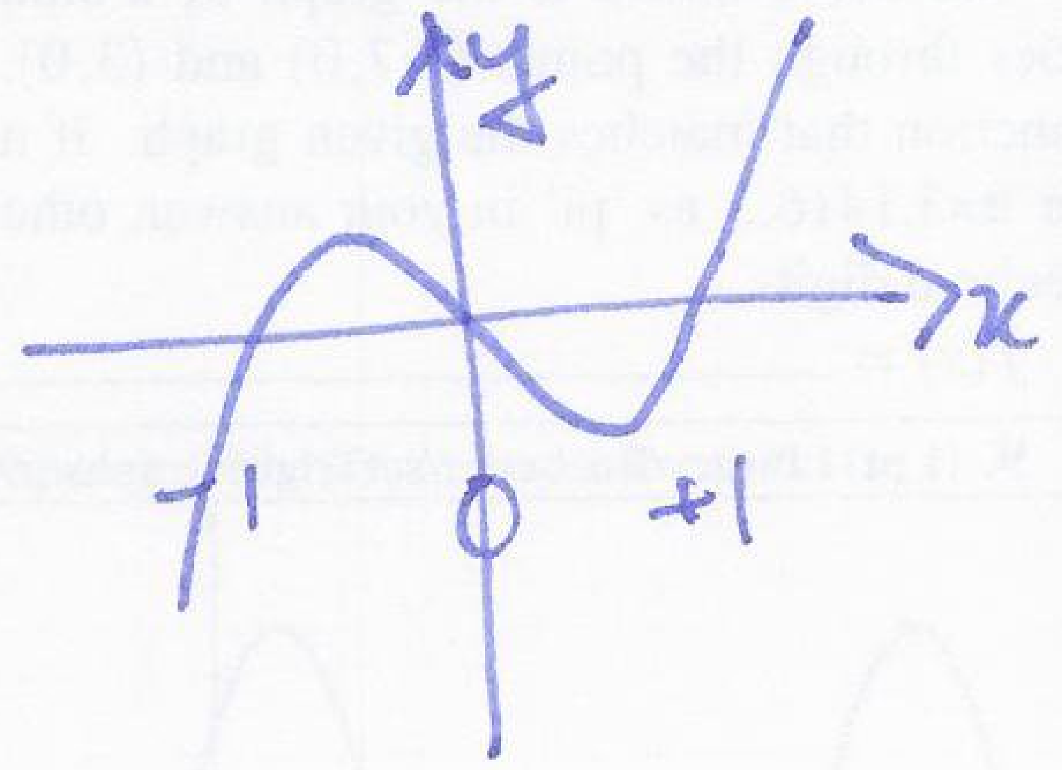
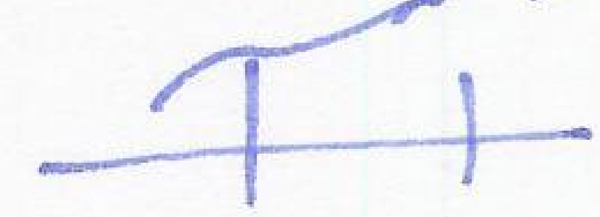
Properties of functions

Example $y = x^3 - x = f(x)$

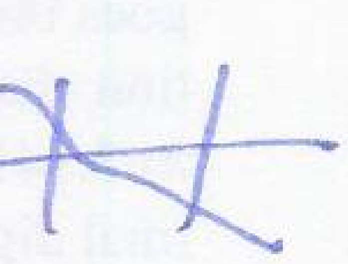


f is <u>increasing</u> on an interval (a,b) if $f(x_1) \le f(x_2)$



whenever $x_1, x_2 \in (a,b)$ with $x_1 \le x_2$.

f is decreasing on (a,b) if $f(x_1) \ge f(x_2)$



whenever $x_1, x_2 \in (a,b)$ with $x_1 \le x_2$.

In example: f is increasing on $[1, \infty)$.

decreasing on $[-\frac{1}{10}, -\frac{1}{10}]$

neither increasing nor decreasing on $[-10, 10]$.

<u>strictly increasing</u> means $f(x_1) < f(x_2)$ if $x_1 < x_2$.

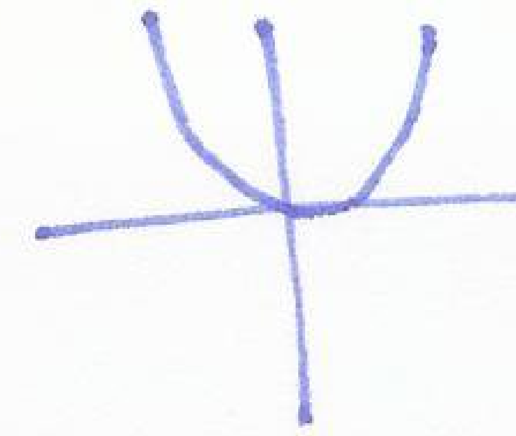
<u>strictly decreasing</u> means $f(x_1) > f(x_2)$ if $x_1 < x_2$.

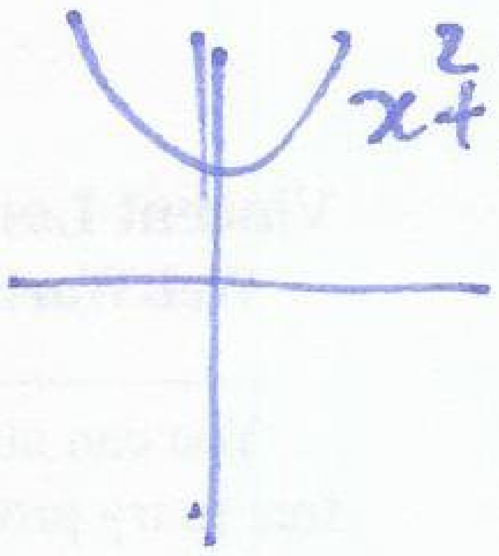
<u>Parity</u> even: $f(-x) = f(x)$ graph has reflectional symmetry in y-axis

odd: $f(-x) = -f(x)$ half turn symmetry about origin $(0,0)$ in $\mathbb{R}^2$.

<u>Examples</u> $x, x^2, x^3, x + x^3, x + x^2$.

