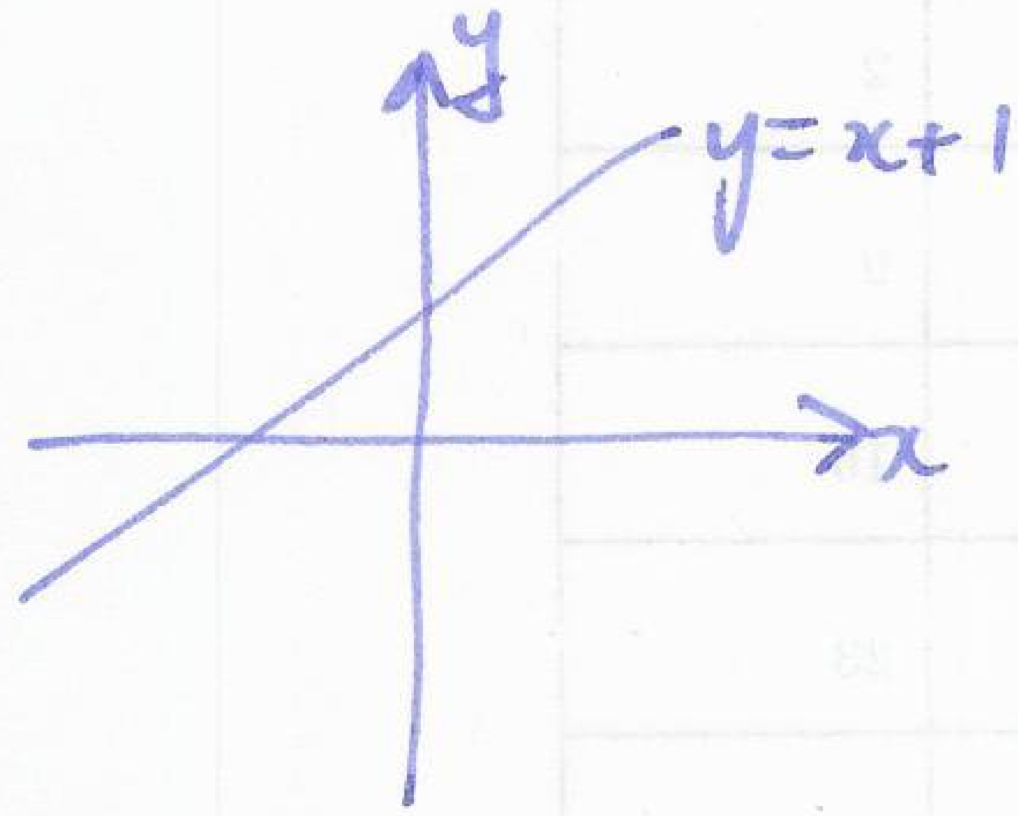
examples $f: \mathbb{R} \to \mathbb{R}$
$x \mapsto x+1$

the graph of this function passes the horizontal line test, i.e. every horizontal line hits the graph exactly once.




equivalently: for every value of $f(x)$ there is a unique $x$ s.t. $x \mapsto f(x)$.

so this function has an inverse.

Q: how do we find a formula for the inverse?

① write down $y = \overset{f(x)}{x+1}$:
② try and solve for $x$ in terms of $y$
③ swap variables.
④ check!

in this example:

$$y = x+1$$

$$x = y-1$$

so

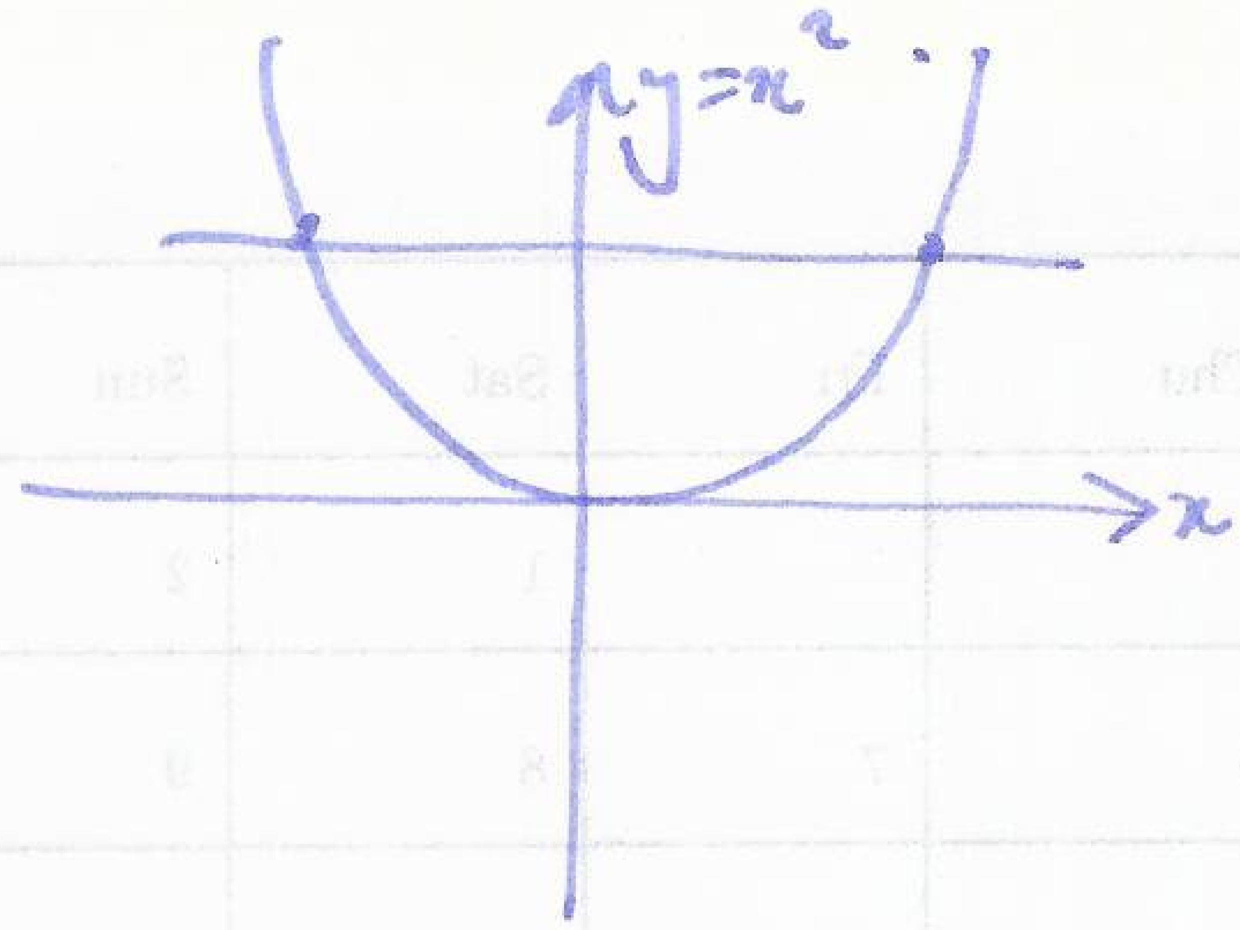
$$f^{-1}(x) = x-1.$$

check. $x \mapsto f(x) \mapsto f^{-1}(f(x))$.

