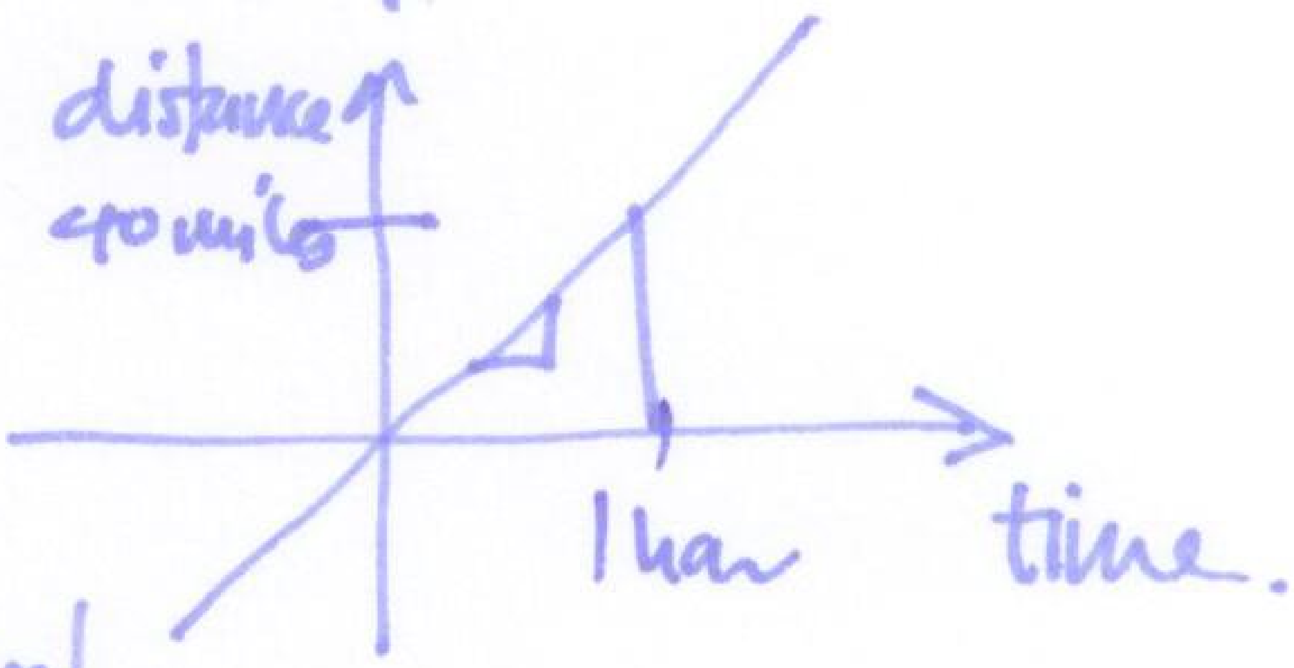
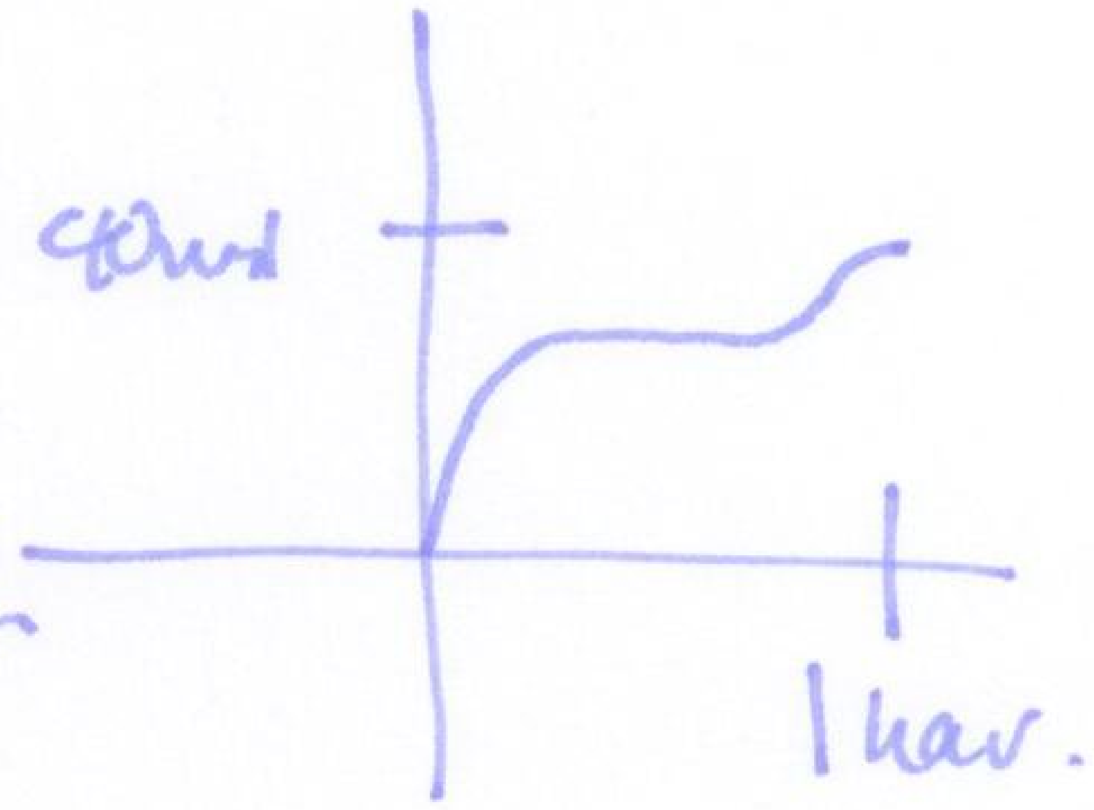
§2.1 Limits, rates of change, tangent lines

motivation: velocity example: driving at constant speed.

velocity $= \dfrac{\text{distance}}{\text{time}}$ = slope of line = 40 mph.

