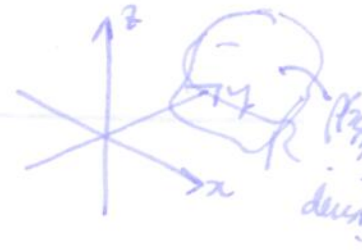
$$\int_{-1}^{1}(1-x^2)\frac{\pi}{2}\,dx = \frac{\pi}{2}\left[x-\frac{1}{3}x^3\right]_{-1}^{1} = \frac{\pi}{2}\left(1-\frac{1}{3}+1-\frac{1}{3}\right) = \frac{2\pi}{3}.$$

Centers of mass, moments of inertia in 3d



mass $m = \iiint_R \rho\, dV$

$$M_{xy} = \iiint_R z\rho\, dV, \quad M_{xz} = \iiint_R y\rho\, dV, \quad M_{yz} = \iiint_R x\rho\, dV$$

center of mass $(\bar{x}, \bar{y}, \bar{z}) = \left(M_{yz}/m,\ M_{xz}/m,\ M_{xy}/m\right)$.

moments of inertia about x-axis : $I_x = \iiint (y^2+z^2)\rho\, dV$ etc.

$$I_y = \iiint (x^2+z^2)\rho\, dV$$
$$I_z = \iiint (x^2+y^2)\rho\, dV.$$

§14.7 Triple integrals in spherical and cylindrical coordinates

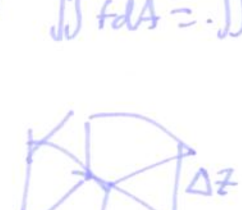
Cylindricals

2d cartesians  polars 

$\sum f(x_i,y_i)\Delta x\Delta y$
$\updownarrow$ $\iint f\,dx\,dy$

$\sum f(r_i,\theta)\, r\Delta r\Delta\theta$.
$\updownarrow$ $\iint f\,dA = \iint f(r,\theta)\, r\,dr\,d\theta$.

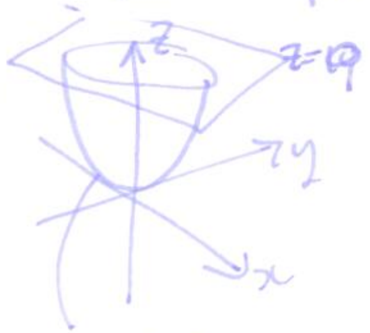
3d. cartesians 

polars 

$\sum f(x_i,y_i,z)\Delta x\Delta y\Delta z \leftrightarrow \iiint f\,dx\,dy\,dz$

$\sum f(x_i,y_i,z)\, r\Delta r\Delta\theta\Delta z$
$\iiint f\, r\,dr\,d\theta\,dz$.

<u>Example</u> shape/region R: above $z=x^2+y^2$, below $z=9$.



$\iiint_R f(x,y,z)\,dV$ by cylindrical use polars

$x = r\cos\theta$

$y = r\sin\theta$

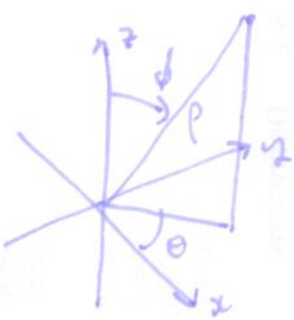
$z = z$.

$$\int_0^{2\pi}\int_0^{3}\int_{r^2}^{9} f(r\cos\theta, r\sin\theta, z)\; r\,dz\,dr\,d\theta$$

$z = x^2+y^2$

$z = r^2$

Spherical coordinates



$x = \rho\cos\theta\sin\phi$

$y = \rho\sin\theta\sin\phi$

$z = \rho\cos\phi$

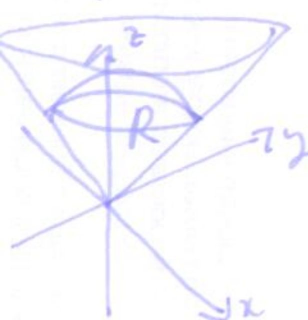
Volume =

$\rho^2\sin\phi\,\Delta\theta\Delta\phi\Delta\rho$.