$$x \mapsto x+1 \mapsto (x+1)-1 = x.$$

**bad example**

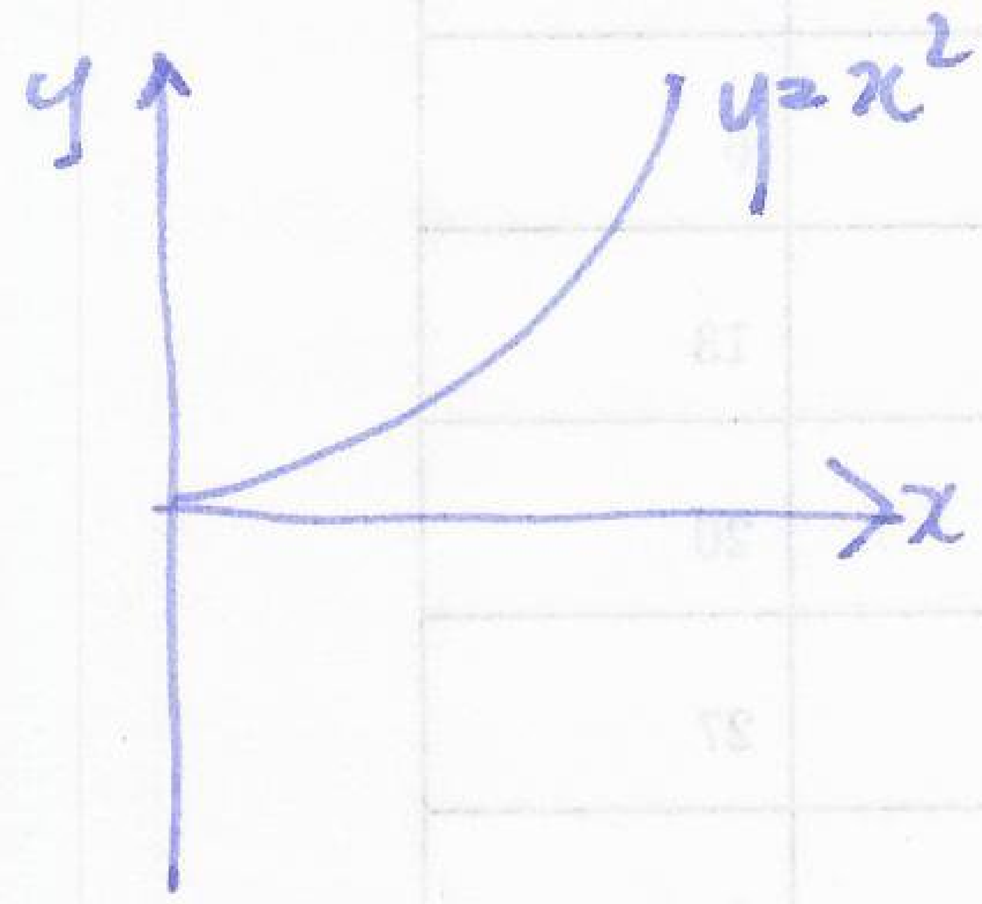
$f(x) = x^2$.




problem: **doesn't** pass horizontal line test so no inverse

**fix**: **restrict domain** consider $f: \cancel{\mathbb{R}}\, [0,\infty) \longrightarrow \mathbb{R}$

$$x \longmapsto x^2.$$




this **does** pass the horizontal line test, so we can define an inverse.

$$f^{-1} : [0,\infty) \longrightarrow [0,\infty)$$

$$x \longmapsto \sqrt{x} = x^{1/2}.$$

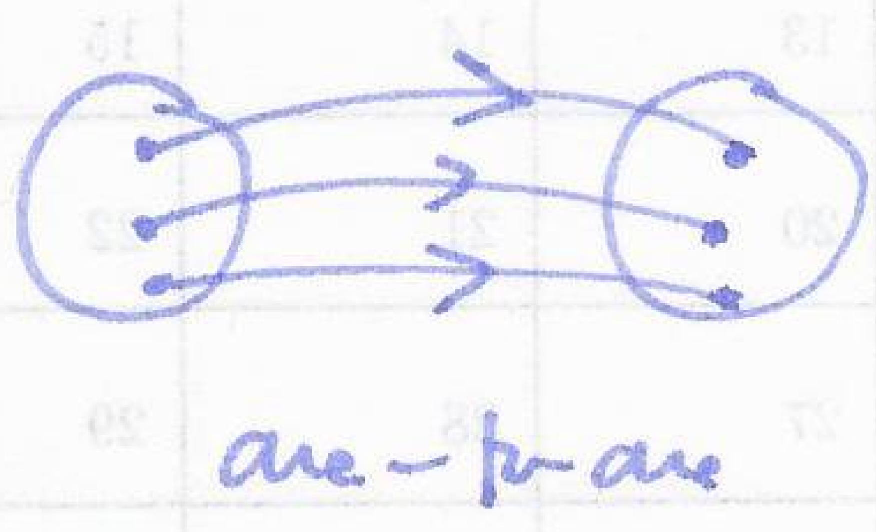
check: $x \longmapsto f(x) \longmapsto f^{-1}(f(x))$.

$$x \longmapsto x^2 \longmapsto \sqrt{x^2} = (x^2)^{1/2} = x$$

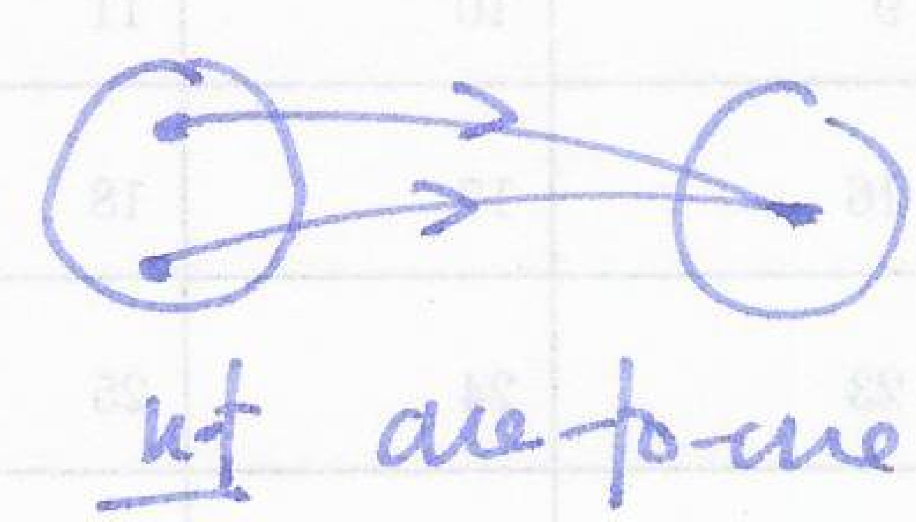
**Defn** $f: D \to R$ $f(x)$ is **one-to-one** or **injective** if for every $c \in R$ there is a **unique** $x \in D$ s.t. $f(x) = c$.

equivalently: graph passes horizontal line test.