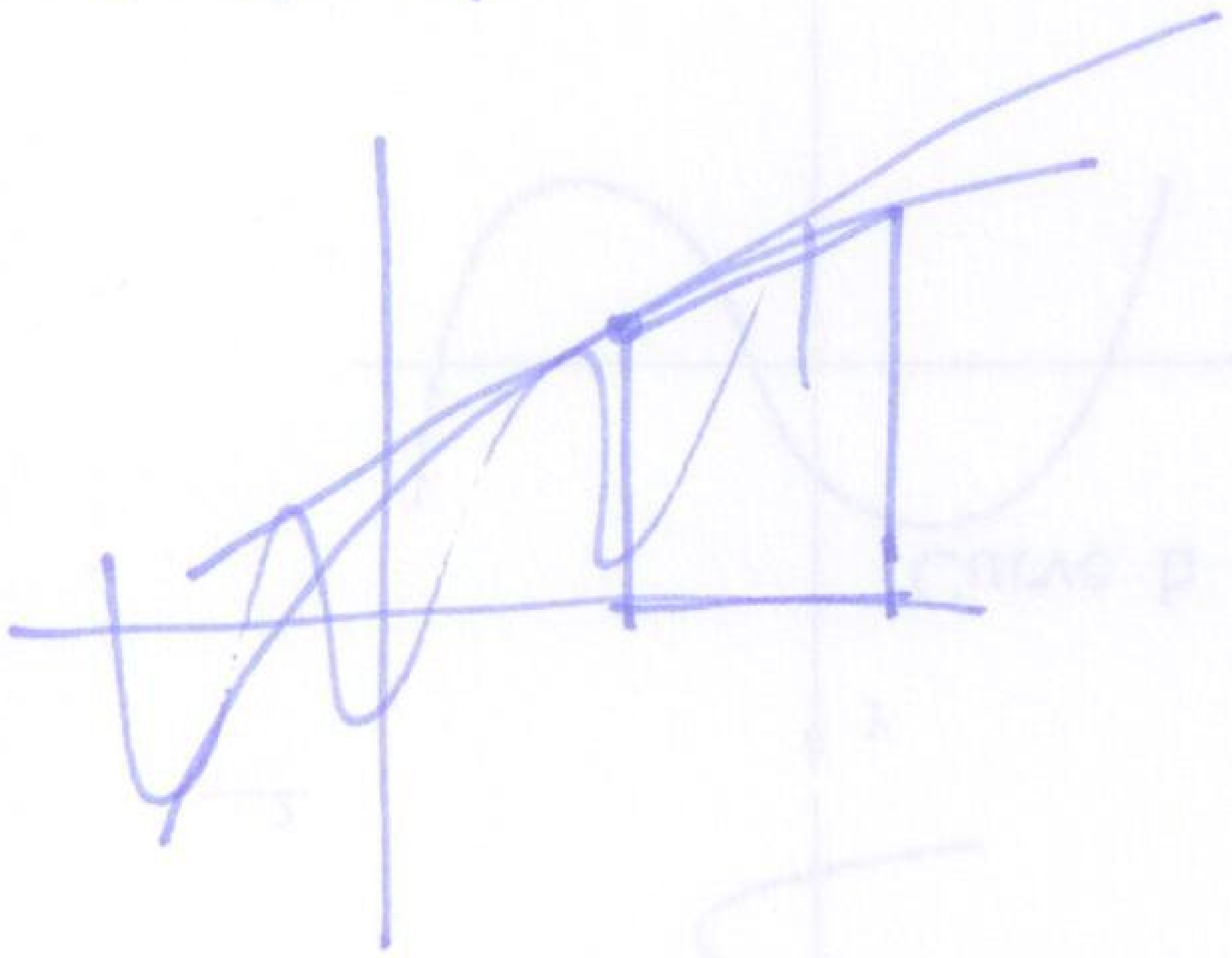


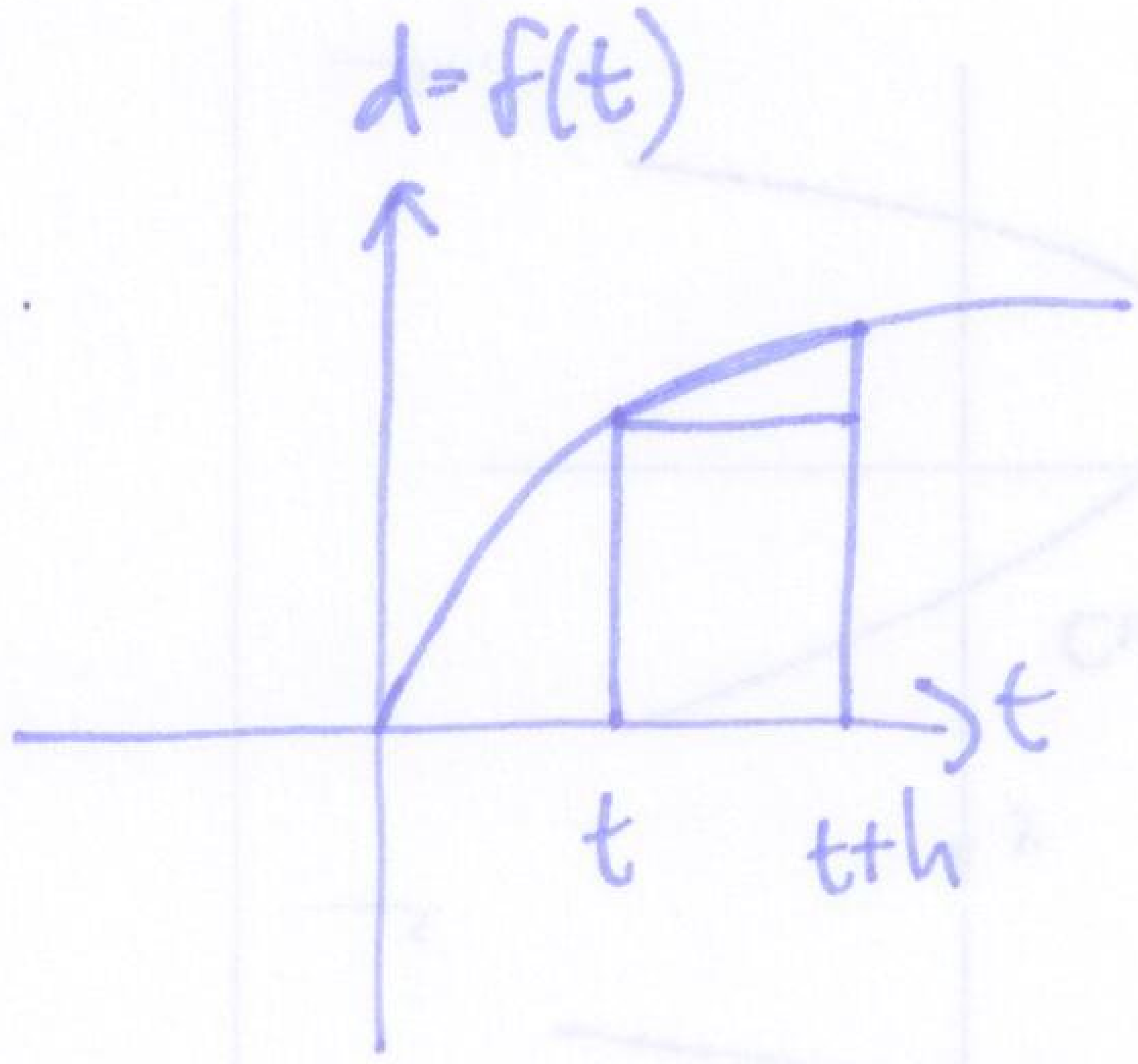
problem: what happens if you don't travel at constant speed?



