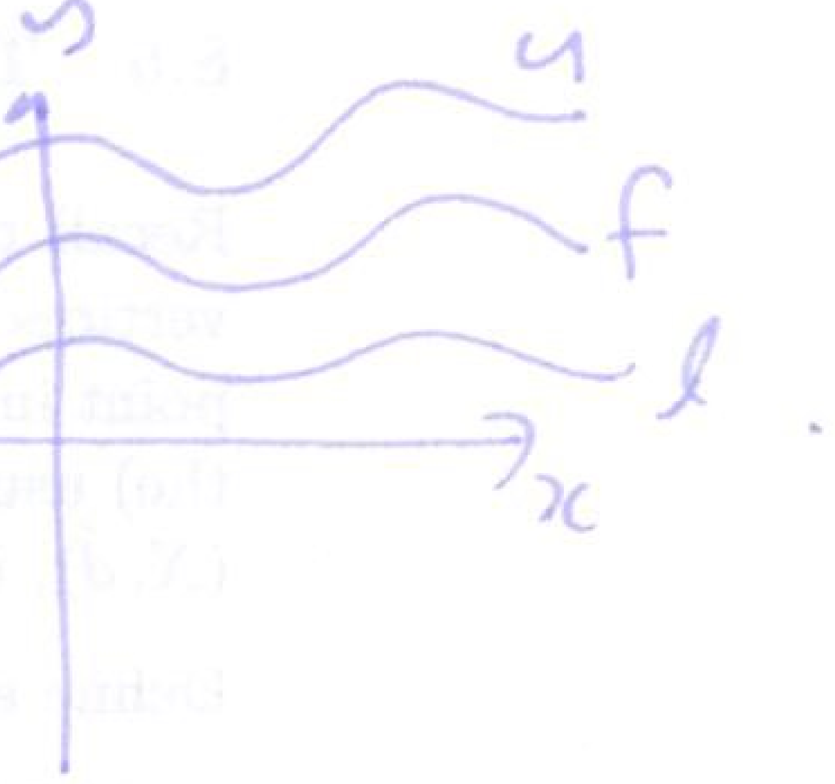
$= \dfrac{1}{4}.$

Example ($\infty - \infty$) $\displaystyle\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right) = \frac{x+1-2}{x^2-1} = \frac{x-1}{x^2-1}$