**picture**.



one-to-one



**not** one-to-one.

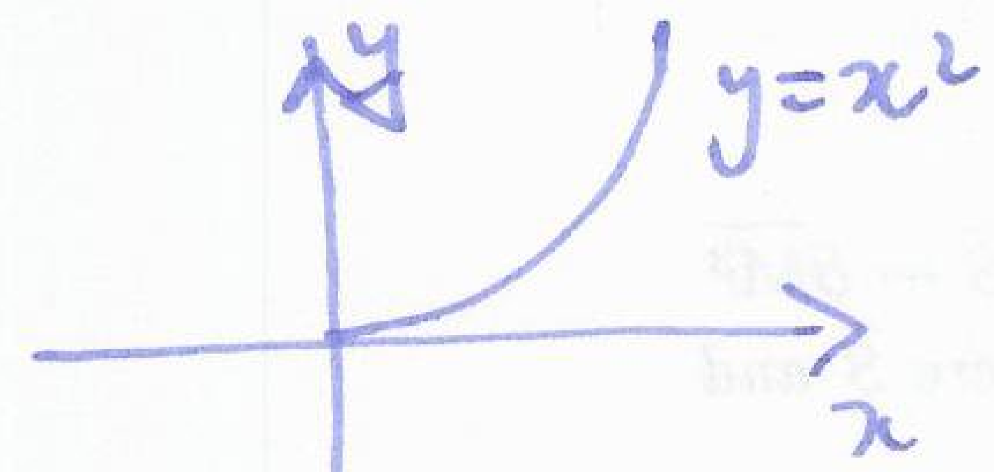
**Thm** If $f$ is one-to-one then $f$ is invertible

and domain of $f$ = range of $f^{-1}$

range of $f$ = domain of $f^{-1}$

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

<u>Drawing the graph of the inverse:</u>




graph of $f$: set of pairs $(x, f(x))$ $x \in$ domain $f$

graph of $f^{-1}$: set of pairs $(y, f^{-1}(y))$ $y \in$ range of $f$

$y \in$ range of $f \Rightarrow \exists\, x \in$ domain s.t. $f(x) = y$. $\quad (f(x), f^{-1}(f(x)))$.