average speed on whole time $=$ $\dfrac{\text{distance travelled in time interval}}{\text{length of time}}$ $=$ 40mph

can look at average speed over any time interval (even very short times).

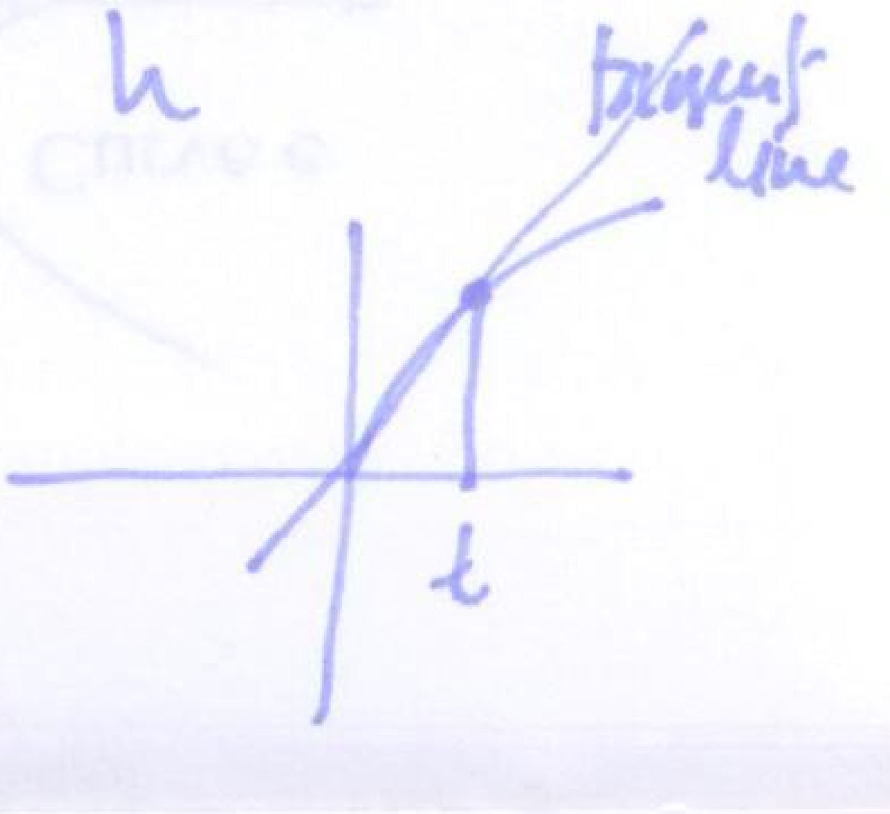




average speed on time interval $[t, t+h]$ is $\dfrac{f(t+h)-f(t)}{h}$

Q: what is the speed at time t?

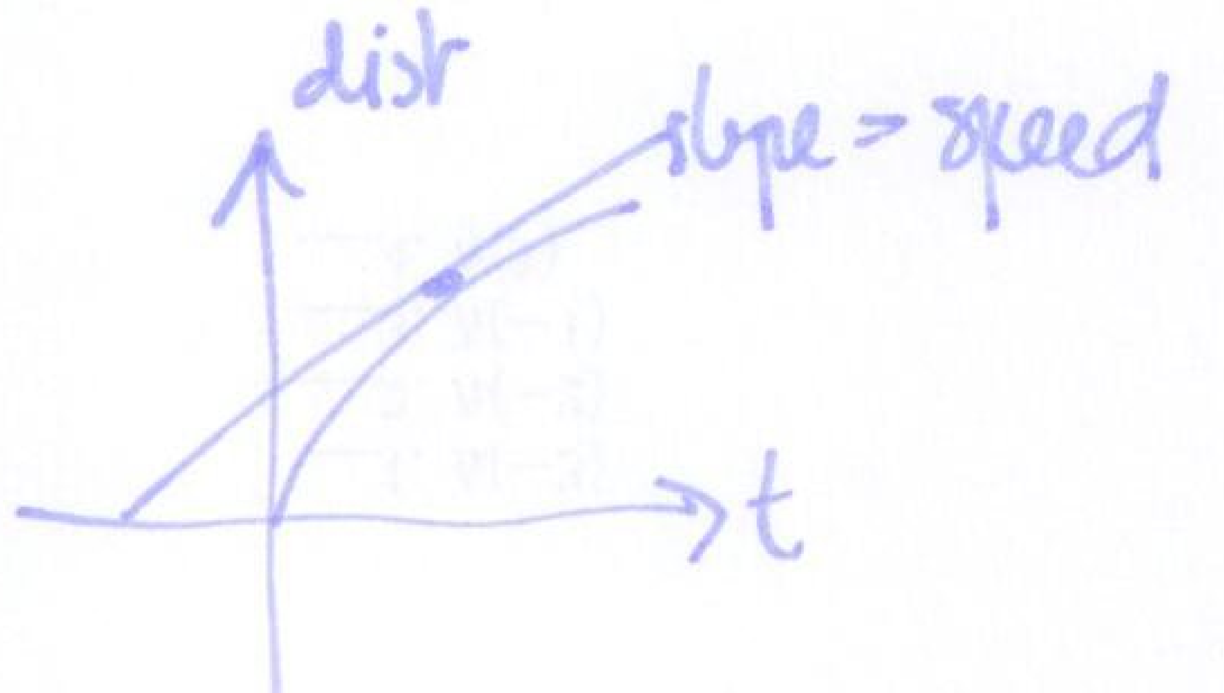
A: given by slope of tangent line at $(t, f(t))$