New graphs from old



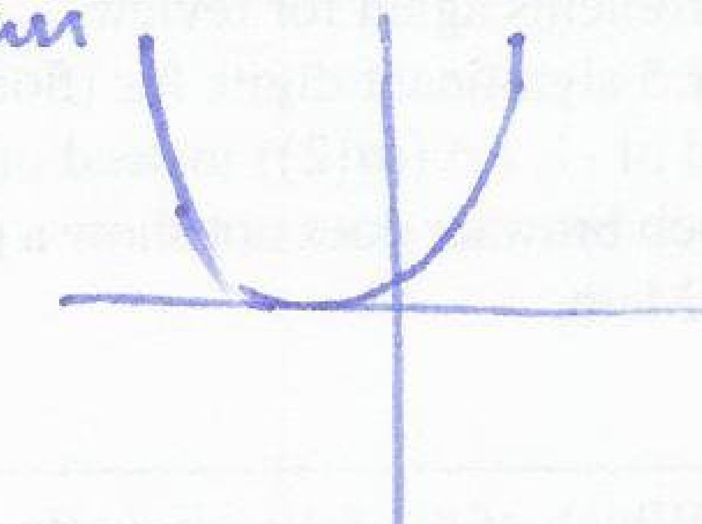


translations · vertical translation : $f(x)+c$

$c>0$ up
$c<0$ down

· horizontal translation : $f(x+c)$

$c>0$ left
$c<0$ right

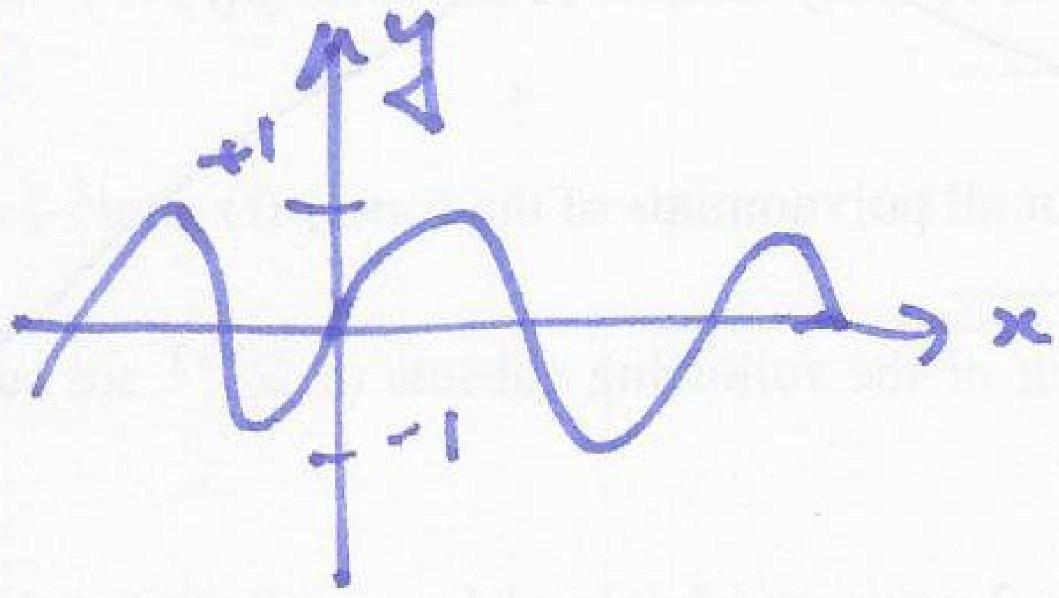


$(x+1)^2$
$= x^2+2x+1$

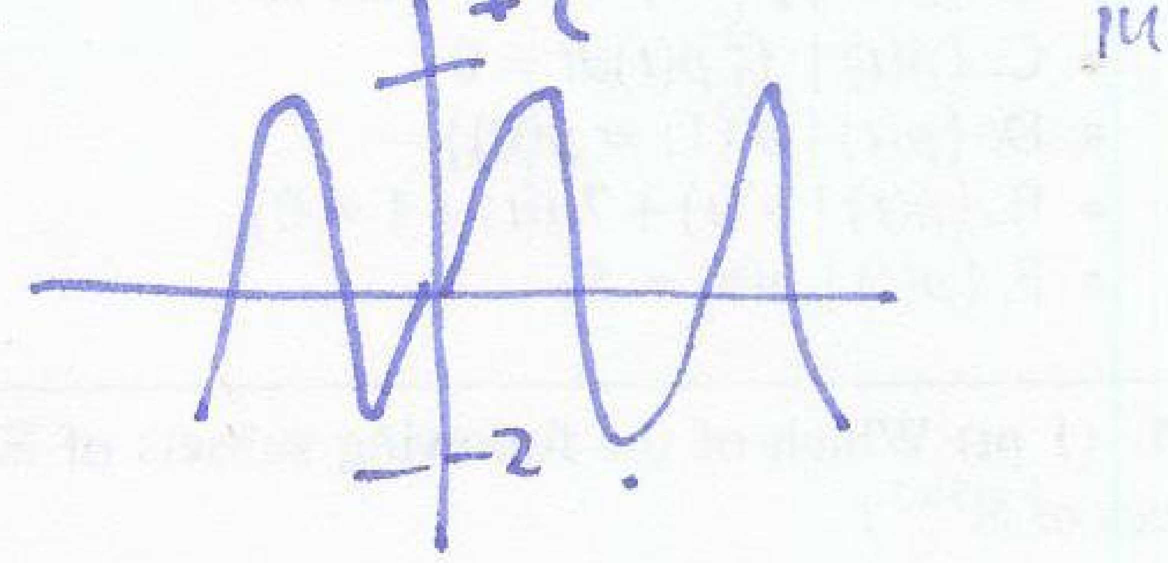
scaling · vertical scaling $kf(x)$ $k>1$ expansion -1 reflection in x-axis
$0<k<1$ contraction

horizontal scaling $f(kx)$ $k>1$ compression
$0<k<1$ contraction
-1 reflection in y-axis.