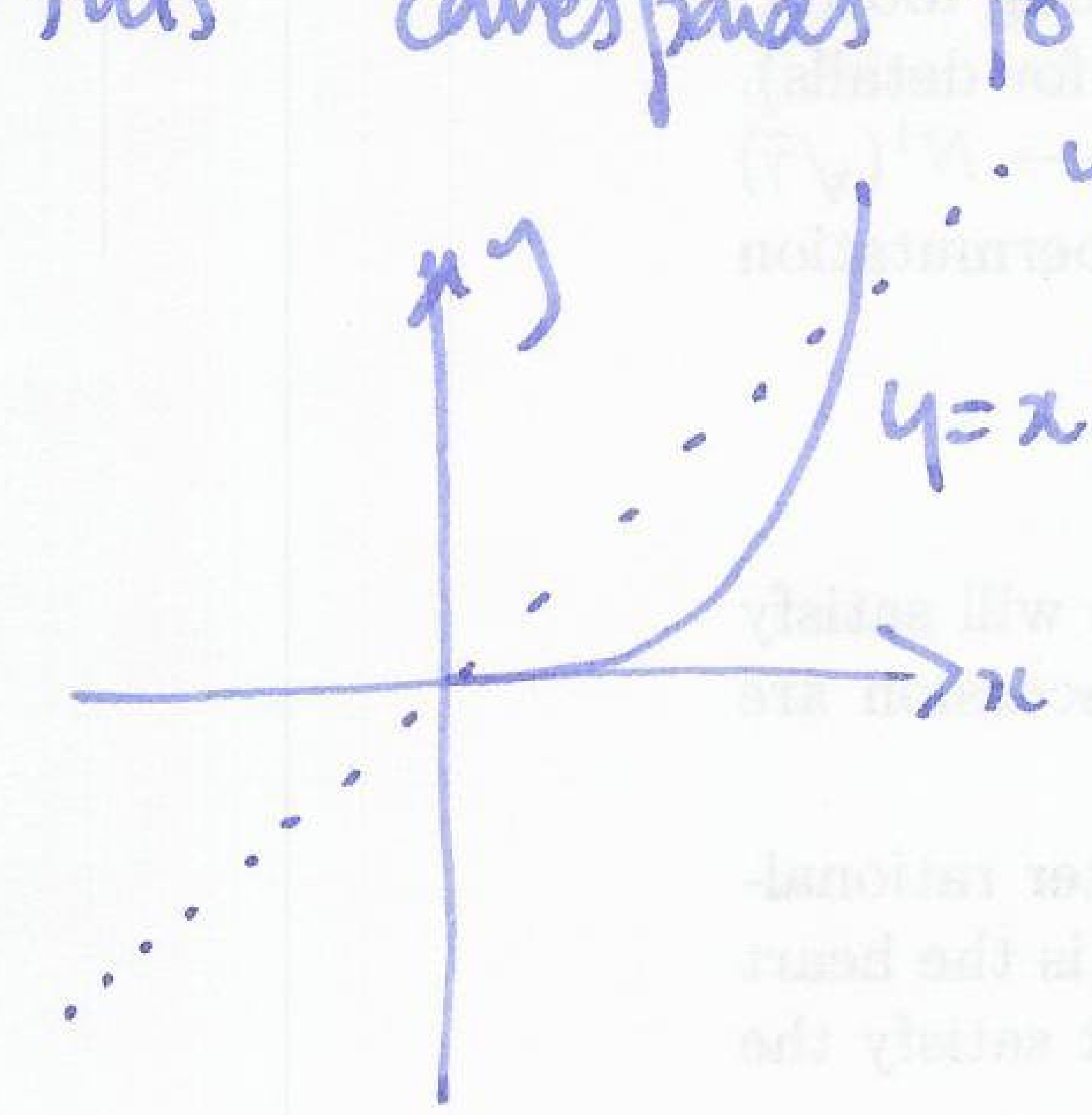
$\qquad = (f(x), x)$

so graph of $f^{-1}$: set of pairs $(f(x), x)$

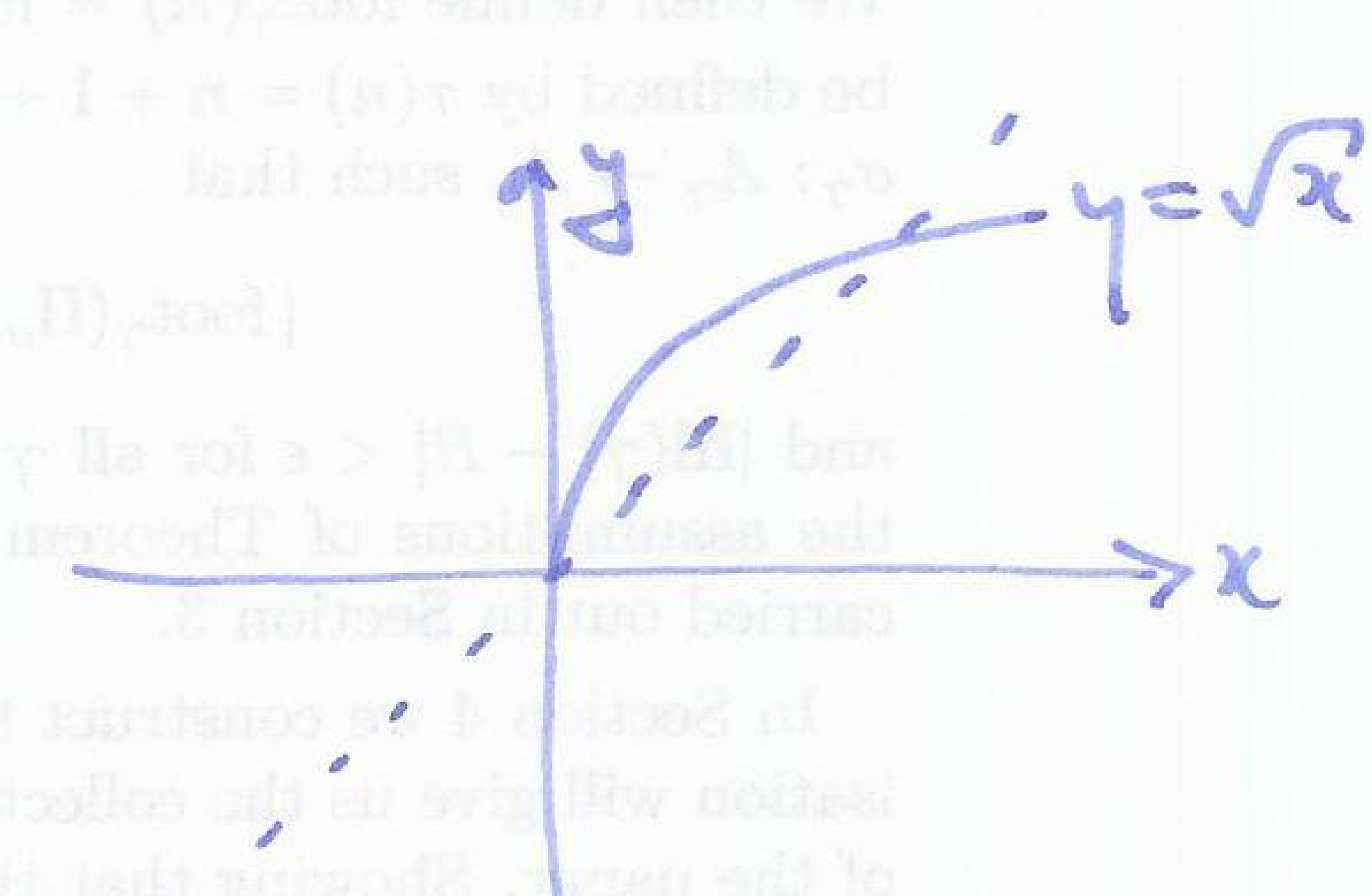
so want to take $xy$ plane and swap $x, y$ values.

i.e. $(x, y) \longrightarrow (y, x)$.

this corresponds to reflection in the line $y = x$.