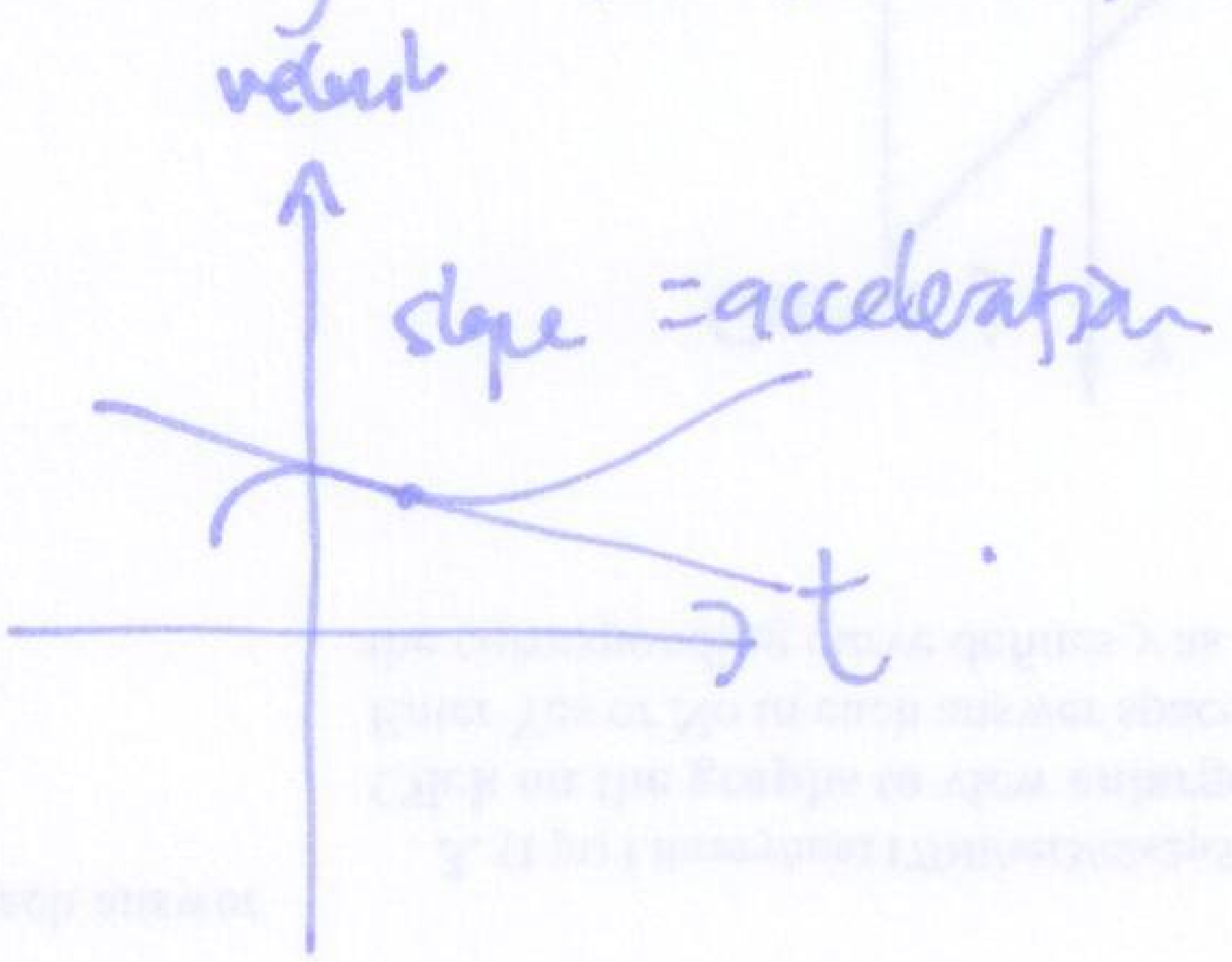


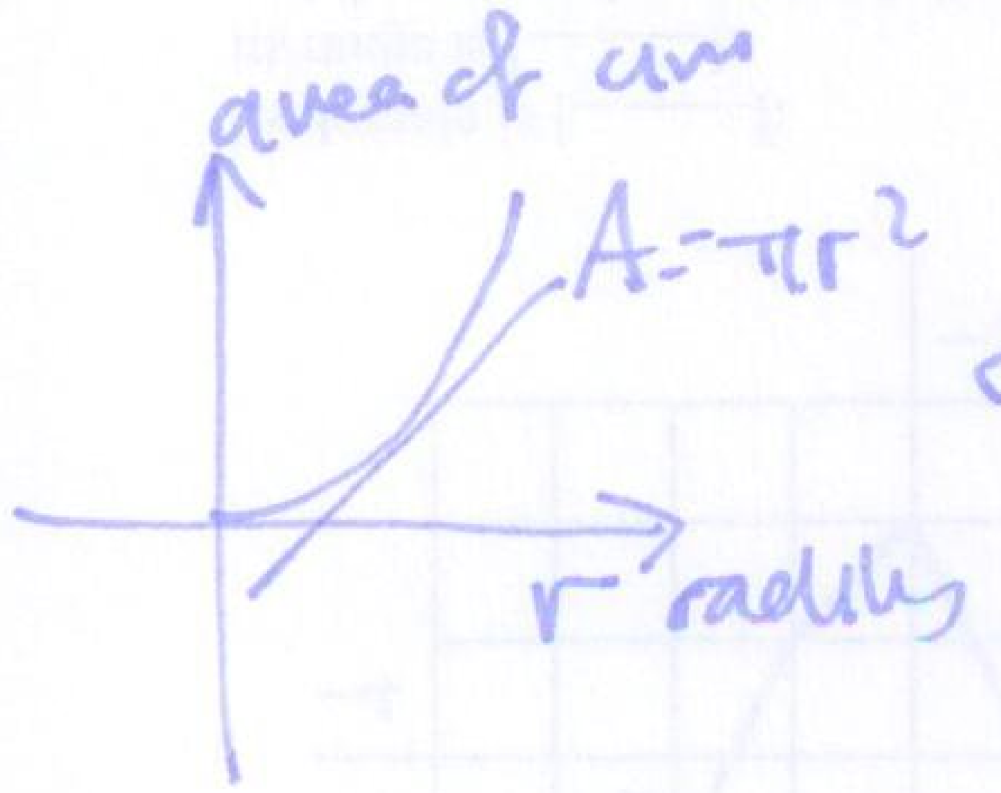
hope: as length of interval $[t, t+h]$ gets small the average speed ~~we~~ gets close to slope of tangent line.

this works for "nice" functions.

Observation: this works for any rate of change / function.



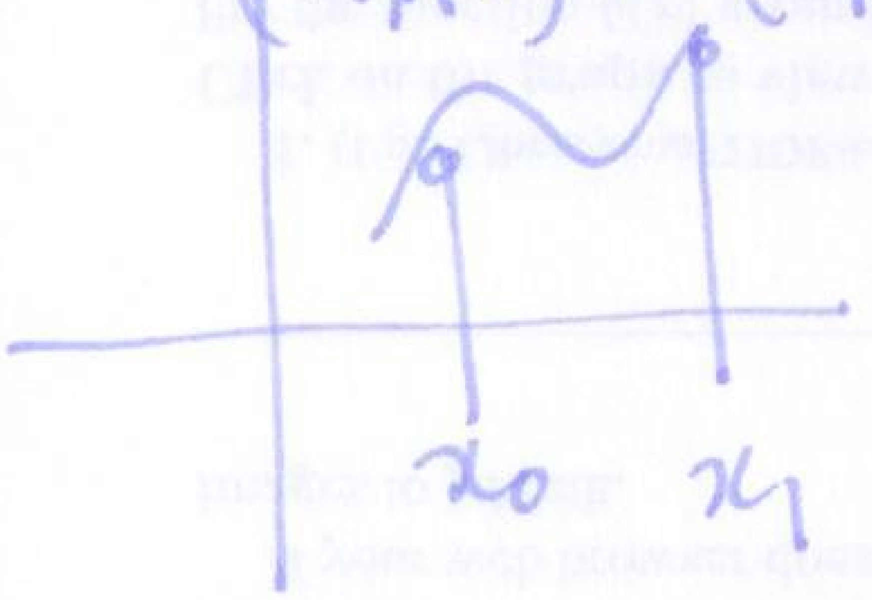