$y=\sin(x)$



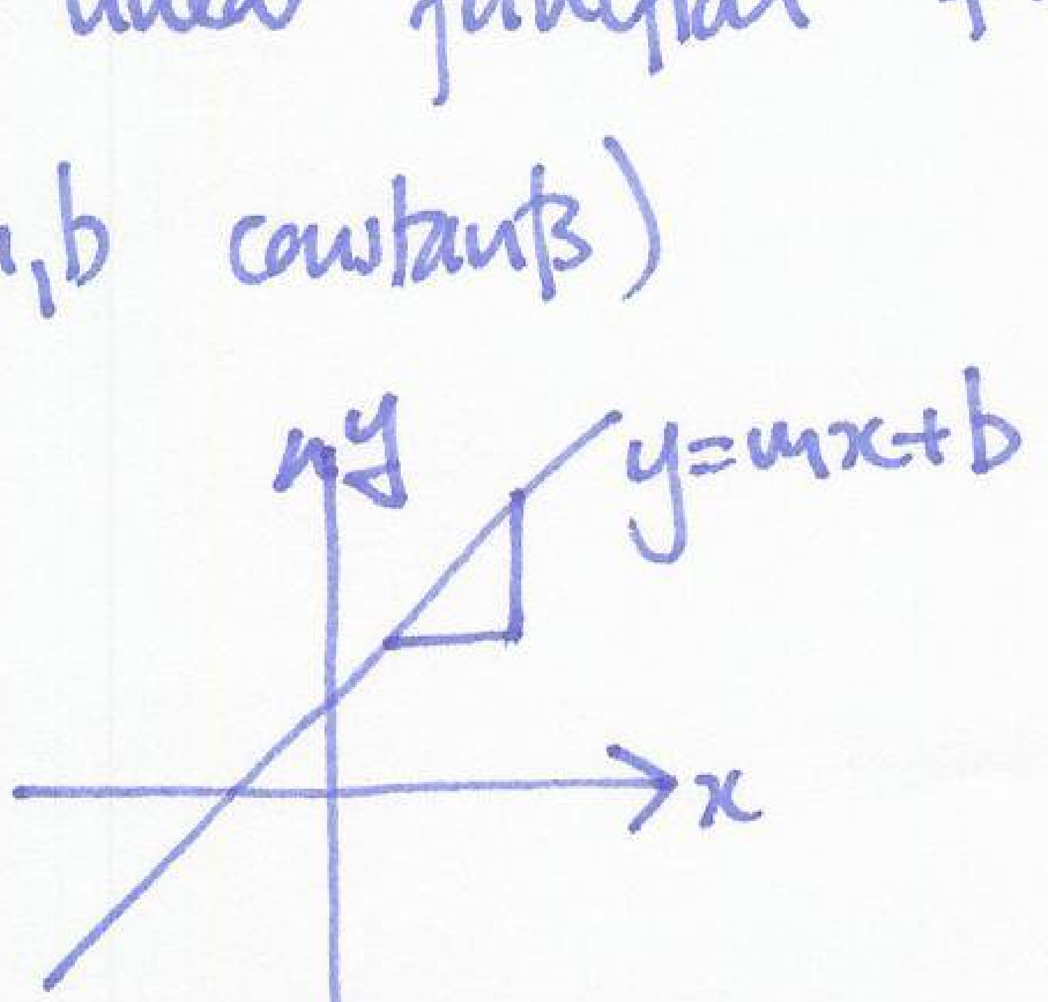
$y=2\sin(x)$



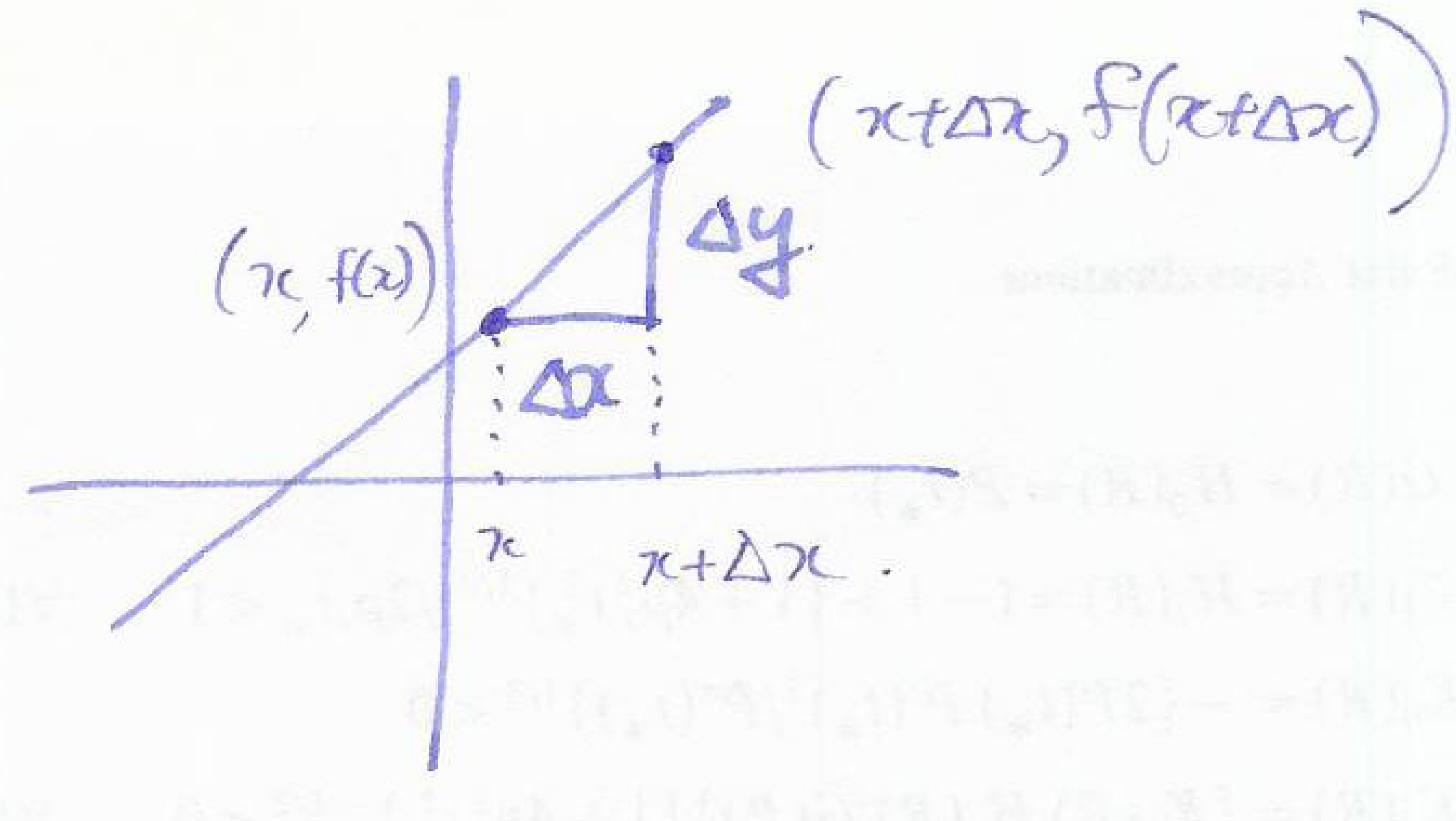
$y=\sin(2x)$

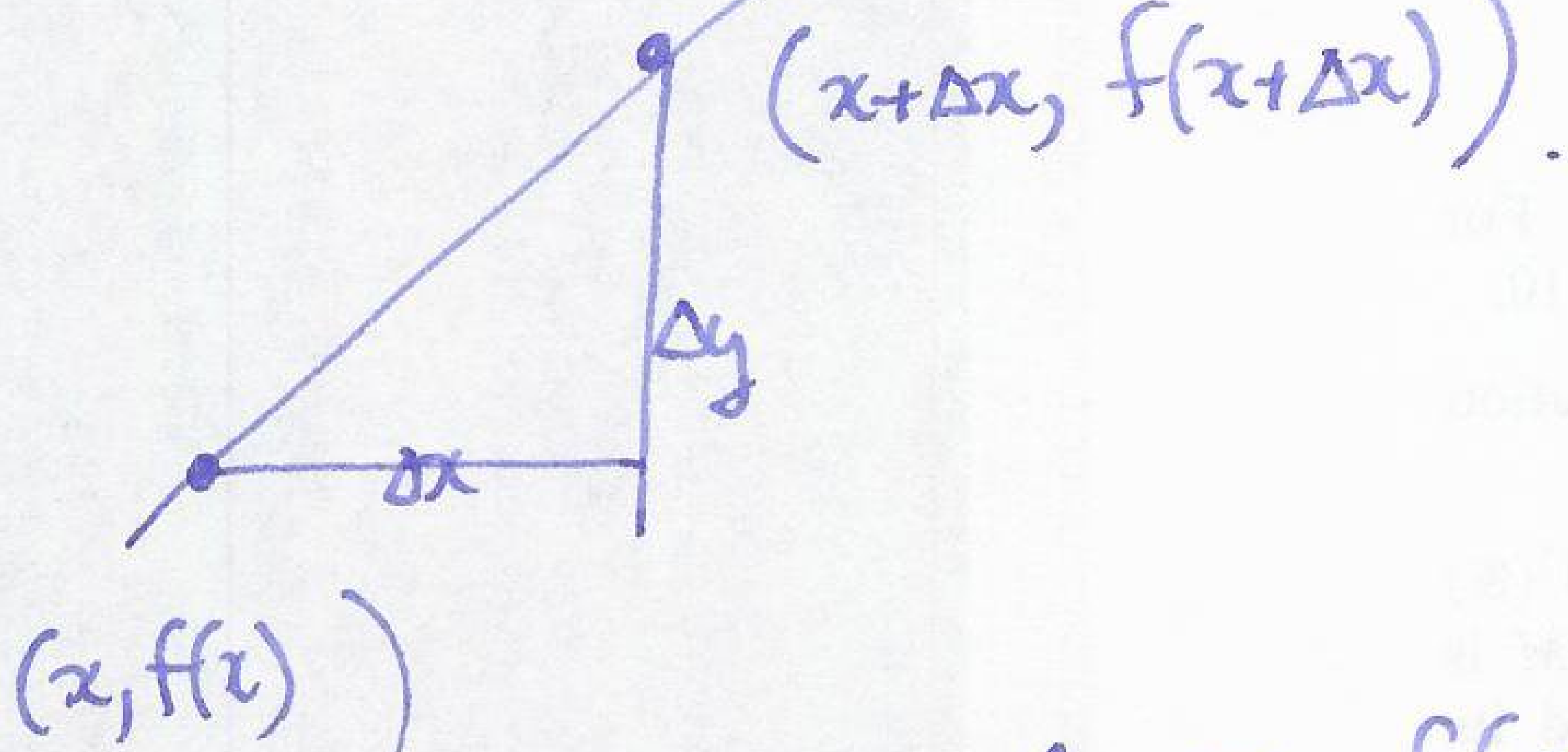
§1.2 Linear and quadratic functions

A linear function $f:\mathbb{R}\to\mathbb{R}$ is of the form $f(x)=mx+b$
(m,b constants) slope $= m$ y-intercept $(0,b)$.



$$\text{slope} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\Delta y}{\Delta x}$$





$$\Delta x = x + \Delta x - x = \Delta x.$$

$$\Delta y = f(x+\Delta x) - f(x).$$

$$\Delta y = m(x+\Delta x) + b - [mx + b].$$

$$\Delta y = \cancel{mx} + m\Delta x + \cancel{b} - \cancel{mx} - \cancel{b} = m\Delta x$$

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{m\Delta x}{\Delta x} = m$$

observations

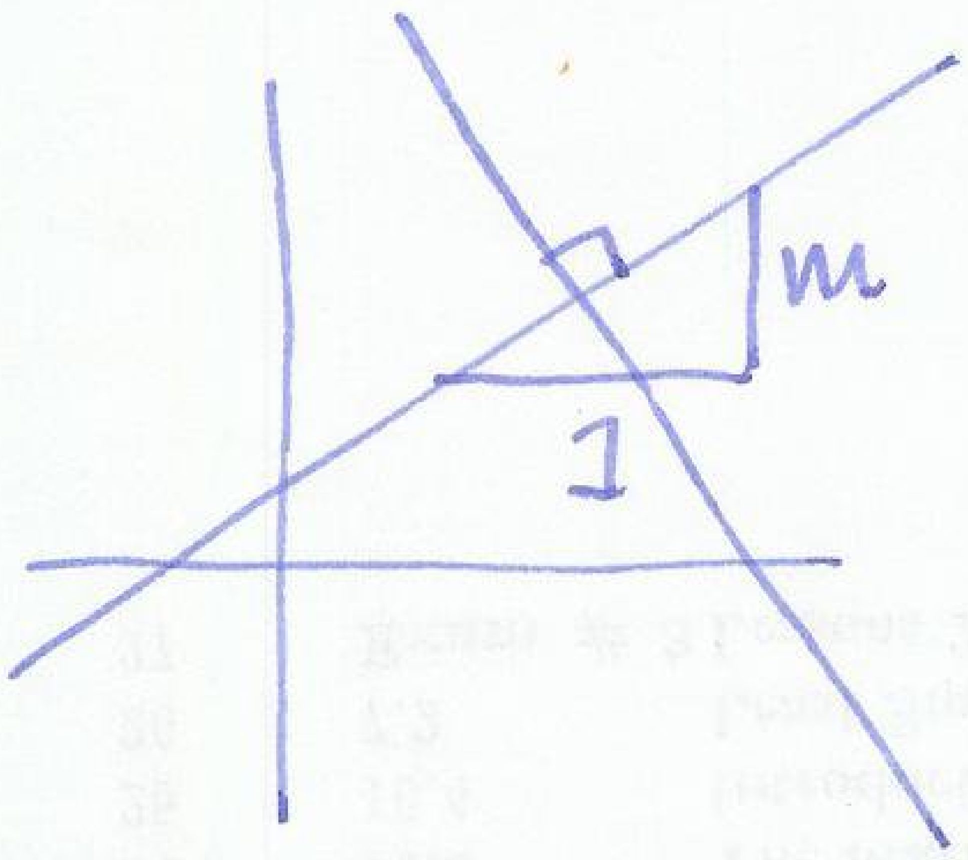
- $|m|$ big then slope is steep.
- $m=0$ horizontal line
- $m<0$ slopes down from left to right