so reflect in $x=y$



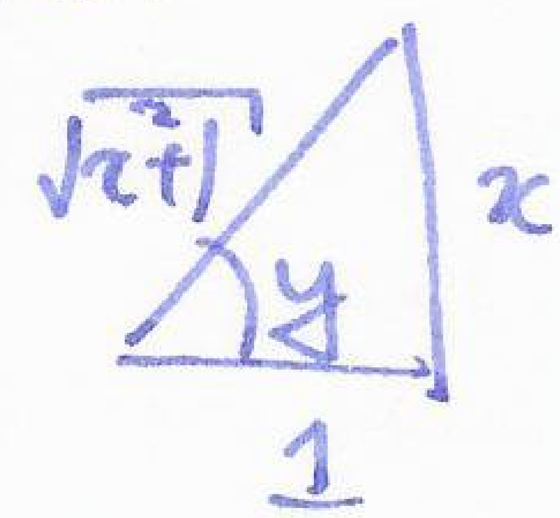
Examples

$$\sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}.$$

$$\sin^{-1}\left(\sin\frac{7\pi}{3}\right) = \frac{\pi}{3}. \quad !!$$

simplify: $\cos(\tan^{-1}x)$  $\cos y$  $y = \tan^{-1}x$

$\tan y = x.$

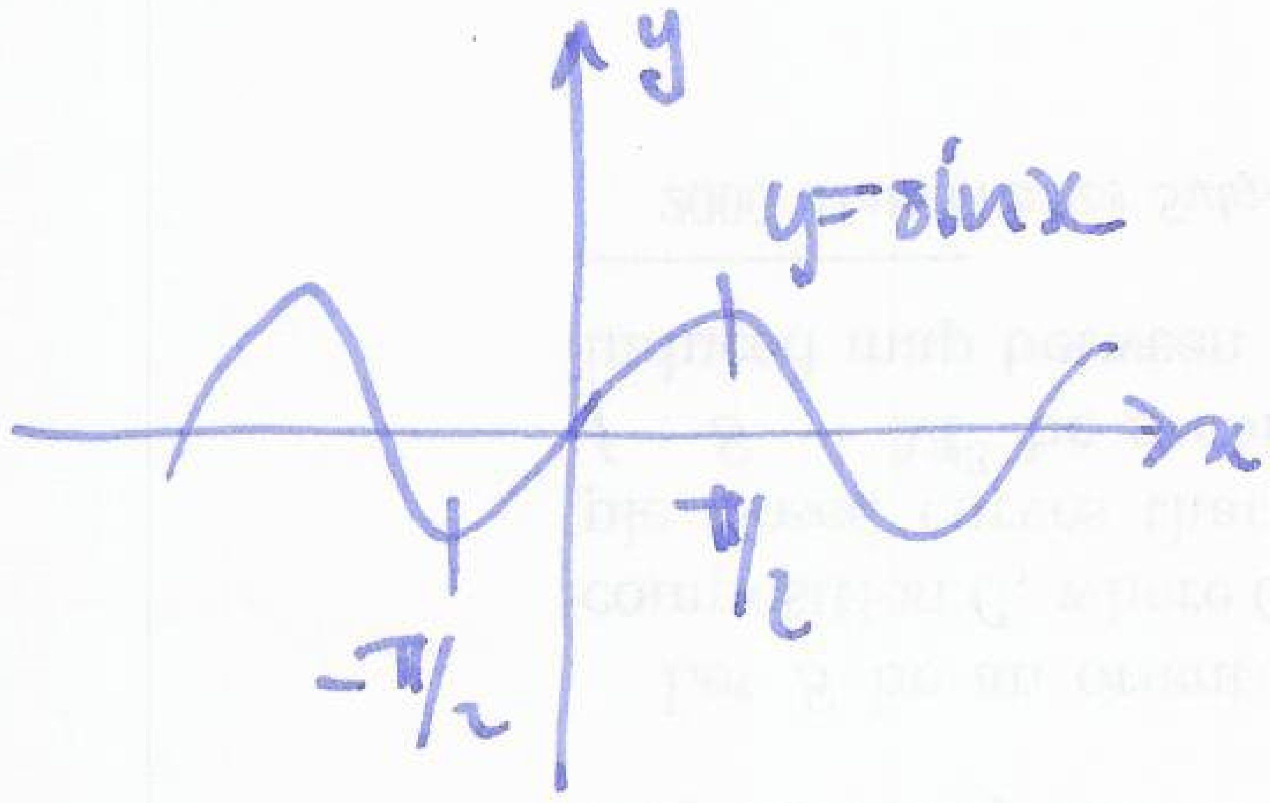



so $\cos y = \dfrac{1}{\sqrt{x^2+1}}.$

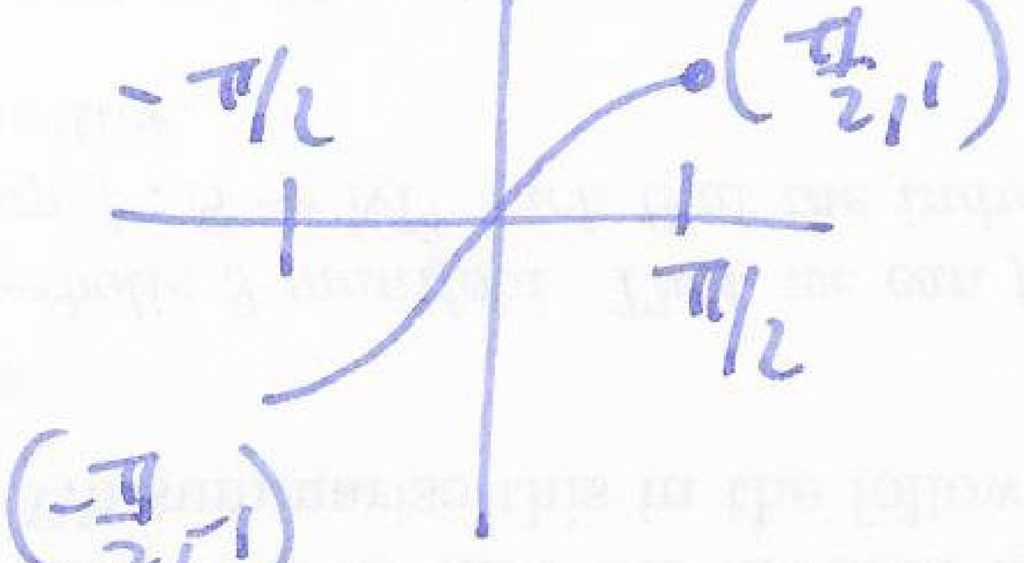
# Inverse trig functions

problem. $y = \sin x$ not one-to-one on $\mathbb{R}$.

fix: restrict domain.




canonical choice: $[-\pi/2, \pi/2]$.




define the inverse trig function: ~~asin~~ $\arcsin(x)$

$\operatorname{asin}(x)$

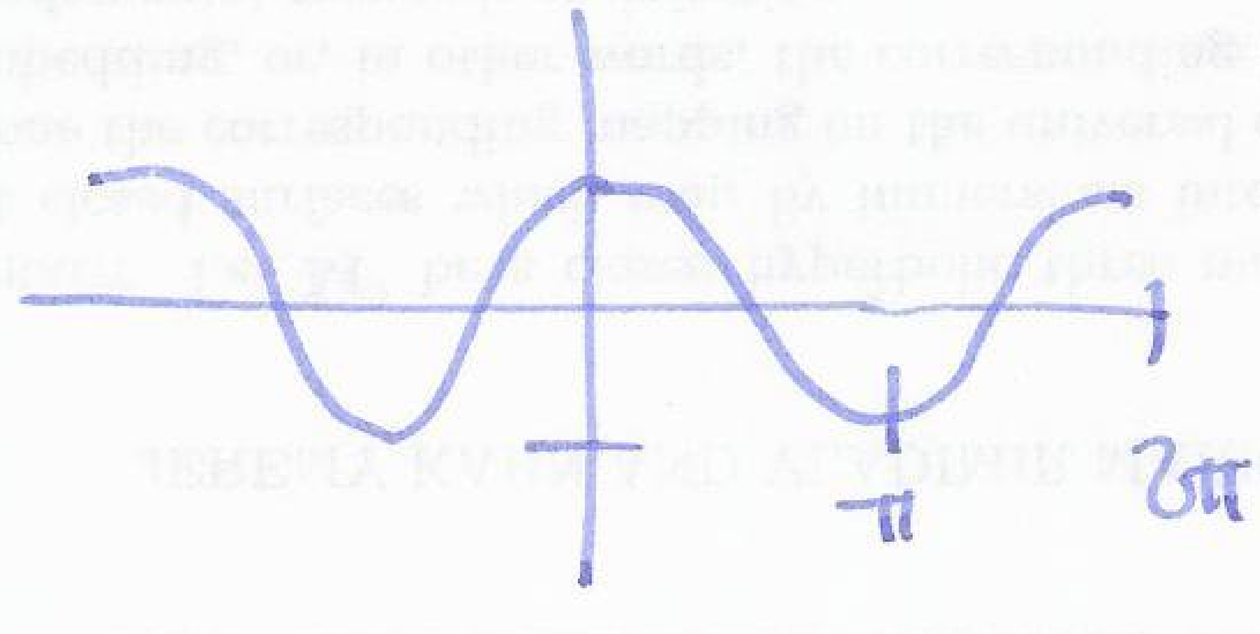
$\sin^{-1}(x)$

to be the inverse of $\sin x : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$