slope = rate of change of area of circle with radius.

summary average rate of change of $f(x)$ on interval $[x_0, x_1]$.

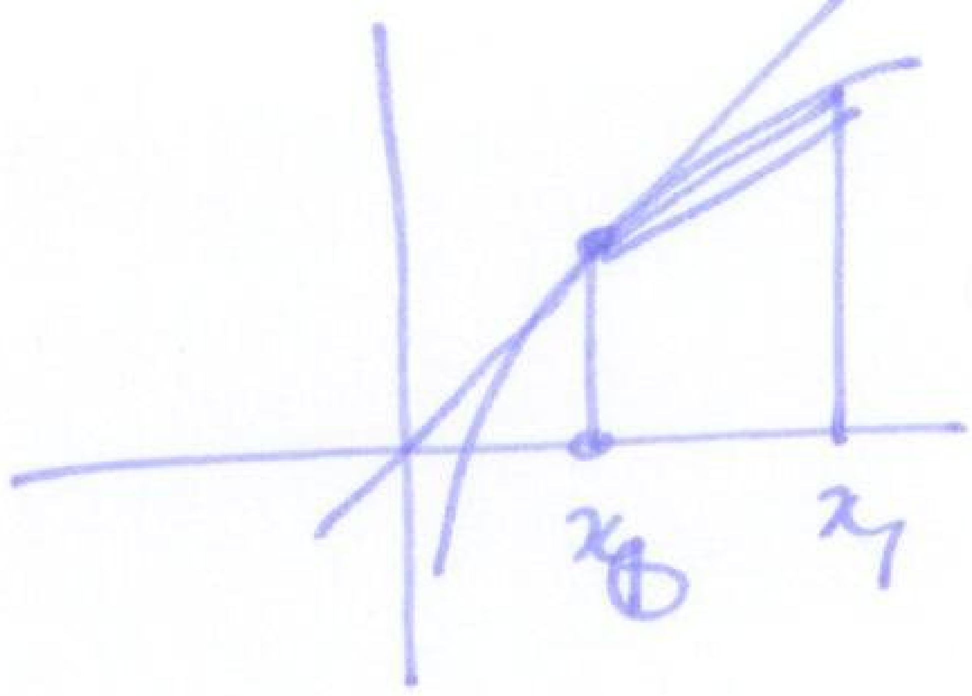
$(x_0, f(x_0))$ $(x_1, f(x_1))$ is



$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

§2.2 Limits

want to find slopes of tangent lines:



average rate of change: $\dfrac{f(x_1)-f(x_0)}{x_1-x_0}$.

