sometimes useful to distinguish right from left limit.

$\lim_{x \to 0+} f(x)$ means right limit

$\lim_{x \to 0-} f(x)$ means left limit

note in order for the limit to exist $\lim_{x \to 0+} f(x)$ must equal $\lim_{x \to 0-} f(x)$

Infinite limits

We say $\lim_{x \to c} f(x) = +\infty$ if $f(x)$ becomes arbitrarily large as $x \to c$

we say $\lim f(x) = -\infty$ if $f(x)$ becomes arbitrarily negative as $x \to c$ (i.e. $-f(x) \to \infty$).

Example $f(x) = \frac{1}{x}$

$\lim_{x \to 0+} f(x) = +\infty$

$\lim_{x \to 0-} f(x) = -\infty$.

} limit does not exist!

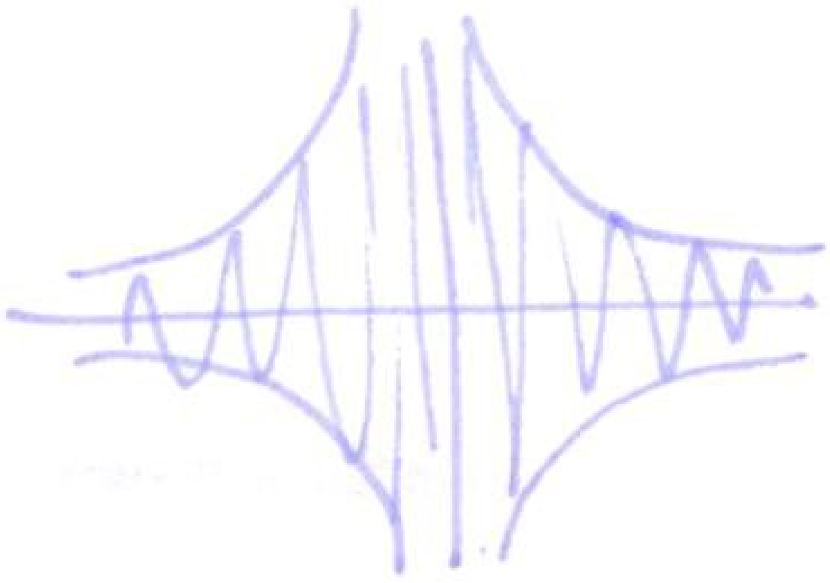
$f(x) = \frac{1}{x^2}$

$\lim_{x \to 0+} f(x) = +\infty$

$\lim_{x \to 0-} f(x) = +\infty$.

} limit exists

Example $f(x) = \dfrac{\sin(\frac{1}{x})}{x}$



no limit as $x \to 0$.

§2.3 Basic limit laws.

Thm assume that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist and are finite.

1) sums: $\lim_{x\to c}\left(f(x)+g(x)\right) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

2) constant multiple: $\lim_{x\to c} k f(x) = k \lim_{x\to c} f(x)$ k constant.

3) products $\lim_{x\to c}\left(f(x)\,g(x)\right) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$.

4) quotients if $\lim_{x\to c} g(x) \neq 0$ then

$$\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

Warning these don't hold if $\lim_{x\to c} f(x)$ or $\lim_{x\to c} g(x)$ does not exist.

Example. $1 = x \cdot \frac{1}{x}$ $\quad \lim_{x\to 0} 1 = 1$ $\quad \lim_{x\to 0} x = 0$

$\lim_{x\to 0} \frac{1}{x}$ DNE.