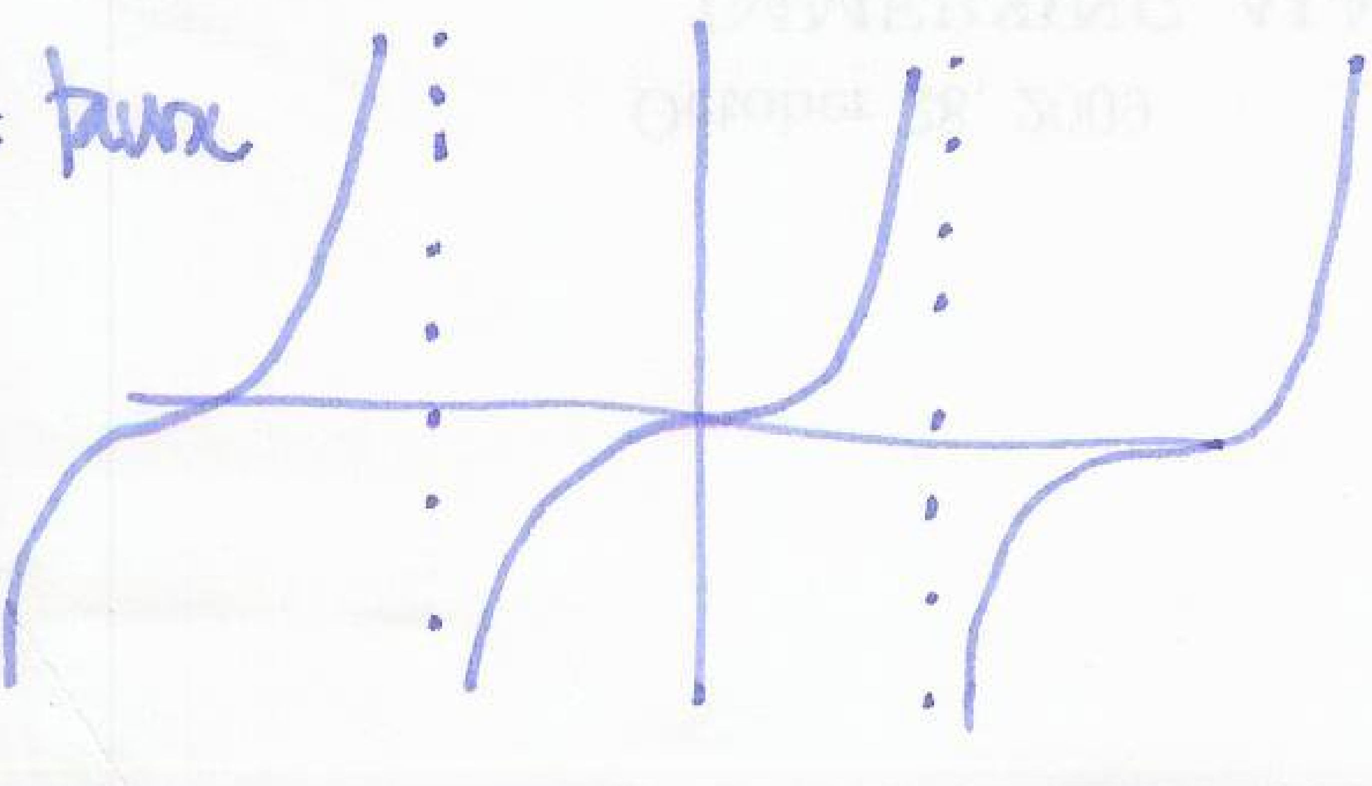
notation warning: $\sin^{-1}(x)$ is the inverse function and it is

not $\dfrac{1}{\sin(x)} = \cancel{\cos}\ \csc(x)$.

similarly $y = \cos x$ restrict to $[0, \pi]$.




$y = \tan x$

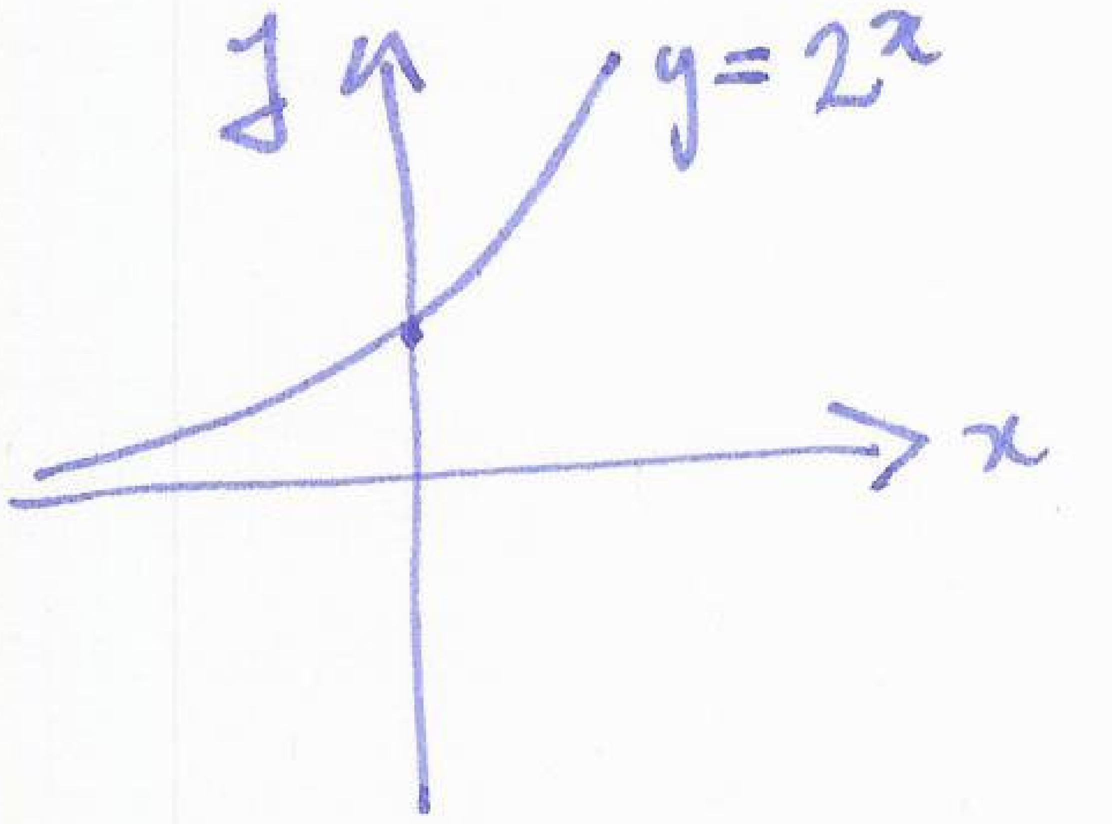


restrict to $\cancel{[-\frac{\pi}{2}]}\ (-\frac{\pi}{2}, \frac{\pi}{2})$.