problem: vertical line has equation $x=c$

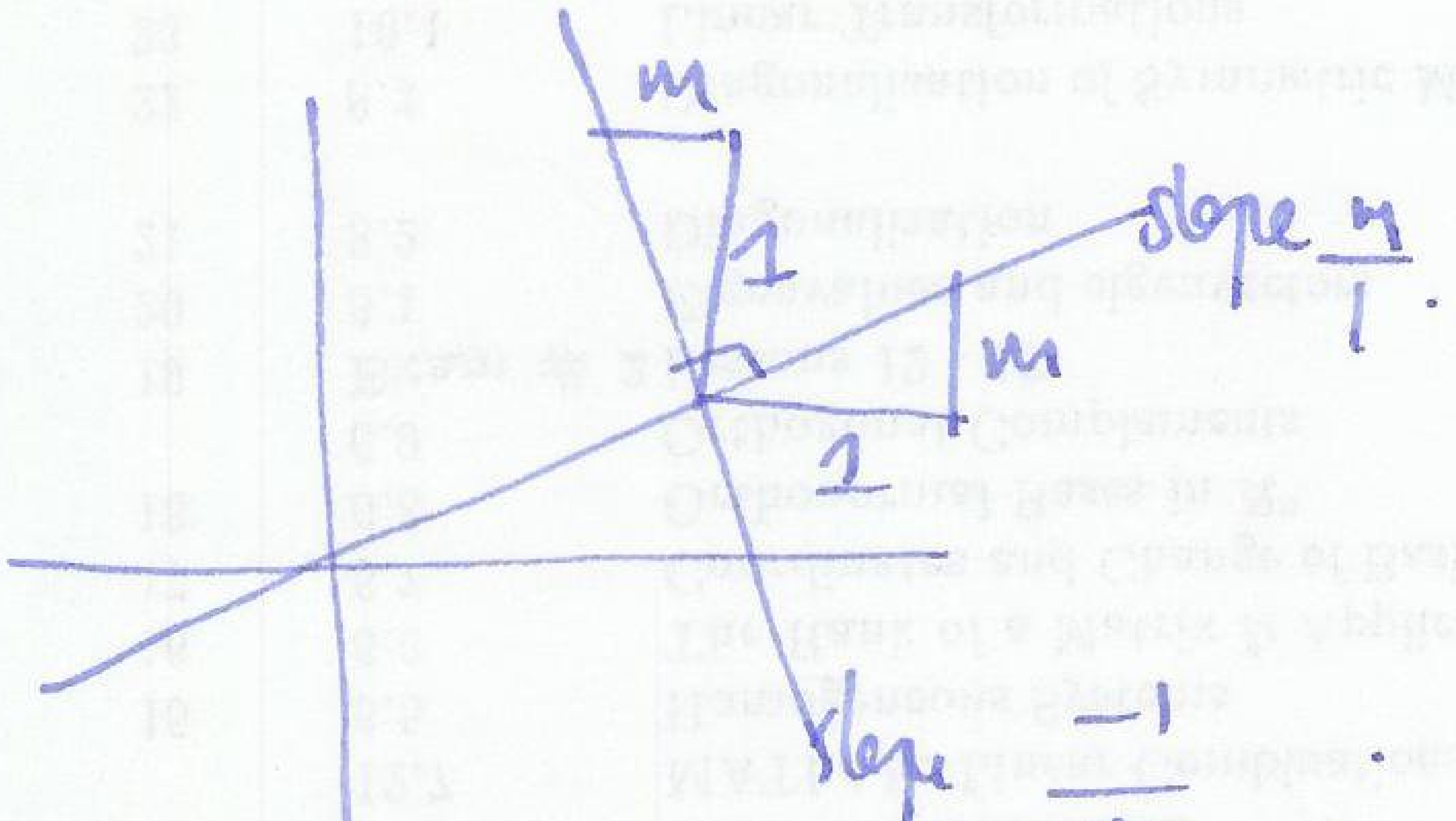
can't write this as a line $y=mx+b$ (sometimes say vertical line has "infinite slope".

perpendicular lines

slope m.



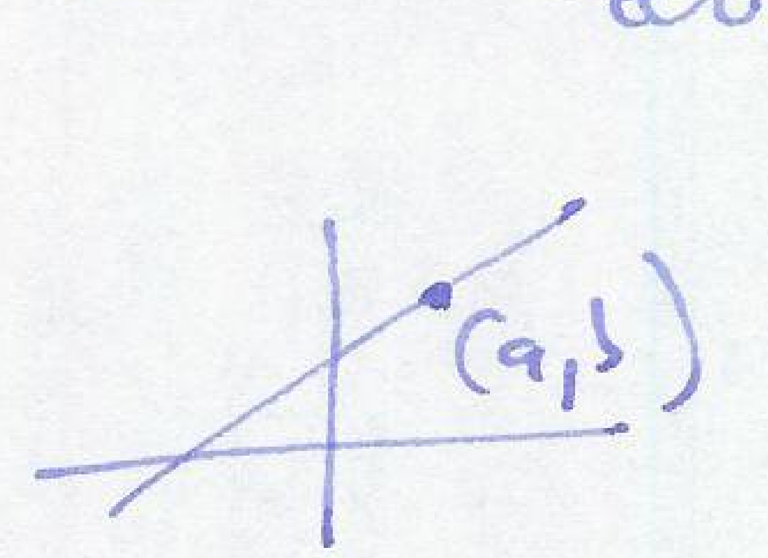
a perpendicular line has slope $-\frac{1}{m}$



note: a vertical line is not the graph of a function.

general linear equation: $ax+by=c$. (both a,b not zero)

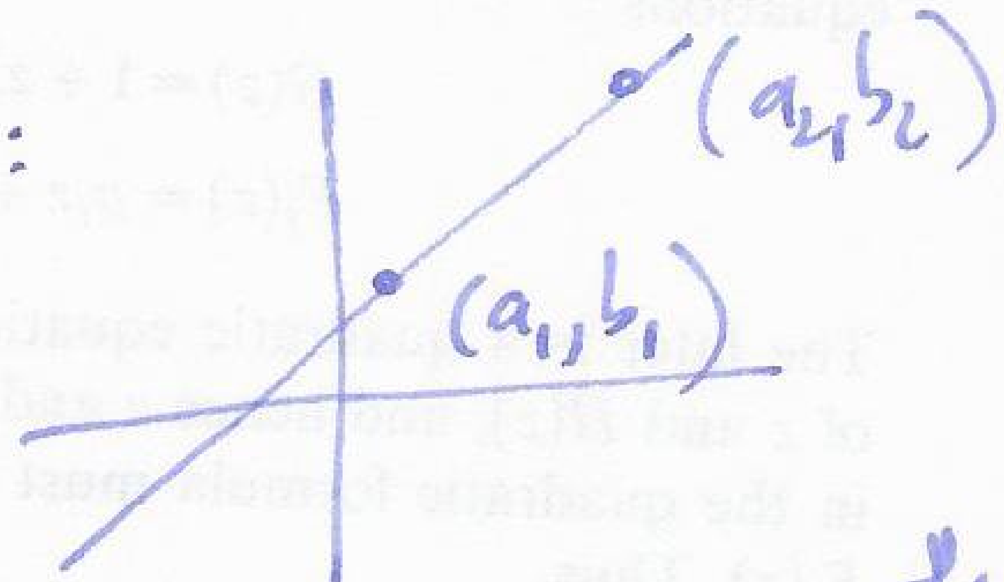
can write any line in this form.