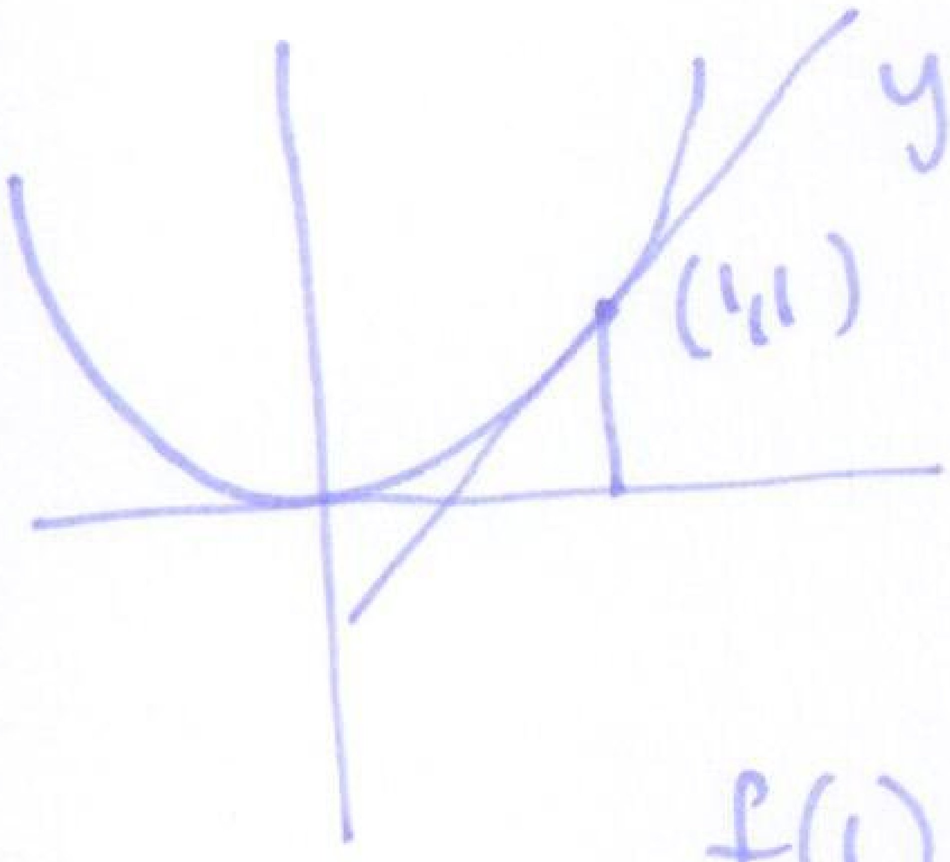
Q: can we just set $x_1=x_0$? A. $\dfrac{f(x_0)-f(x_0)}{x_0-x_0} = \dfrac{0}{0}$ undefined

no.

observation

① if we draw pictures the average slopes seem to get closer to the actual slope as length of interval gets small.

② seems to work for sample calculations too:



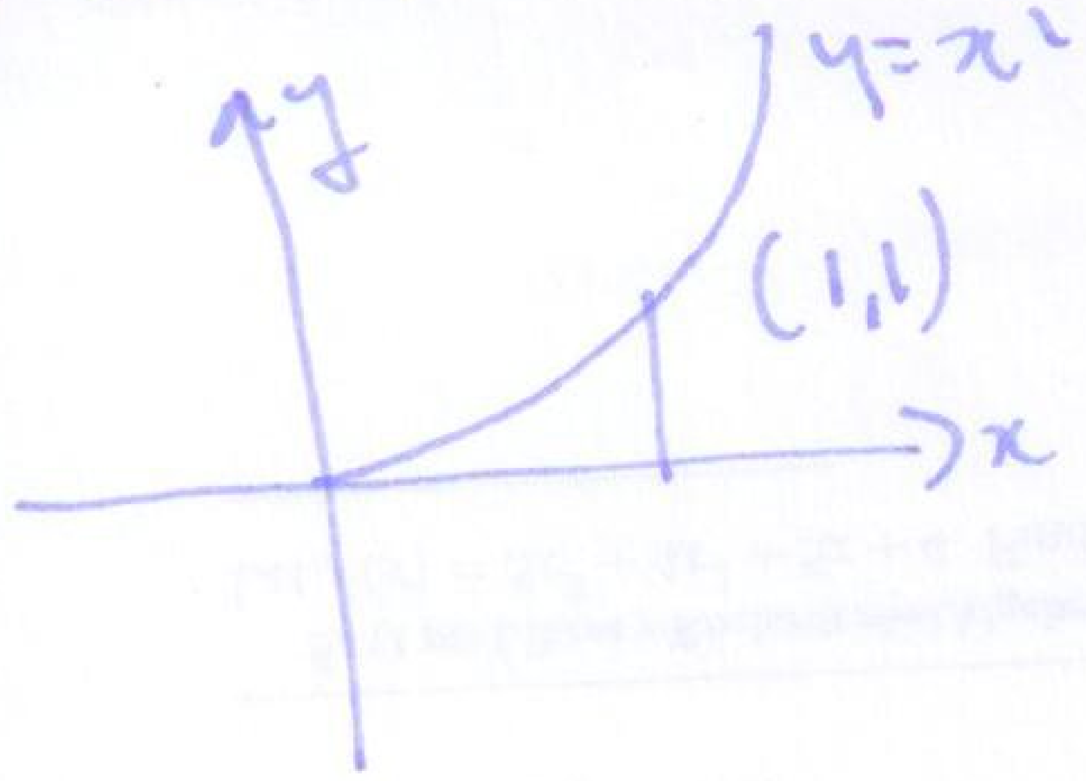
~~$f(1)=1$~~

~~$f(1+h) = (1+h)^2 = 1+2h+h^2$~~.