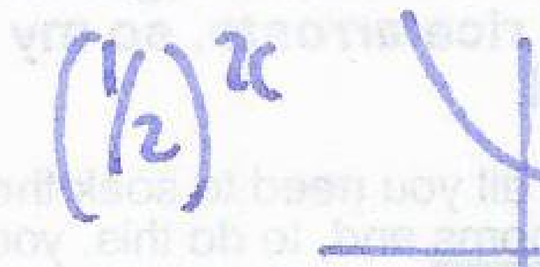
# §1.6 Exponential and Logarithm functions

**Example.** $x \mapsto 2^x$

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|
| $f(x)$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $1$ | $2$ | $4$ | $8$ |




can use any (+ve) number instead of 2: $x \mapsto b^x$.




useful properties

- positive: $b^x > 0$ for all $x$.
- $f(x) = b^x$ increasing if $b > 1$, decreasing if $0 < b < 1$
- range of $b^x$ is $(0, \infty)$
- rapid increase: $b^x$ $(b > 1)$ grows faster than every polynomial $x^n$.

exponent laws: $(b > 0)$

$$b^0 = 1 \qquad b^{-x} = \frac{1}{b^x}$$

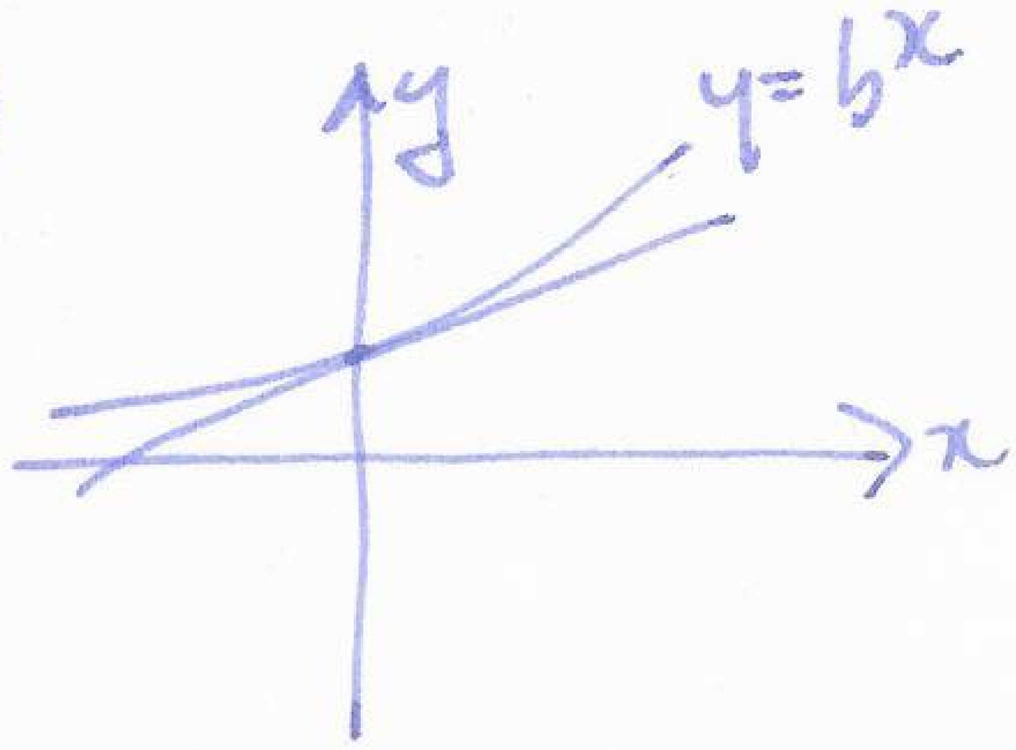
$$b^x b^y = b^{x+y} \qquad (b^x)^y = b^{xy}$$

$$\frac{b^x}{b^y} = b^{x-y} \qquad b^{1/n} = \sqrt[n]{b}$$

there is a special exponential function $e^x$, where

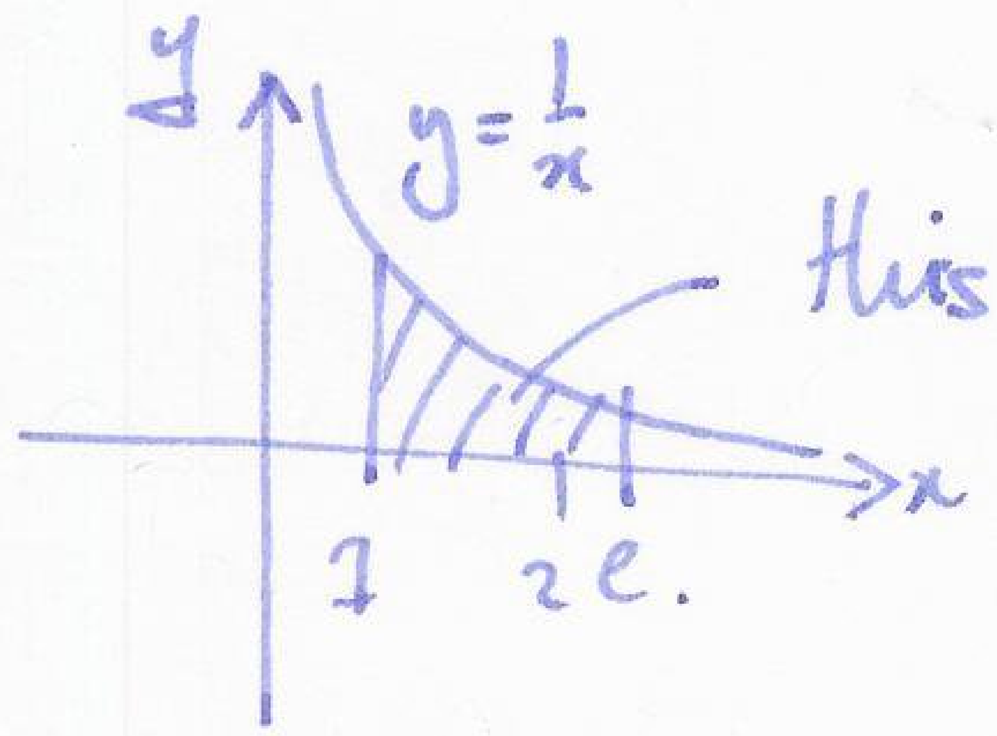
$e = 2.71828\ldots$

<u>properties</u>. ①




$e$ is the unique ~~number~~ choice of $b$ such that the slope of $y=b^x$ is 1 at $x=0, y=1$.

②




this region has area 1 .

## <u>Logarithms</u>

The logarithm is the <u>inverse</u> function for the exponential function.

note : $b^x : \mathbb{R} \to (0,\infty)$

so inverse function : $(0,\infty) \to \mathbb{R}$.

name for inverse of $b^x$ is $\log_b x$

special logarithm base e is called <u>natural</u> logarithm.