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<u>Observation</u>

- sum and product laws hold for ~~arbitrary~~ multiple functions.

e.g. $\lim\limits_{x\to c} f_1(x)+f_2(x)+f_3(x) = \lim\limits_{x\to c} f_1(x) + \lim\limits_{x\to c} f_2(x) + \lim\limits_{x\to c} f_3(x)$

<u>Exer.</u> difference $\lim\limits_{x\to c} f(x)-g(x) = \lim\limits_{x\to c} f(x) - \lim\limits_{x\to c} g(x)$

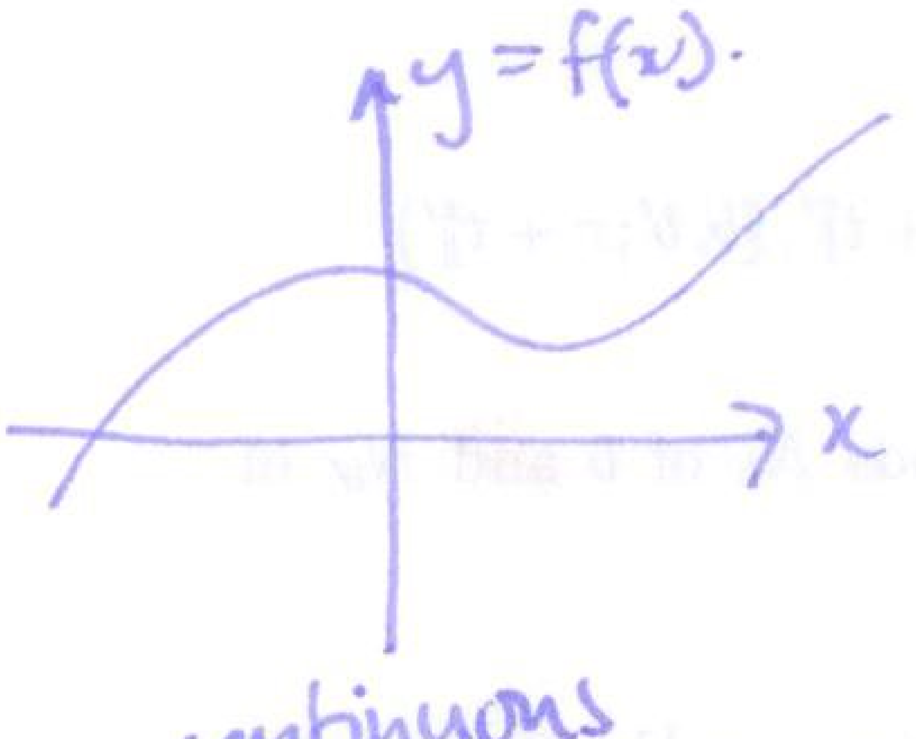
(use sum and constant multiple)

<u>Example</u> $\lim\limits_{x\to 3} x^2 = \lim\limits_{x\to 3} x \lim\limits_{x\to 3} x = 3.3. = 9$.

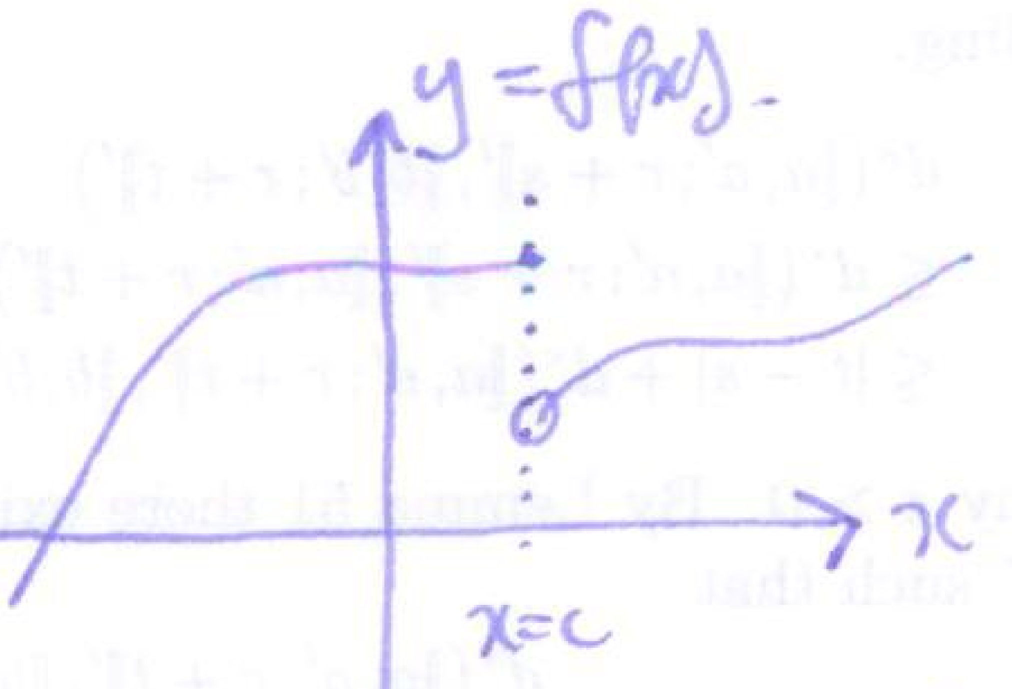
$$\lim_{t\to 2} \frac{t+5}{3t} = \frac{\lim\limits_{t\to 2} t+5}{\lim\limits_{t\to 2} 3t} = \frac{7}{6}.$$