other useful ways of writing down lines:

line with slope m through (a,b) $\quad y-b=m(x-a)$

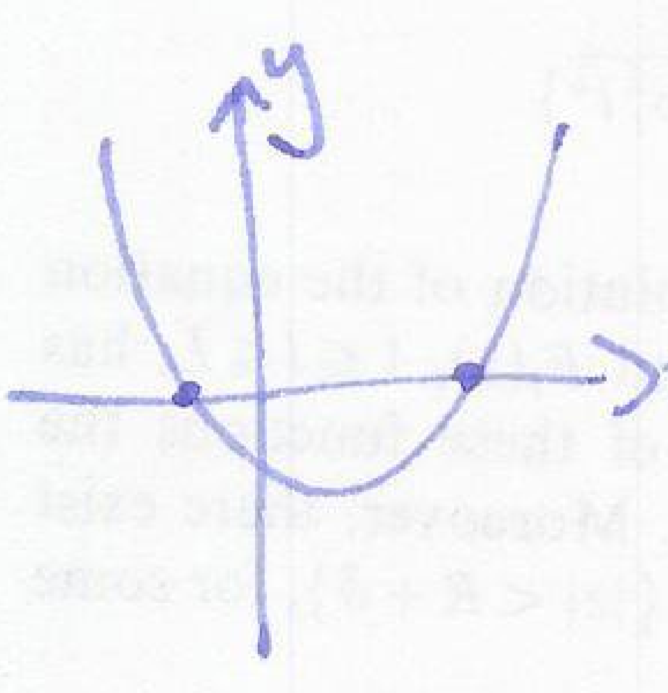
the line through two points:



slope $m=\dfrac{b_2-b_1}{a_2-a_1}$

then $y-b_1=m(x-a_1)$.

Quadratic functions $\quad f(x)=ax^2+bx+c$



graphs are parabolas

at most two distinct real solutions to $f(x)=0$

given by $x=\dfrac{\pm b\pm\sqrt{b^2-4ac}}{2a}$

no real sol^ns if $b^2-4ac<0$ $\quad$ are distinct real sol^n if $b^2-4ac=0$.

always two complex sol^ns (might not be distinct) if coeffs complex.

factorization: $ax^2+bx+c=a(x-r_1)(x-r_2)$ if r_1,r_2 are sol^ns/roots.

completing the square

example ~~$4x^2+12x$~~ $2x^2+2x+1 = 2(x^2+x+\frac{1}{2})$.

$= 2\left((x+\frac{1}{2})^2 - \frac{1}{4} + \frac{1}{2}\right) = 2\left(\underbrace{(x+\frac{1}{2})^2}_{} - \frac{1}{4}\right)$.

$x^2+x+\frac{1}{4}$

minimum value when this is zero.

1.3 Basic classes of functions

we're interested in all functions $f: \mathbb{R} \to \mathbb{R}$

all such functions: very badly behaved

example: $f(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$ ⟸ yuck.

nice functions: polynomials $a_n x^{n} + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$.

degree

leading term. a_i coeffs.

rational functions $\frac{p(x)}{q(x)}$ p, q polynomials.

algebraic functions: things with roots / fractional powers in them $\sqrt{x^2+1}$ $\sqrt{x}+3$. etc.

exponential functions: $y = 2^x$, $y = e^x$ etc.

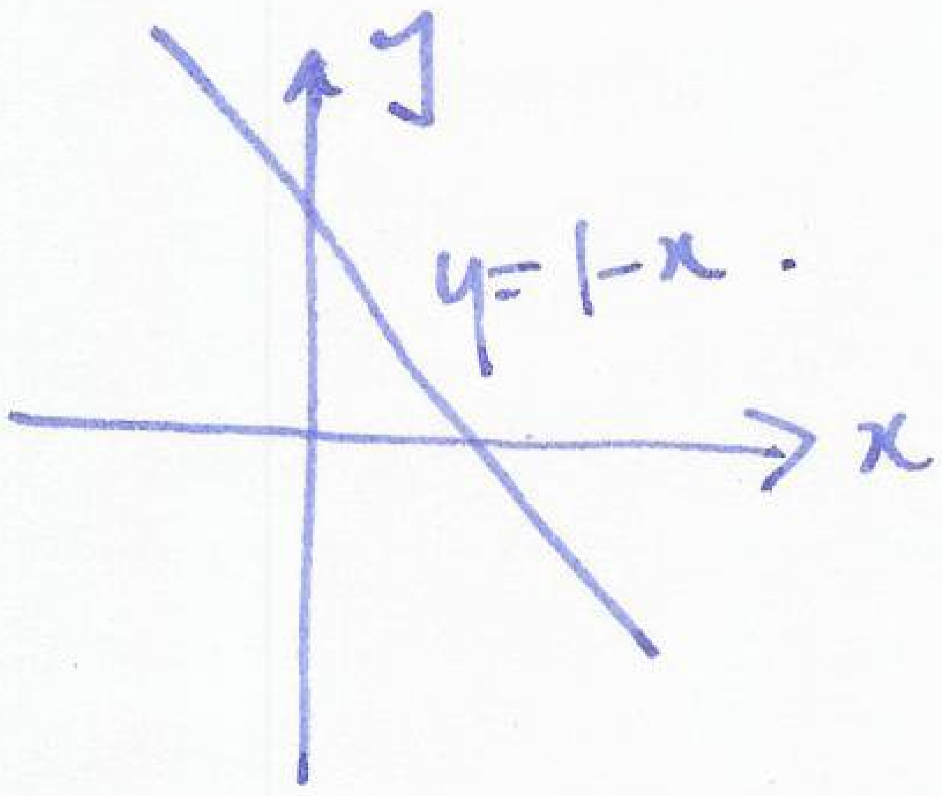
trigonometric functions: functions built from $\sin(x)$ $\cos(x)$ ....

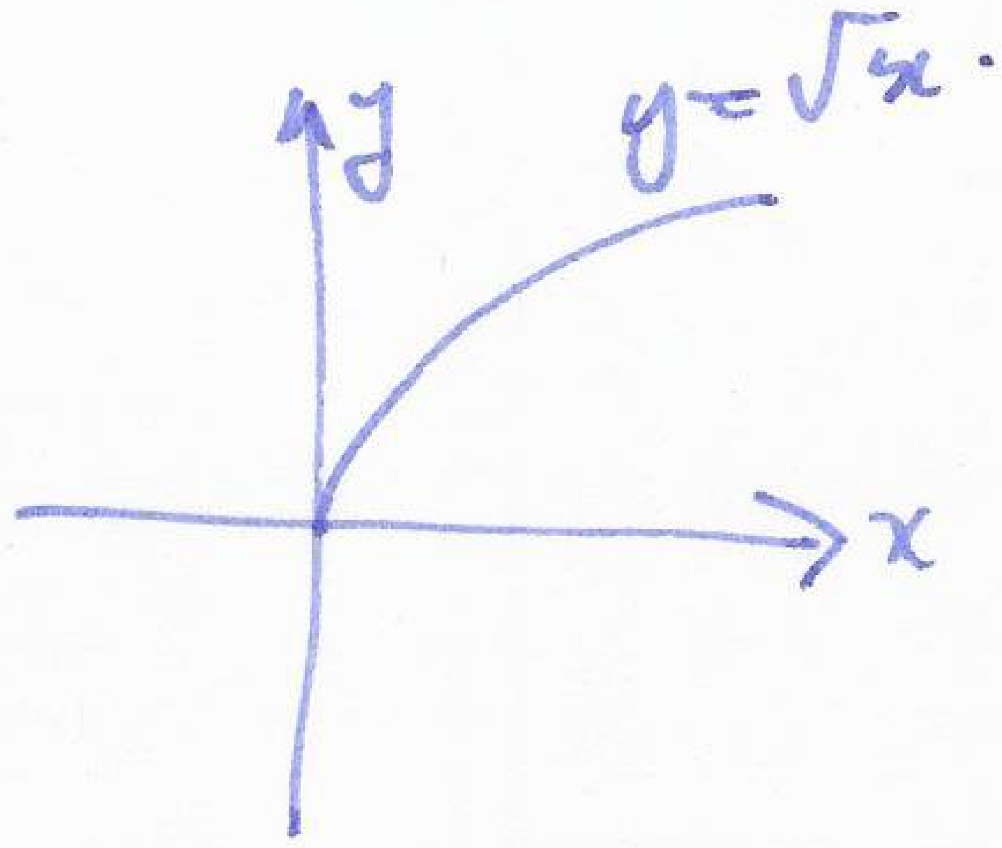
elementary functions: functions made from these by addition/multiplication/composition/inverses eg $\frac{1+e^{\sqrt{x}}}{\sin(\cos(x))}$.