$\log_e(x) = \ln(x)$
$= \log(x)$.

inverse function property

$$\log_b(b^x) = x = b^{\log_b(x)}$$

.

laws of exponents $\longleftrightarrow$ laws of logarithms.

$$b^0 = 1 \qquad\qquad \log_b(1) = 0.$$

$$b^1 = b \qquad\qquad \log_b(b) = 1$$

$$b^x b^y = b^{x+y} \qquad\qquad \log(st) = \log(s+t).$$

$$\hookrightarrow \log_b(b^x b^y) = \log_b(b^{x+y})$$

$$\qquad\qquad\qquad\qquad \parallel$$

$$x+y = \log_b(b^x) + \log_b(b^y).$$

$$\frac{b^x}{b^y} = b^{x-y} \qquad\qquad \log_b(s/t) = \log_b(s) - \log_b(t).$$

$$\frac{1}{b^x} = b^{-x} \qquad\qquad -\log_b(s) = \log_b\left(\frac{1}{s}\right).$$

$$(b^x)^y = b^{xy} \qquad\qquad s\log_b(t) = \log_b(t^s).$$

converting between bases:

$$\log_b x = \frac{\log_a x}{\log_a b} \quad \text{for any } a.$$

including $a = e$ :

$$\log_b x = \frac{\ln x}{\ln b}.$$

# Hyperbolic functions

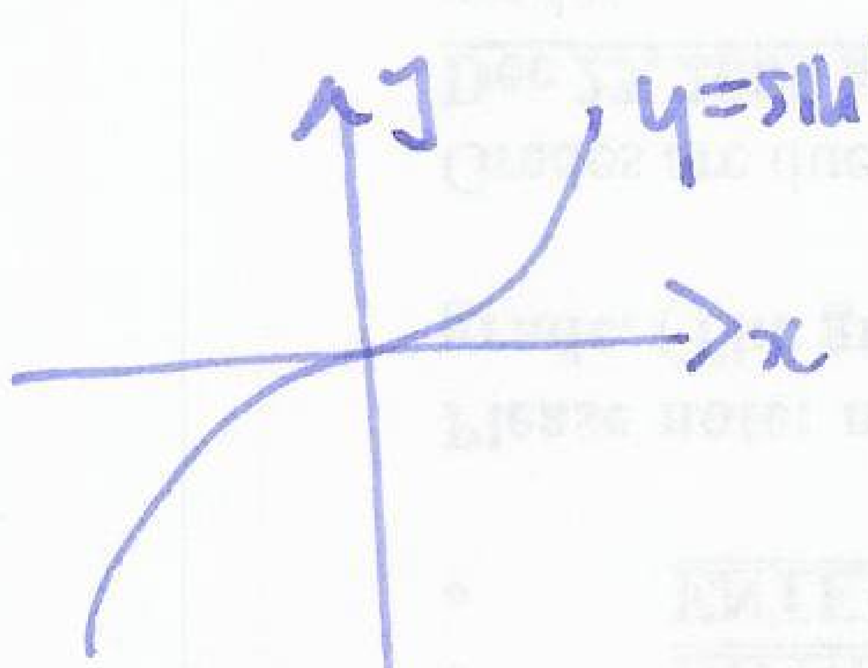
$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

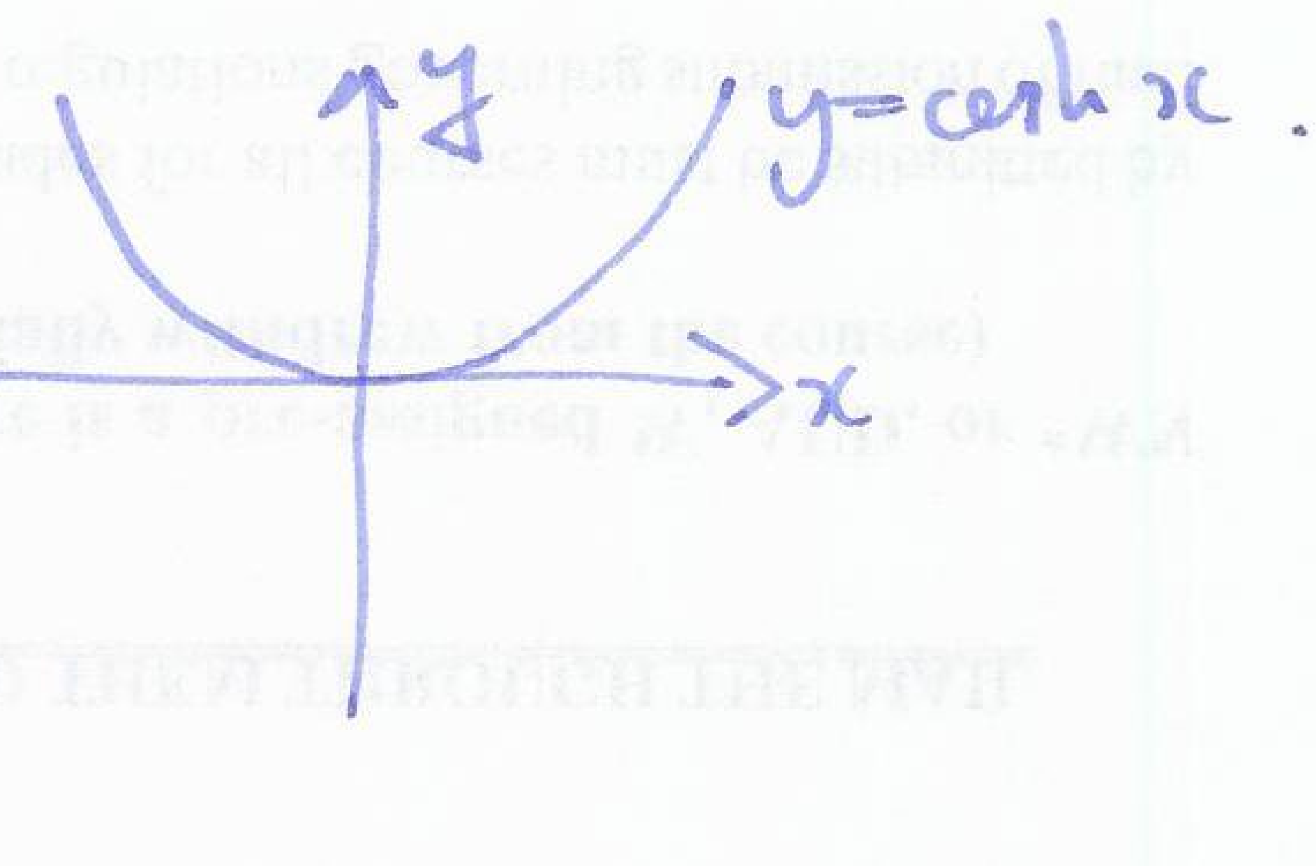
~~even~~ odd

$$\sinh(-x) = \sinh(x)$$

even ~~odd~~

$$\cosh(-x) = \cosh(x)$$




identities : ① $\cosh^2 x - \sinh^2 x = 1$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

check ① : $\cosh^2 x - \sinh^2 x = \left[\frac{e^x + e^{-x}}{2}\right]^2 - \left[\frac{e^x - e^{-x}}{2}\right]^2.$

$$\frac{1}{4}\left[e^{2x} + 2 + e^{-2x}\right] - \frac{1}{4}\left[e^{2x} - 2 + e^{-2x}\right] =$$

$$= \frac{1}{4}\left[e^{2x} - e^{-2x} + 2 + 2 + e^{-2x} - e^{-2x}\right] = \frac{4}{4} = 1.$$

inverse functions $\operatorname{asinh}(x)$

$\operatorname{acosh}(x)$ defined on $[0, \infty)$.