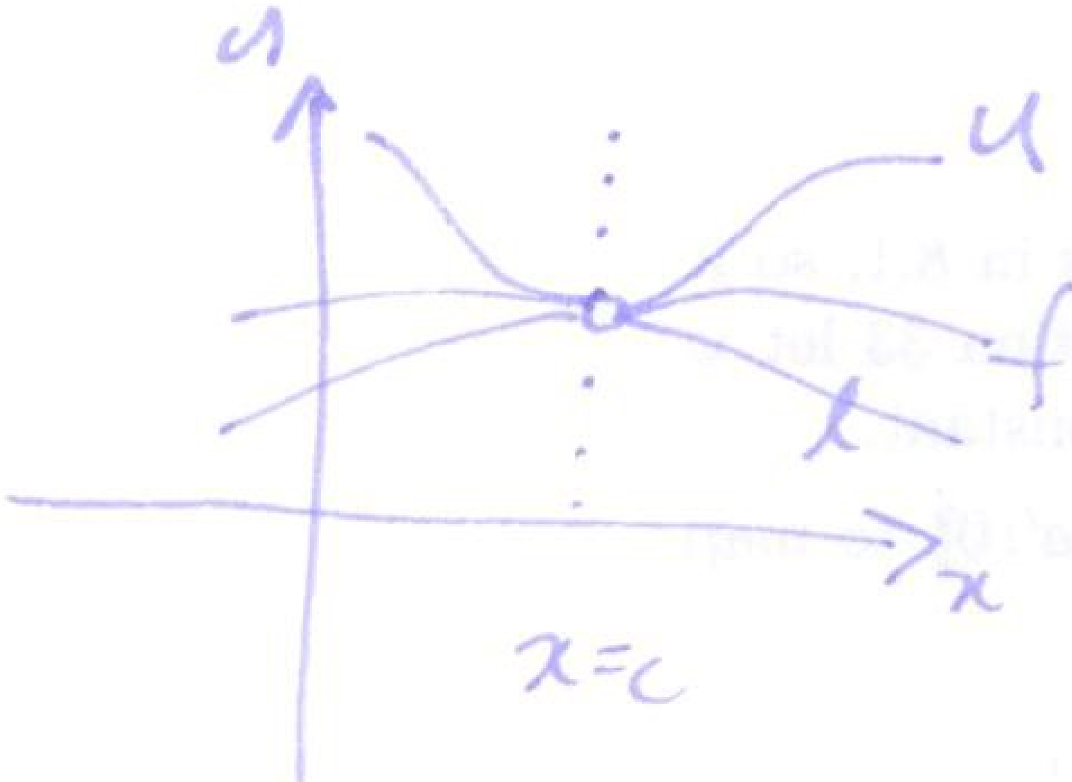
$$= \frac{1}{x+1} = \frac{1}{2}.$$

§2.6 Trigonometric limits

Q: what is $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$?



squeeze theorem: suppose $l(x) \le f(x) \le u(x)$

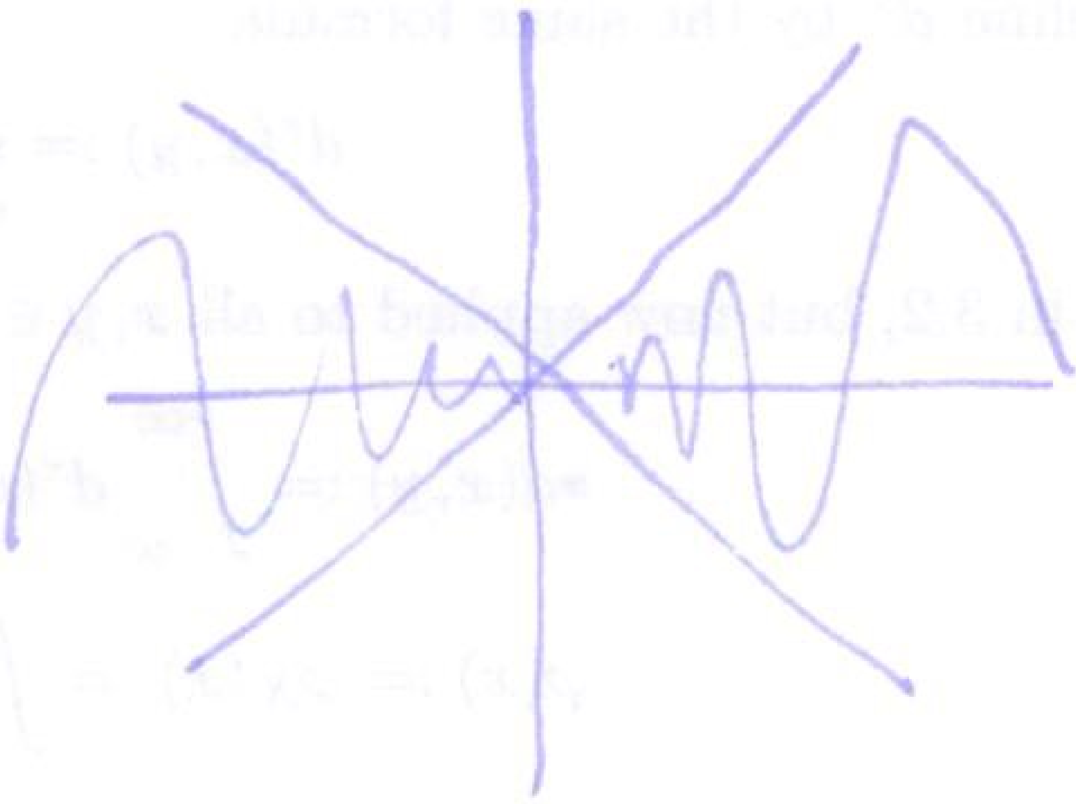


Thm suppose $l(x) \le f(x) \le u(x)$

and $\displaystyle\lim_{x\to c} l(x) = \lim_{x\to c} u(x) = L$

then $\displaystyle\lim_{x\to c} f(x) = L$.