§ 2.4 Limits and continuity

<u>Example</u>



continuous



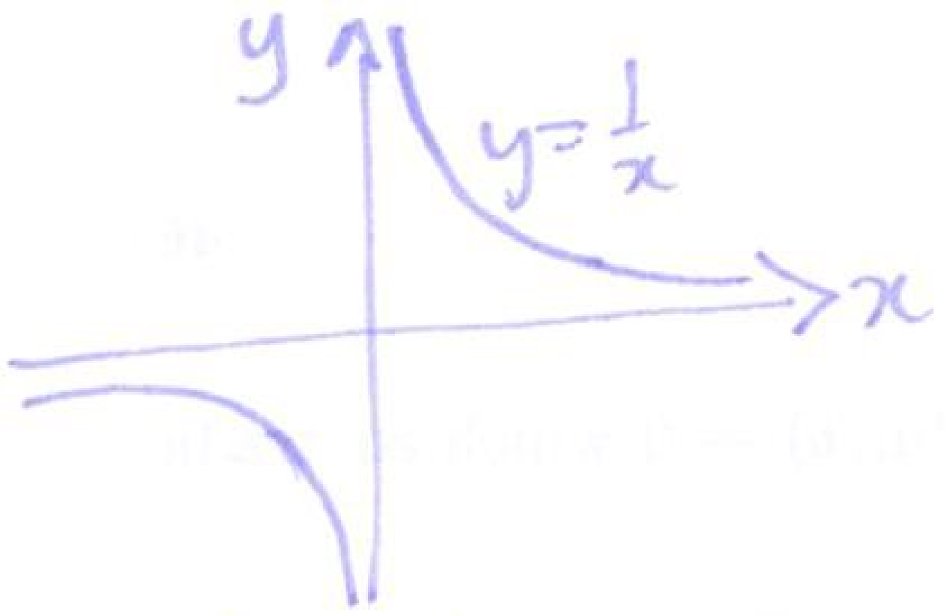
not continuous. at $x=c$

<u>Defn</u> we say $f(x)$ is continuous at $x=c$ if

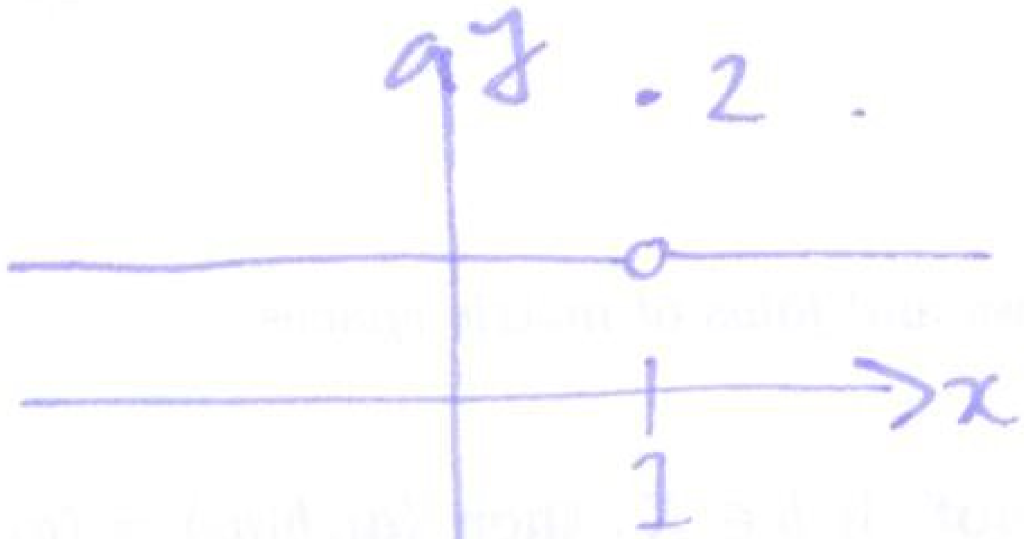
$$\lim_{x\to c} f(x) = f(c)$$

If the limit does not exist, or is not equal to $f(c)$, then f is not continuous (discontinuous) at c.

Example



not cts at 0.