~~$f(1+h) =$~~

		interval	slope.
$f(1)=1$.	$f(2)=4$.	$[1,2]$	$\frac{4-1}{1} = 3$.
	$f(1.5) = 9/4$ ($3/2$)	$[1, 1\frac{1}{2}]$	$\frac{9/4-1}{1} = 5/4 = 1.25$
	$f(1.1) = 1.21$	$[1,1.1]$	$\frac{1.21-1}{0.1} = 2.1$
	$f(1.01) = 1.0201$.	$[1,1.01]$	$\frac{1.02\ldots}{0.01} \approx 2$.

(b) seems to work algebraically:



$f(1) = 1 \quad f(1+h) = (1+h)^2 = 1+2h+h^2.$

$$\frac{f(1+h)-f(1)}{h} = \frac{1+2h+h^2-1}{h} = \frac{2h+h^2}{h}.$$

if $h \neq 0$! $= 2+h$

Def^n Let f be a function defined on an interval containing c, but not necessarily at c. We say the limit of $f(x)$ as x approaches c is equal to L written

$$\lim_{x \to c} f(x) = L$$

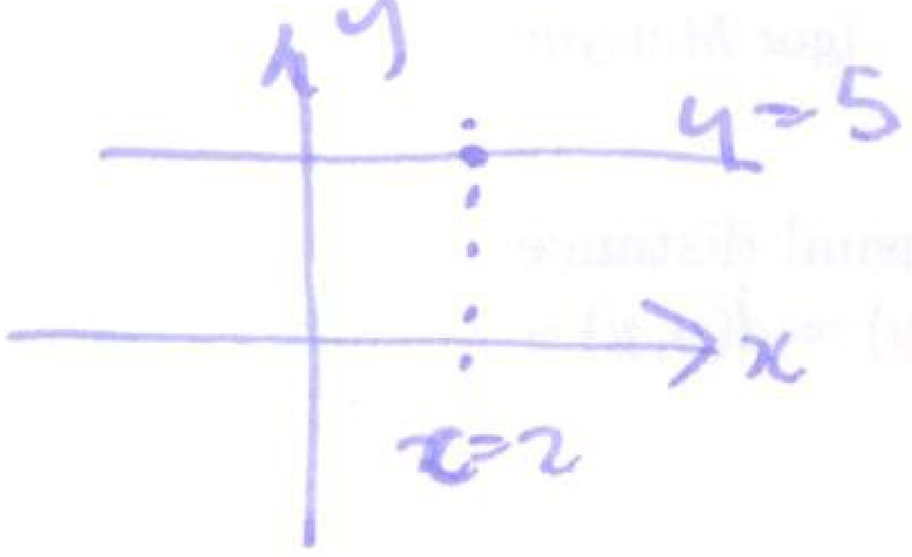
if $|f(x) - L|$ ~~can be made~~ becomes arbitrarily small when x is chosen sufficiently close to c.

also say $f(x)$ converges to L as $x \to c$

$f(x) \to L$ as $x \to c$.

Examples a) $f(x) = 5$. $c = 2$.

b) $f(x) = \frac{5x}{x}$ $c = 2$

a) 

want to show $|f(x) - 5|$ is close to zero for x close to $c = 2$.

but $|f(x) - 5| = |5 - 5| = 0$ so true for all x, in particular all x close to zero.

c) $\lim_{x \to 2} 2x+1 = 5$

want to show $|f(x) - 5|$ is close to zero for x close to 2.

$$|f(x) - 5| = |2x + 1 - 5| = |2x - 4| = 2|x - 2|$$

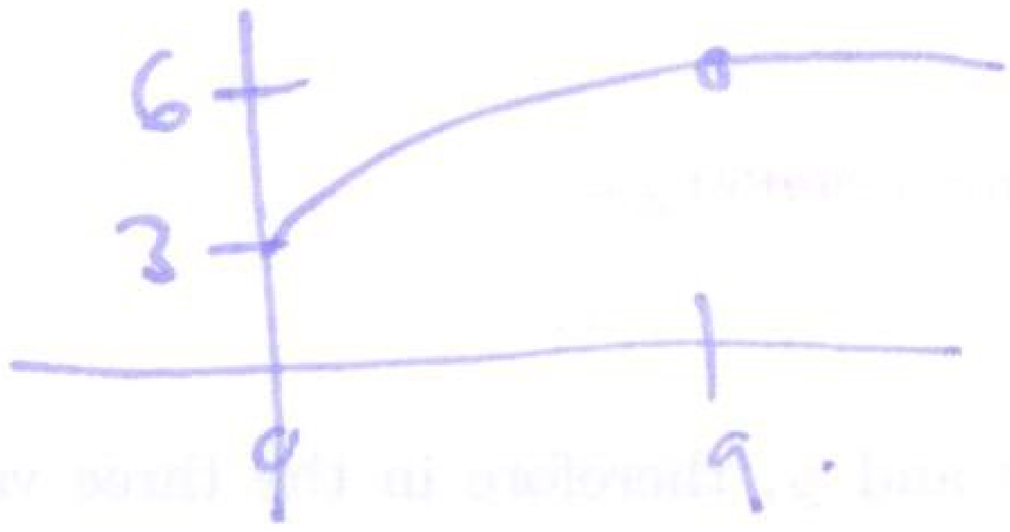
when x is close to 2 $|x-2|$ is small $\Rightarrow 2|x-2|$ is small □. we'll be more precise later on (2.8)

Thus for any constants k, c a) $\lim_{x \to c} k = k$ b) $\lim_{x \to c} x = c$.

Investigating limits: · try drawing a picture.
· estimating close values.