Example $\displaystyle\lim_{x\to 0} x\sin\left(\tfrac{1}{x}\right)$.



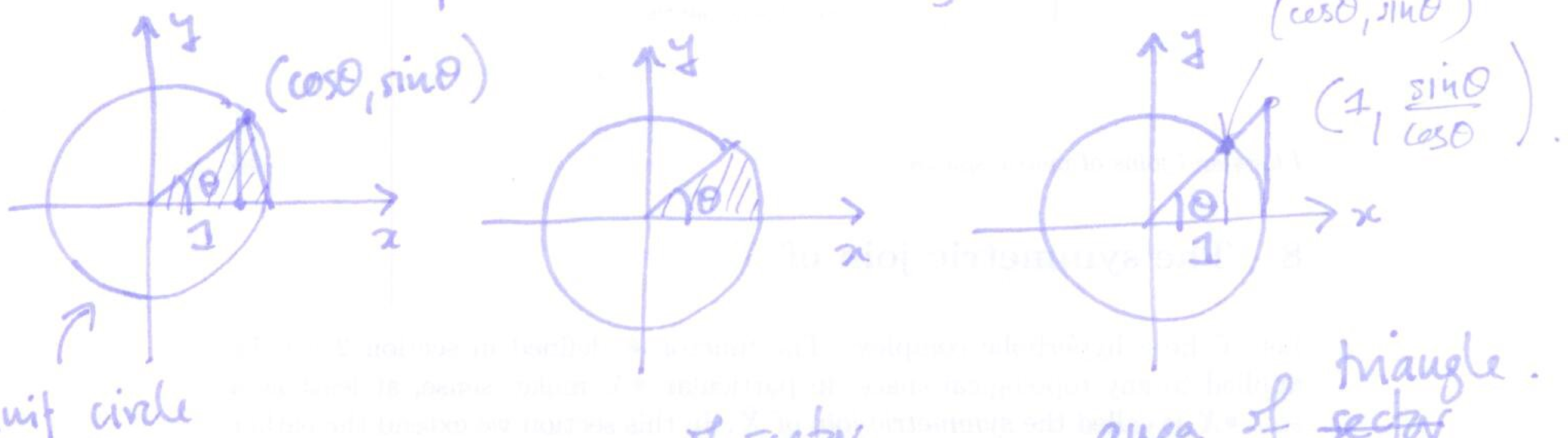
$-|x| \le x\sin\left(\tfrac{1}{x}\right) \le |x|$

$\displaystyle\lim_{x\to 0}|x| = \lim_{x\to 0} -|x| = 0.$ so $\displaystyle\lim_{x\to 0} x\sin(1/x) = 0.$

Thm $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ $\quad\displaystyle\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0$

Proof (of $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$) ~~for $0 < \theta \le \frac{\pi}{2}$~~ .

assume $0 < \theta < \frac{\pi}{2}$, consider the following three areas:



area of sector

area of triangle. ~~sector~~

$\frac{1}{2} b.h.$

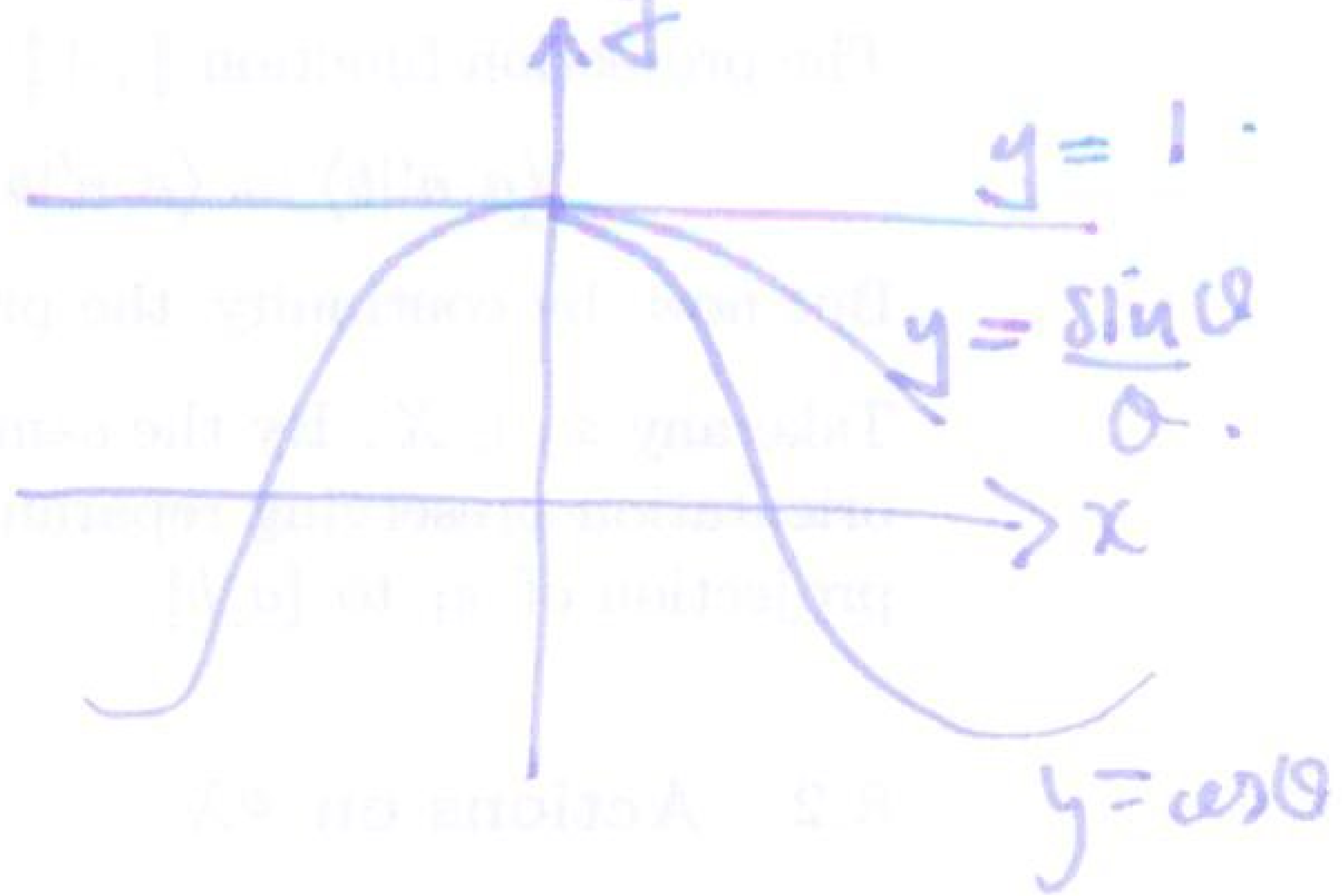
area of triangle:

$$\underbrace{\frac{1}{2}bh = \frac{1}{2} 1 . \sin\theta \le \frac{1}{2}\theta}_{} = \frac{\theta/2\pi}{\pi r_1^2} \quad \underbrace{\frac{1}{2}\theta \le \frac{1}{2} . 1 . \frac{\sin\theta}{\cos\theta}}_{}$$

$$\sin\theta \le \theta \qquad \theta \le \frac{\sin\theta}{\cos\theta}$$

$$\frac{\sin\theta}{\theta} \le 1 \qquad \cos\theta \le \frac{\sin\theta}{\theta} .$$

$$\cos\theta \le \frac{\sin\theta}{\theta} \le 1$$



$$\left.\begin{array}{l} \lim_{\theta \to 0_+} \cos\theta = 1 \\ \lim_{\theta \to 0_+} 1 = 1 \end{array}\right\} \text{ so squeeze theorem implies}$$