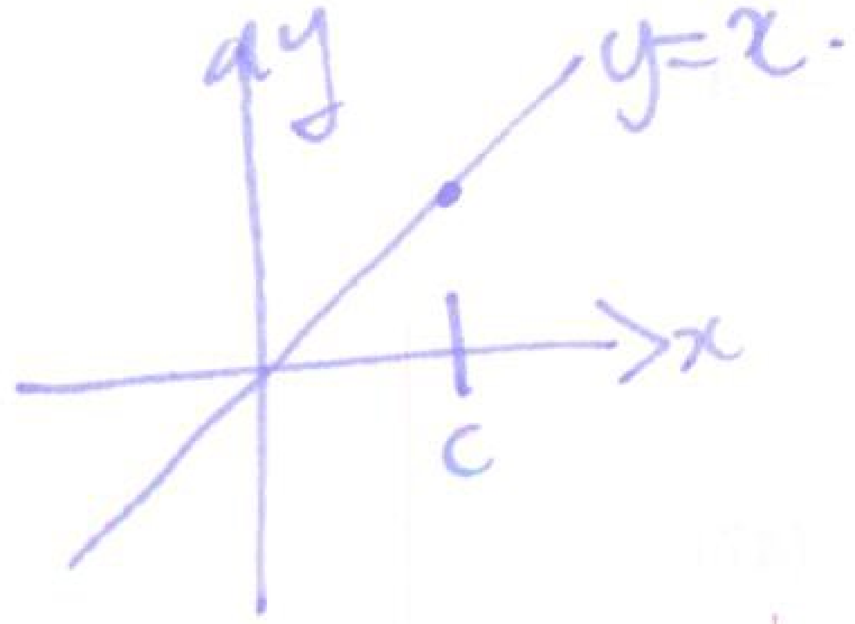


$$f(x) = \begin{cases} 1 & x \neq 1 \\ 2 & x = 1 \end{cases}$$

not cts at $x=1$.

(cts everywhere else though).

Example show $f(x) = x$ is cts.



$f(c) = c$ so want to show $\lim_{x \to c} f(x) = c$.

claim

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c.$$

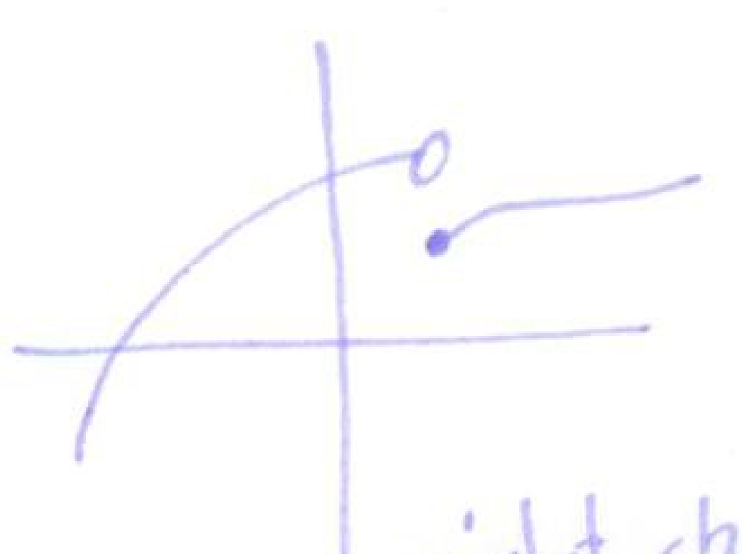
proof need $|f(x) - L|$ to get small as $x \to c$.