Example composition of functions.

Q: find domain / range of ~~$\sqrt{1-x}$~~ $f(x) = 1-x$ composed with $g(x) = \sqrt{x}$.

$$x \overset{f}{\longmapsto} 1-x \overset{g}{\longmapsto} \sqrt{1-x}.$$





$f: \mathbb{R} \to \mathbb{R}$.

$g: [0,\infty) \to [0,\infty)$.

only takes the values.

so domain: values of $1-x$ which are positive

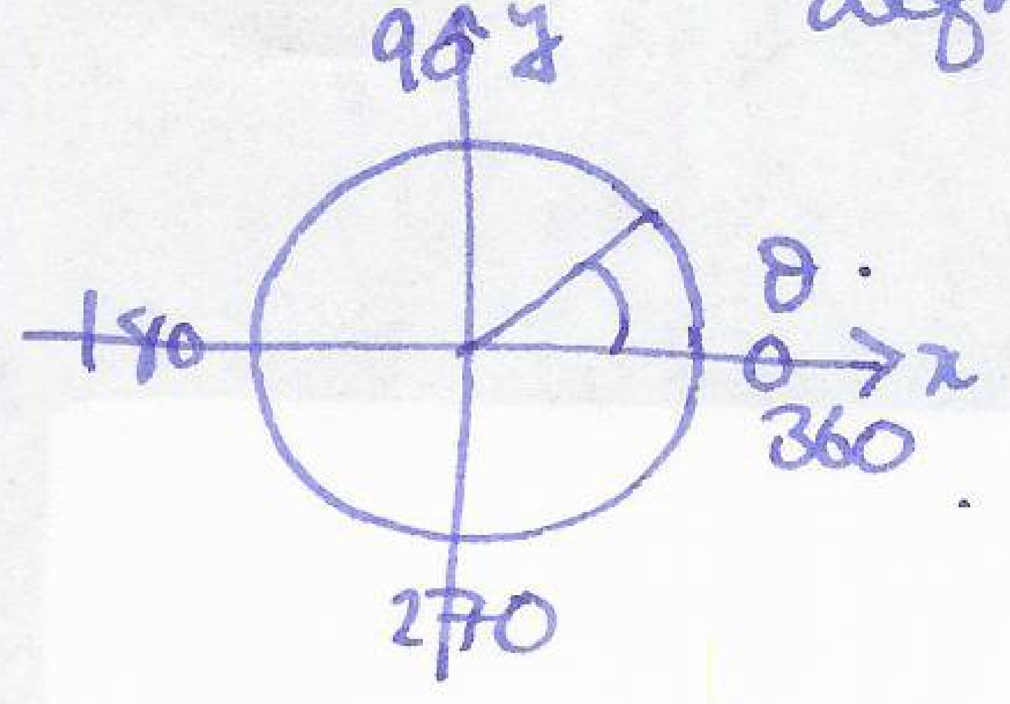
$1-x \geq 0$. $\quad 1 \geq x \quad x \leq 1$. $\quad (-\infty, 1]$.

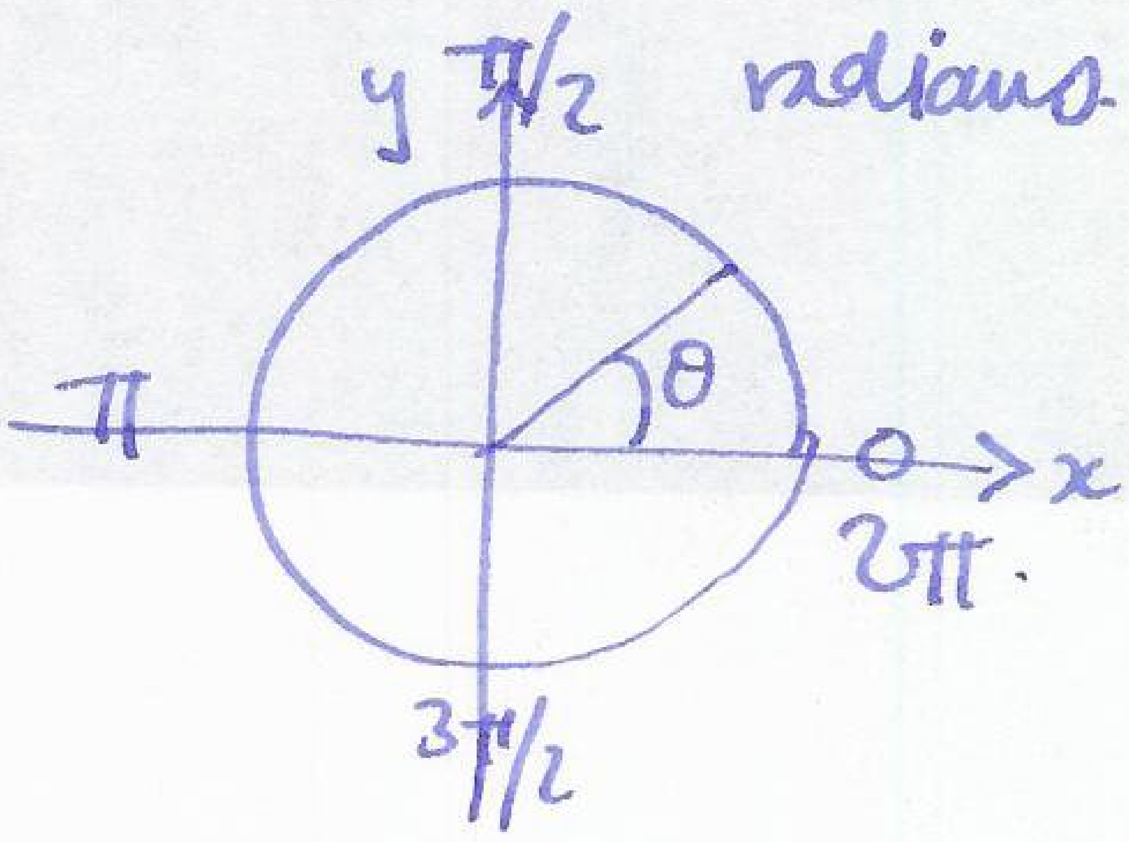
range: $[0,\infty)$.

so $\sqrt{1-x} : (-\infty, 1] \to [0,\infty)$.

§1.4 Trigonometric functions

· angles / radians.



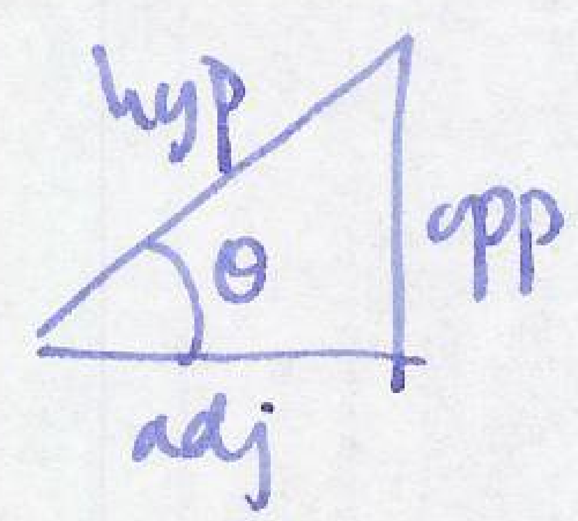


· angle (degrees) $= \frac{360}{2\pi}$ angle (radians) | angle (radians) $= \frac{2\pi}{360}$ angle (degrees)