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 .$$

Example

1. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{2x}$ Know: $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

write this as: $3x = y$ $x = \dfrac{y}{3}.$

$$\lim_{\frac{y}{3}\to 0} \frac{\sin y}{2(y/3)} = \lim_{y\to 0} \frac{3}{2}\,\frac{\sin y}{y} = \frac{3}{2}\underbrace{\lim_{y\to}\frac{\sin y}{y}}_{1} = \frac{3}{2}.$$

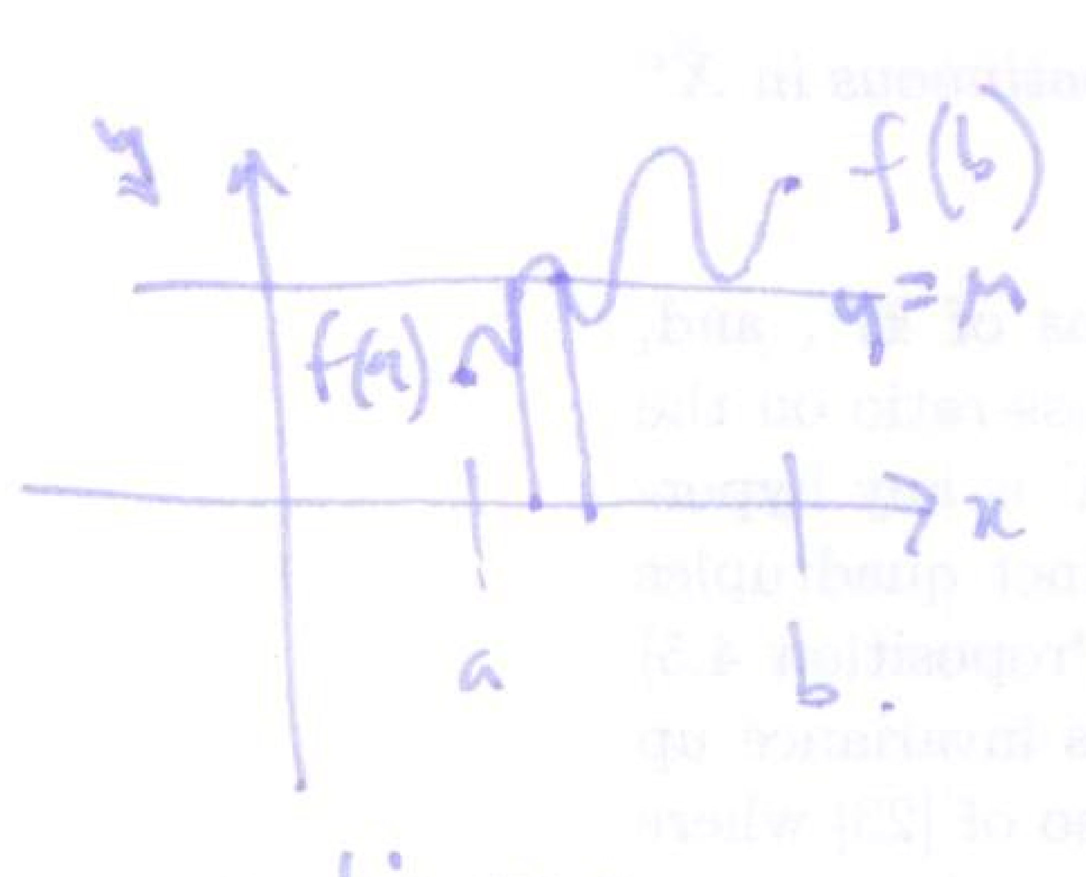
2. $\lim\limits_{t\to 0} \dfrac{1-\cos t}{\sin t} = \lim\limits_{t\to 0} \dfrac{1-\cos t}{t}\,\dfrac{t}{\sin t}$

note: $\lim\limits_{t\to 0} \dfrac{1-\cos t}{t} = 0$ $\lim\limits_{t\to 0}\dfrac{t}{\sin t} = 1$