$$|x - c| \to 0 \text{ as } x \to c. \quad ✓.$$

Defn $f(x)$ is left continuous at $x=c$ if $\lim_{x \to c-} f(x) = f(c)$

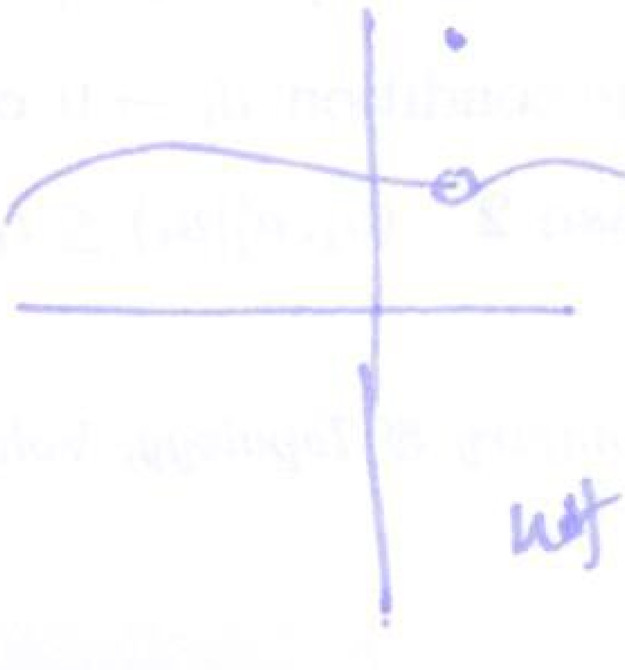
right continuous at $x=c$ if $\lim_{x \to c+} f(x) = f(c)$.

examples.



right cts

left cts



not left or right cts.

if at least one of the left or right limits is ~~discontinuous~~ infinite, at $x=c$

we say $f(x)$ has an infinite discontinuity at $x=c$.

Building continuous functions

Thm 1 Suppose that $f(x)$ and $g(x)$ are both cts at $x=c$ then the following functions are cts at $x=c$

1) $f(x)+g(x)$
2) $kf(x)$ for any constant k.
3) $f(x)\,g(x)$
4) $f(x)/g(x)$ if $g(c)\neq 0$.

Proof: these follow directly from the limit laws.

e.g. if $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist then

$$\lim_{x\to c} f(x)+g(x) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$$ which gives 1).

etc. $\square$.

Thm 2 Polynomials $p(x)$ are continuous.

Rational functions $\frac{p(x)}{q(x)}$ are continuous except for values of x when $q(x)=0$.

Proof $f(x)=x$ is cts.

so $f(x)f(x)=x^2$ is cts $\Rightarrow x^3, x^4 \ldots$ are continuous.

so $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ is cts. $\boxtimes$

so $\dfrac{p(x)}{q(x)}$ is cts at x as long as $q(x) \neq 0$ $\square$.

Useful facts.

Thm 3 Basic functions. • $\sin(x)$, $\cos(x)$ are continuous • $b>0$ b^x cts • $x^{1/n}$ cts $n \in \mathbb{N}$ • $b>0$, $b \neq 1$ $\log_b(x)$ cts

Thm 4 Inverse functions If $f: D \to R$ is continuous with inverse $f^{-1}: R \to D$ then f^{-1} is cts.

Thm 5 Composition $f(g(x))$ if g is cts at $x=c$ and f is cts at $x=g(c)$ then $f(g(x))$ is cts at $x=c$.