Example $\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3}$. problem can't plug in 9.

draw picture :



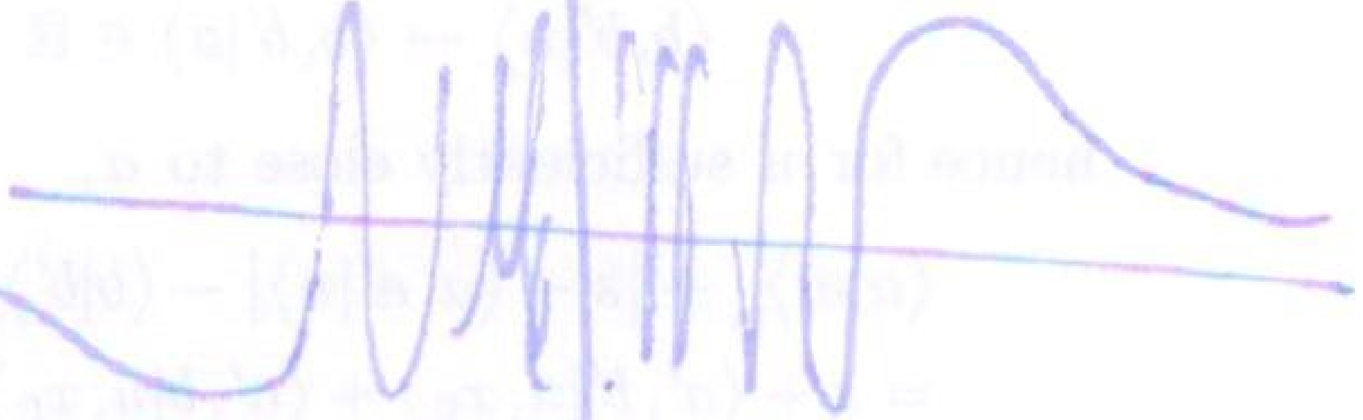
looks like 6.

compute :

x	$\frac{x-9}{\sqrt{x}-3}$	x	$\frac{x-9}{\sqrt{x}-3}$
8.9	5.98329	8.99	5.99833.
9.1	6.016..	9.01	6.00166..

Example no limit.

$f(x) = \sin\left(\frac{1}{x}\right)$ near $x=0$



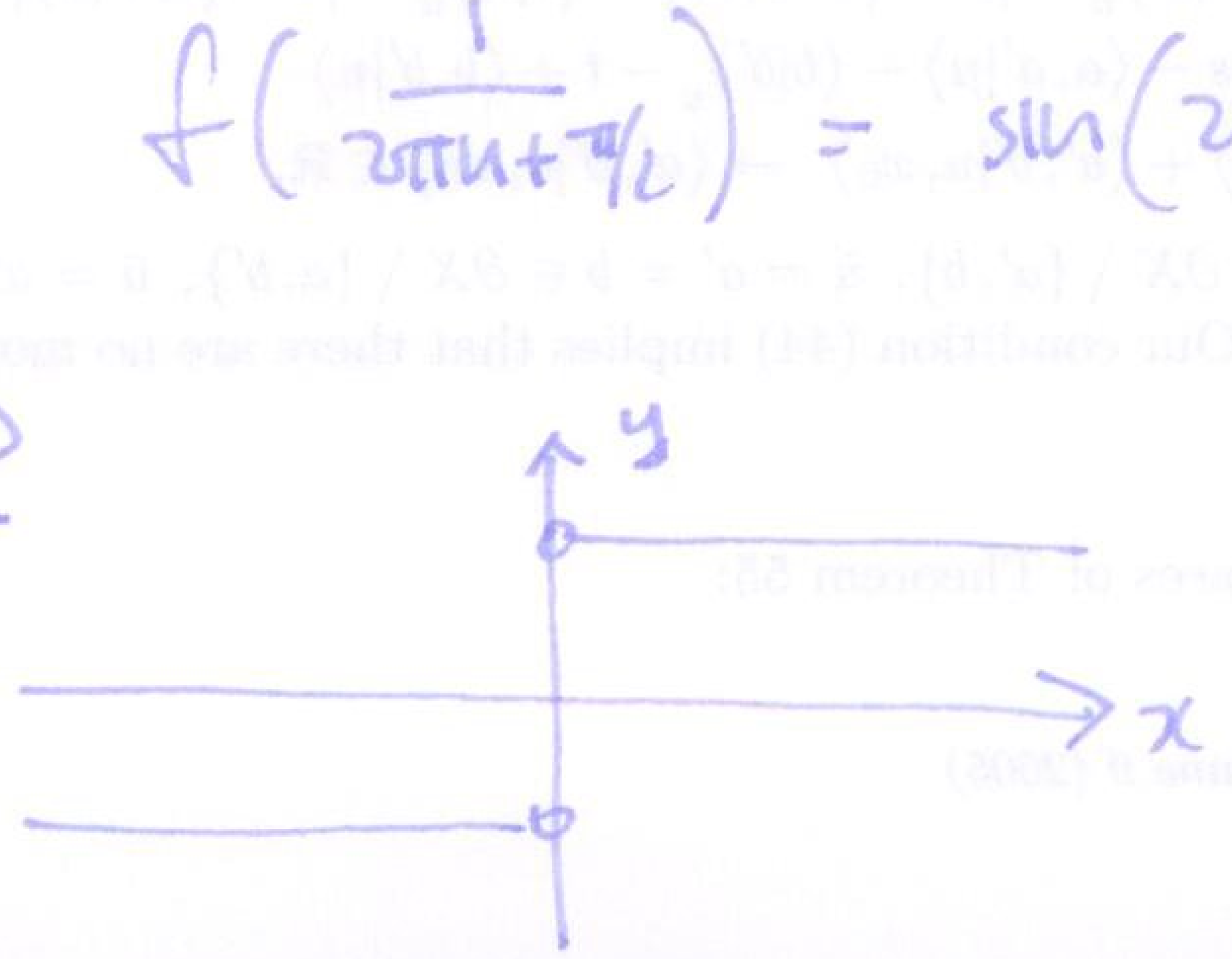
note ~~sin~~

$$f\left(\frac{1}{2\pi n}\right) = \sin(2\pi n) = 0.$$

$$f\left(\frac{1}{2\pi n+\pi/2}\right) = \sin\left(2\pi n+\frac{\pi}{2}\right) = 1.$$

One sided limits

Example.



$$f(x) = \frac{x}{|x|}. = \begin{cases} +1 & x>0 \\ -1 & x<0 \end{cases}$$