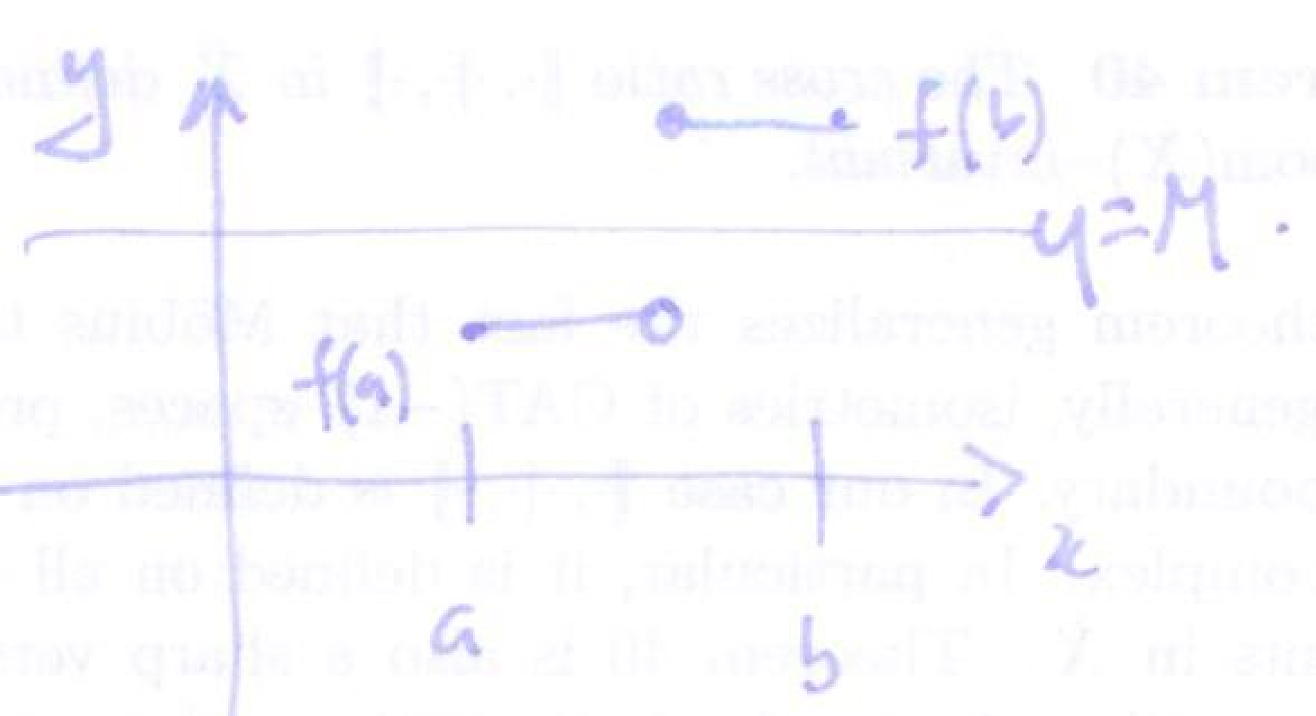
so $\lim\limits_{t\to 0} \dfrac{1-\cos t}{t}\,\dfrac{t}{\sin t} = \lim\limits_{t\to 0}\dfrac{1-\cos t}{t}\,\lim\limits_{t\to 0}\dfrac{t}{\sin t} = 0.1 = 0.$

§2.7 Intermediate value theorem

"continuous functions can't skip values."



continuous function.



discontinuous function