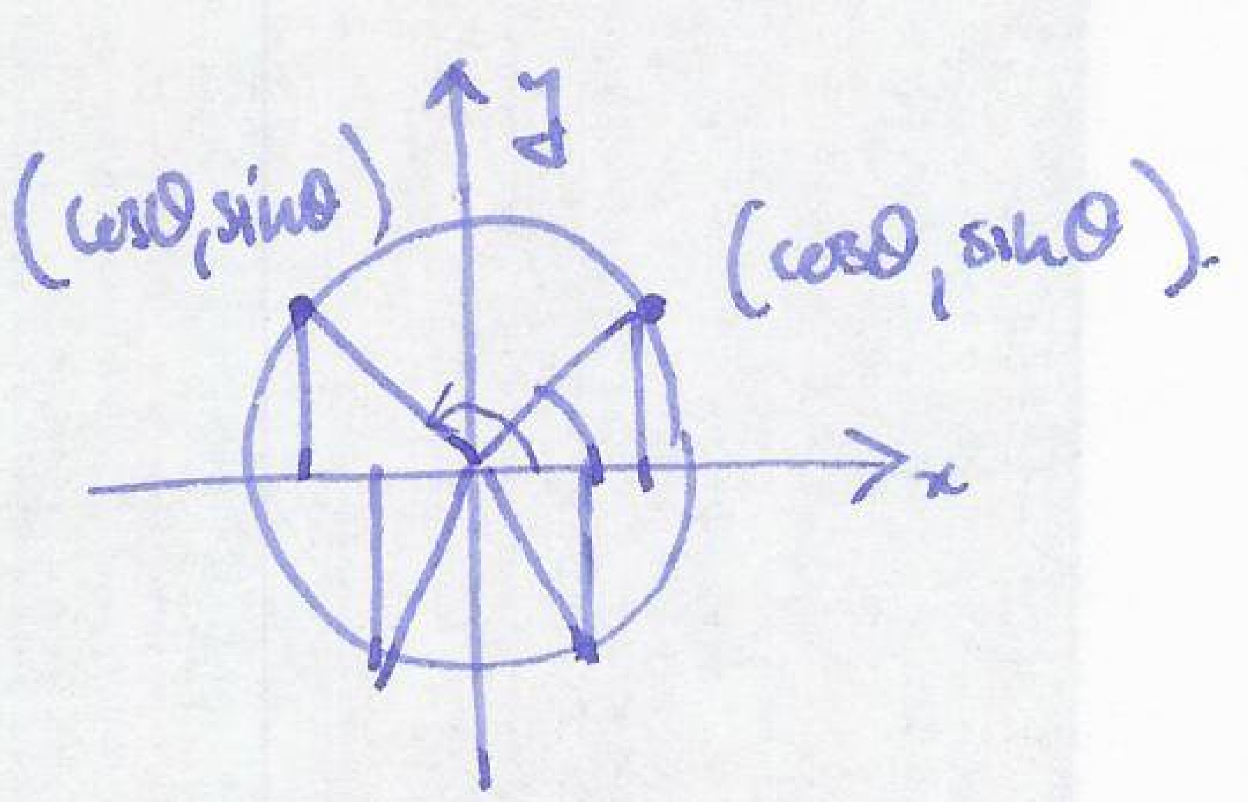
· angle in radians = distance travelled around the unit circle.

· on the unit circle, angle θ is same as angle $\theta + 2\pi$
or $\theta + 2\pi n$ $\quad n \in \mathbb{Z}$.

· recall: right angled triangles.



$\sin\theta = \frac{\text{opp}}{\text{hyp}}$
$\cos\theta = \frac{\text{adj}}{\text{hyp}}$ } can extend these definitions to work for all values of θ.

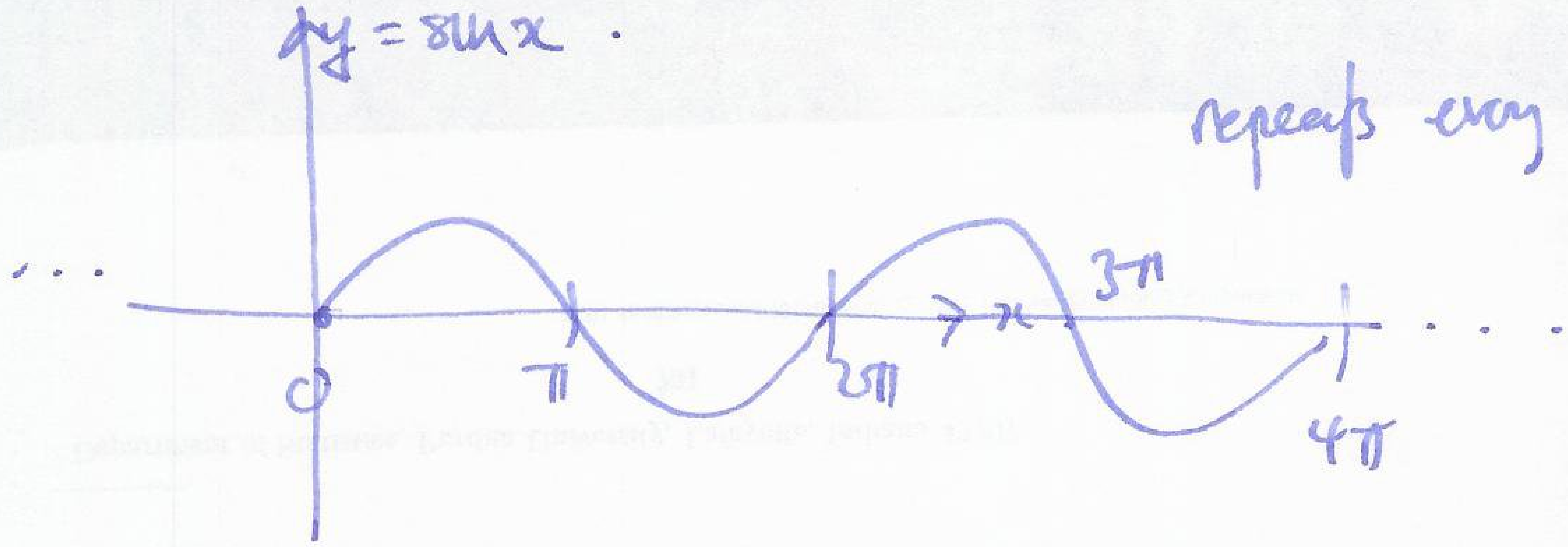


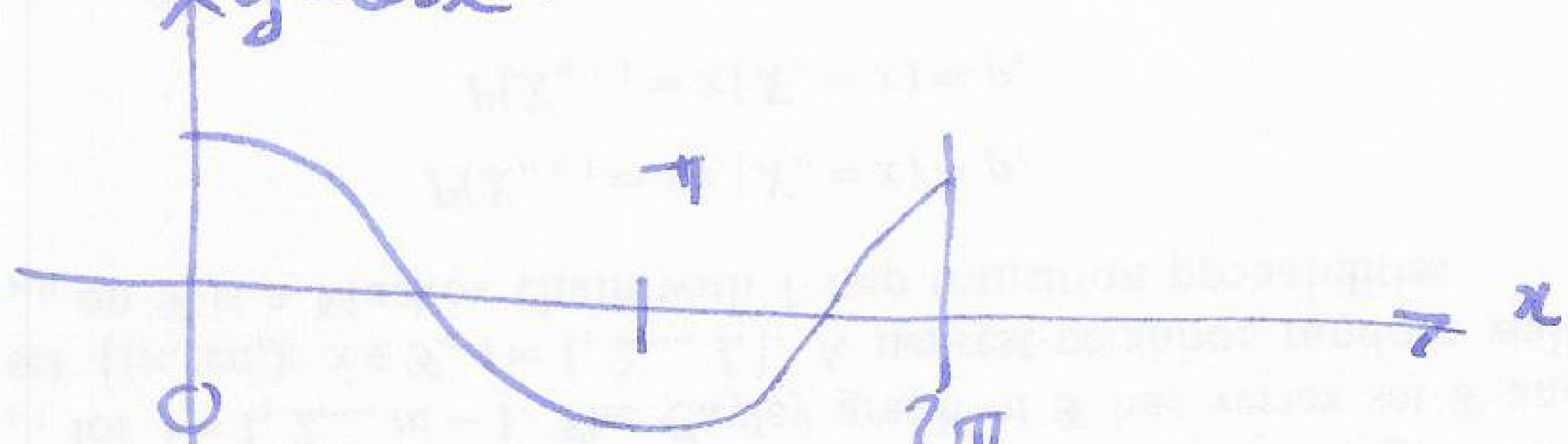
note: $\sin(-\theta) = \sin\theta$
$\cos(-\theta) = \cos\theta$

special values:

θ	0	$\pi/2$	$\pi/4$	$\pi/3$	$\pi/6$
$\sin\theta$	0	1	$\sqrt{2}/2$	$\frac{\sqrt{3}}{2}$	$1/2$
$\cos\theta$	1	0	$\sqrt{2}/2$	$\frac{1}{2}$	$\sqrt{3}/2$

The graphs of $\sin x$ and $\cos x$ are periodic with period 2π.