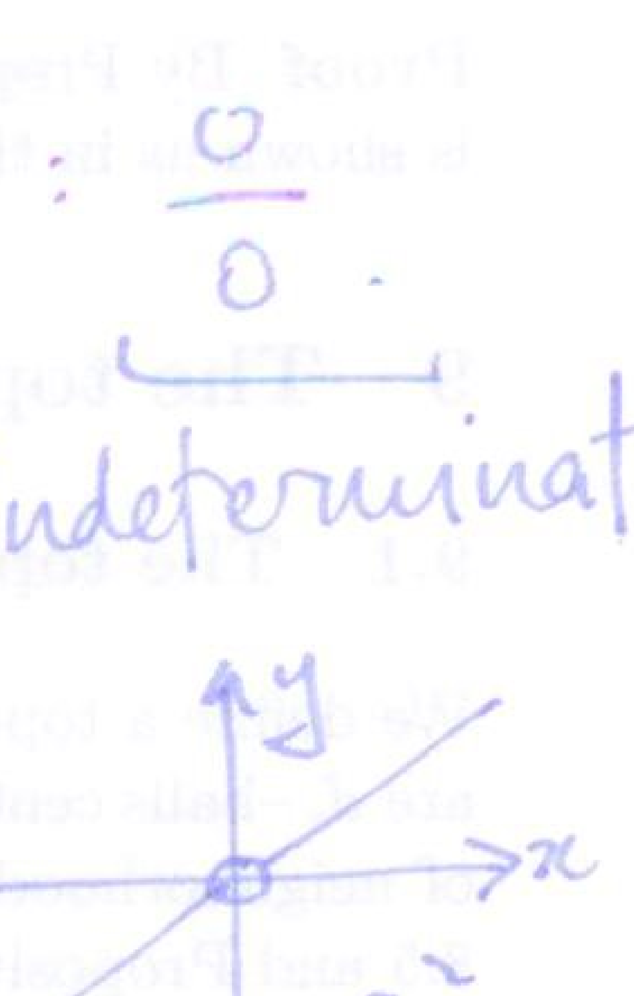
Limits of continuous functions are easy to calculate: just substitute.

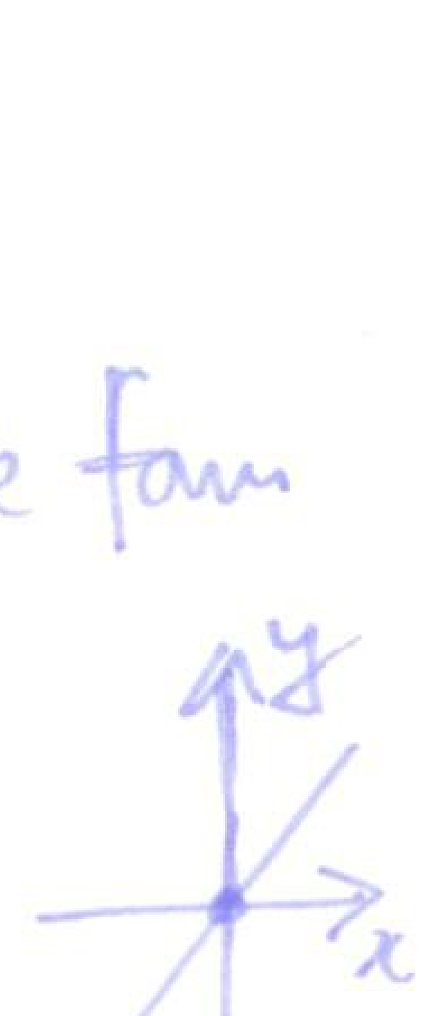
Example $f(x) = \dfrac{2^x + \sin(x)}{\sqrt{x^2+x}}$ $\lim_{x \to 1} f(x) = \dfrac{2^1 + \sin(1)}{\sqrt{1+1}}$.

§2.5 Evaluating limits algebraically

Example $\lim_{x \to 0} \dfrac{x^2}{x}$ $\dfrac{x^2}{x}$ undefined at 0: $\dfrac{0}{0}$ indeterminate form

but limit does not depend on value at 0.





so $\dfrac{x^2}{x} = \dfrac{x}{1} = x$ for $x \neq 0$.

so $\lim_{x \to 0} \dfrac{x^2}{x} = \lim_{x \to 0} x = 0$.