$\iiint_R f(x,y,z)\,dV$.

$\iiint_R f(\rho\cos\theta\sin\phi, \dots)\,\rho^2\sin\phi\,d\theta\,d\phi\,d\rho$

<u>Example</u> R = region inside cone $z^2=x^2+y^2$ and sphere $x^2+y^2+z^2=1$.
$z>0$

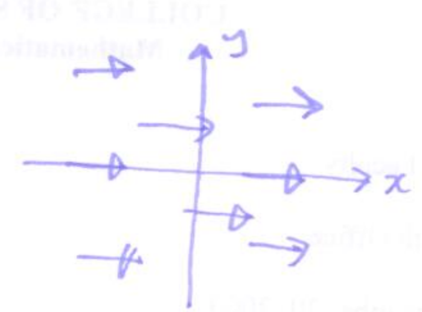


$$\int_0^{\pi/4}\int_0^{2\pi}\int_0^{1} f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\,\rho^2\sin\phi\; d\rho\,d\theta\,d\phi.$$

§15.1 Vector fields

Def$^{\underline{n}}$: A vector field is a function which assigns a vector to each point in the domain.

Example (2d) · $\underline{F}(x,y) = \langle 1, 0 \rangle$



· $F(x,y) = \langle x, y \rangle$

· any gradient vector field. $\underline{F} = \nabla f$, $f(x,y)$.

Def$^{\underline{n}}$ A vector field is conservative if $\underline{F} = \nabla f$ for some scalar function f called the potential function for $\underline{F}$.

Example (3d) (inverse square field).

$$\underline{F}(x,y,z) = \underline{F}(\underline{x}) = \frac{1}{\|\underline{x}\|^3}\,\underline{x}$$

this is conservative with potential function $f(\underline{x}) = -\frac{1}{\|\underline{x}\|} = -\frac{1}{\sqrt{x^2+y^2+z^2}}$.

check:

∇f: $\frac{\partial f}{\partial x} = +\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2} \cdot 2x = \frac{x}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$. so $\nabla f = \frac{1}{\|\underline{x}\|^3}\,\underline{x}$.

not all vector fields are conservative! How to tell?

2d: Th$^{\underline{m}}$ $\underline{F}(x,y) = \langle M(x,y), N(x,y) \rangle$ is conservative iff $\boxed{\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}}$.

3d: Def$^{\underline{n}}$ curl $\underline{F}$ = $\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$

$\underline{F}(\underline{x}) = \langle M, N, P \rangle$

note $\nabla \times \underline{F}$ is also a 3d vector field!

$$= \langle P_y - N_z, -P_x + M_z, N_x - M_y \rangle.$$

Thm $\underline{F}(x,y,z)$ is conservative iff $\underline{\nabla} \times \underline{F} = \underline{0}$.

How to find the potential function:

example (2d) $\underline{F}(x,y) = \langle y, x \rangle = \underline{\nabla} f$ for some f.

so
$$\left.\begin{array}{l} \dfrac{\partial f}{\partial x} = y \Rightarrow f(x,y) = xy + c_1(y) \\ \dfrac{\partial f}{\partial y} = x \Rightarrow f(x,y) = xy + c_2(x) \end{array}\right\} \begin{array}{l} \text{but } \dfrac{\partial}{\partial y} c_1 = 0, \dfrac{\partial}{\partial x} c_2 = 0 \\ \Rightarrow f(x,y) = xy + c \end{array}$$

c constant

Divergence

2d: $\underline{F}(\underline{x}) = \langle M, N \rangle$, then $\underline{\nabla} \cdot \underline{F} = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}$

3d: $\underline{F}(\underline{x}) = \langle M, N, P \rangle$, then $\underline{\nabla} \cdot \underline{F} = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y} + \dfrac{\partial P}{\partial z}$

if $\underline{\nabla} \cdot \underline{F} = 0$ say $\underline{F}$ is divergence free (fluid flows $\leftrightarrow$ incompressible)

Useful fact: $\underline{\nabla} \cdot \underline{\nabla} \times \underline{F} = 0$ (div curl $\underline{F} = 0$).