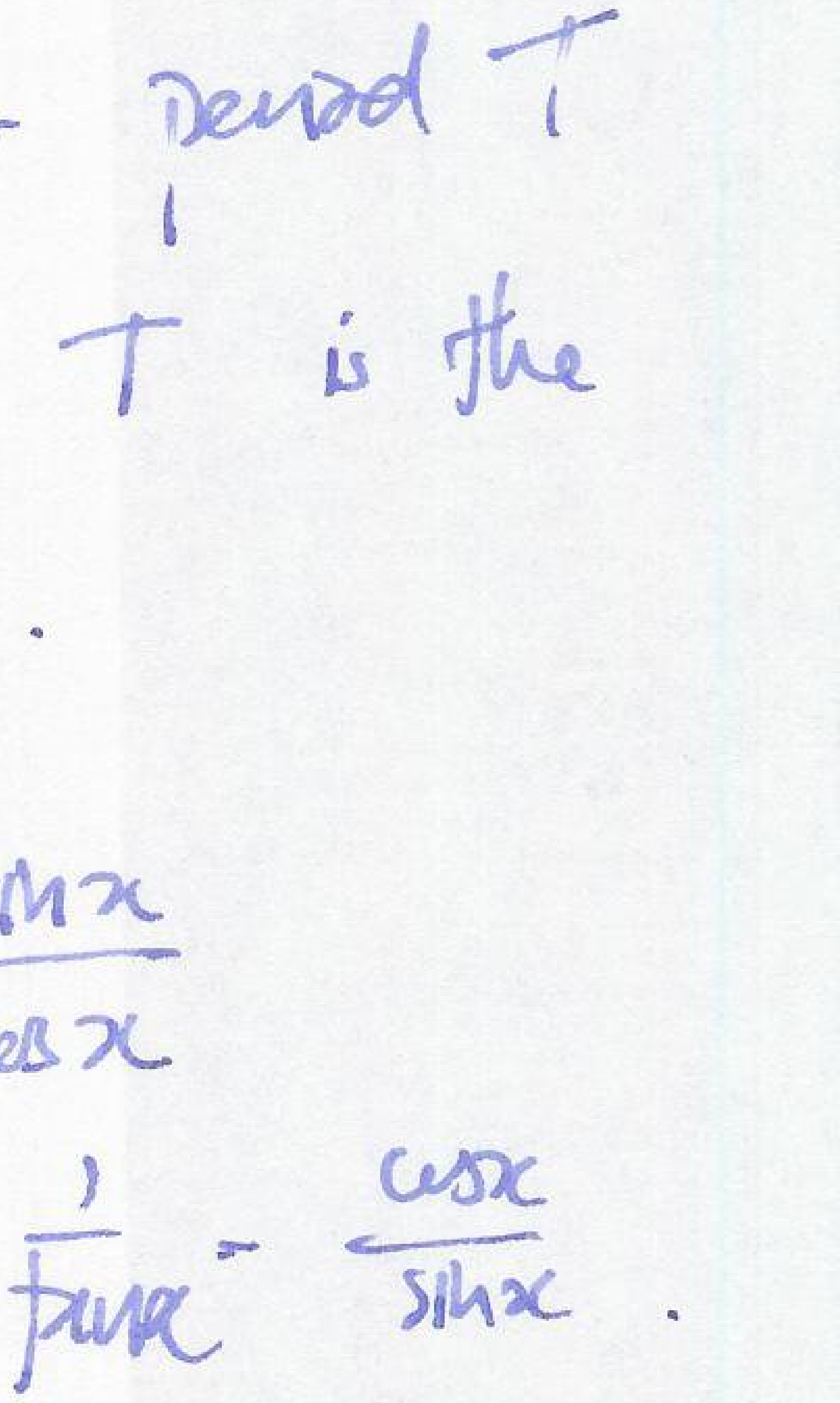


In general, a function $f(x)$ is periodic with period T if $f(x+T) = f(x)$ for all x, and T is the smallest such number for which this is true.

Other standard trig functions : $\tan x = \dfrac{\sin x}{\cos x}$

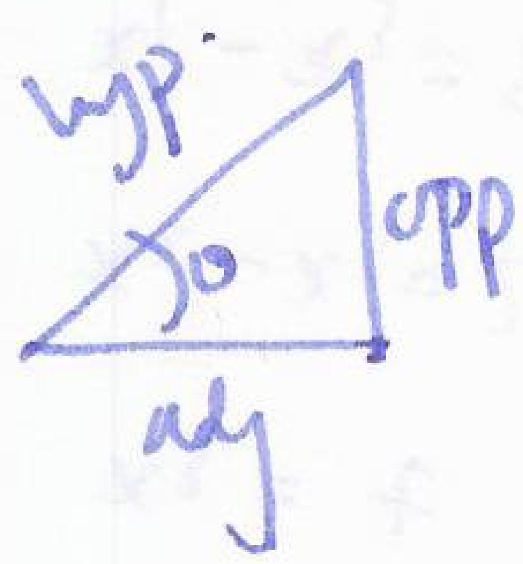
$\operatorname{cosec} x = \dfrac{1}{\sin x}$ $\quad \sec x = \dfrac{1}{\cos x}$ $\quad \cot x = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$.

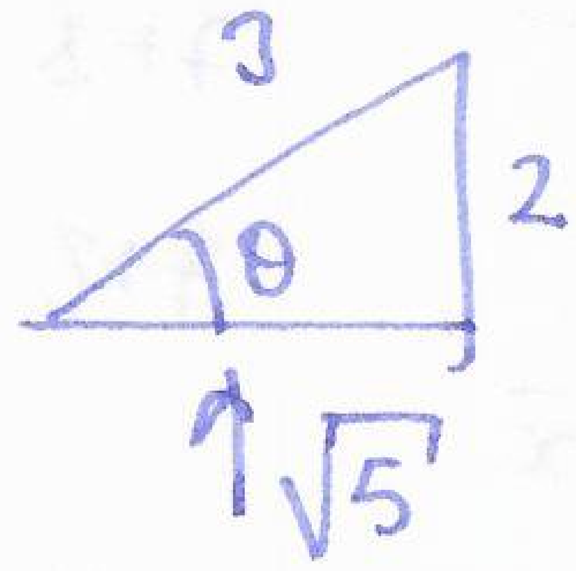
Trig identities



Pythagorean identity : $\sin^2\theta + \cos^2\theta = 1$

Examples spose $\frac{\sin\theta}{\cancel{\cos\theta}}$ ... more precisely: spose $\sin\theta$ ($\cancel{\cos\theta}$) $= 2/3$ find $\cos\theta$ ($\cancel{\sin\theta}$), $\tan\theta$.





$\cos\theta = \frac{\sqrt{5}}{3}$ $\tan\theta = \frac{2}{\sqrt{5}}$.

what about $\sin 2\theta$? $\sin 2\theta = 2\sin\theta\cos\theta = 2\,\frac{2}{3}\cdot\frac{\sqrt{5}}{3} = \frac{4\sqrt{5}}{9}$.

gives: $1+\frac{\cos^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$ $\qquad$ $\frac{\sin^2\theta}{\cos^2\theta}+1=\frac{1}{\cos^2\theta}$

$1+\cot^2\theta=\operatorname{cosec}^2\theta$ $\qquad$ $\tan^2\theta+1=\sec^2\theta$.

more useful identities:

<u>addition</u>:

$\sin(x+y)=\sin x\cos y+\cos x\sin y$

$\cos(x+y)=\cos x\cos y-\sin x\sin y$

double angle: $\cos 2x=\cos^2 x-\sin^2 x=1-2\sin^2 x$
$=2\cos^2 x-1$

$\sin 2x=2\sin x\cos x$.

shift: $\sin(x+\frac{\pi}{2})=\cos x$ $\qquad$ $\cos(x+\frac{\pi}{2})=-\sin x$.

§1.5 Inverse functions

$f: X\to Y$.
$x\mapsto f(x)$

$X\leftarrow Y: f^{-1}$.
$x \leftarrow\!\shortmid f(x)$

<u>want</u>:
the inverse function should be the <u>reverse</u> function

<u>problem</u> the inverse is usually <u>not</u> a function.

suppose there is $a,b\in X$ with $f(a)=f(b)$ what is $f^{-1}(f(a))$?