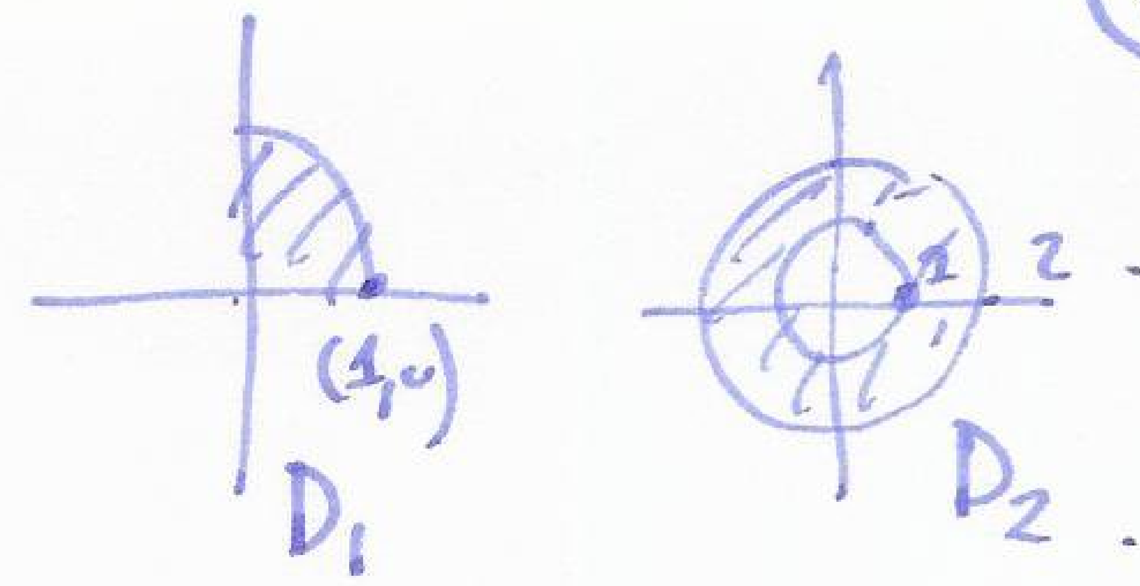
§14.3 Polar coordinates

double integrals over regions $\iint_D f(x,y)\,dA$



some regions hard to describe in cartesian coordinates, but easy to describe in polars.

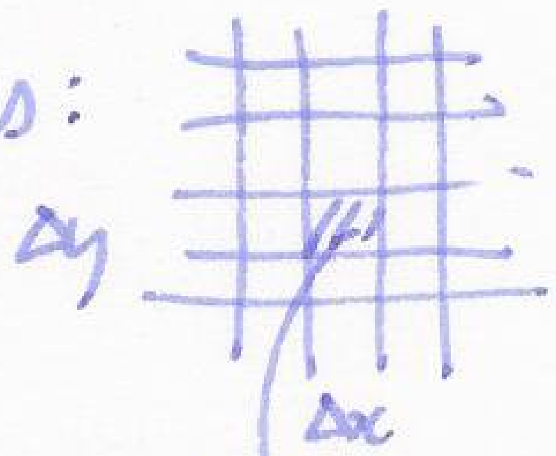
D_1 : $0 \le \theta \le \pi/4$ $\quad 0 \le R \le 1$

D_2 : $0 \le \theta \le 2\pi$ $\quad 1 \le R \le 2$

__Useful fact__ : in polar coords $\iint_D f(x,y)\,dA = \iint_D f(x(r,\theta), y(r,\theta))\, r\,dr\,d\theta$

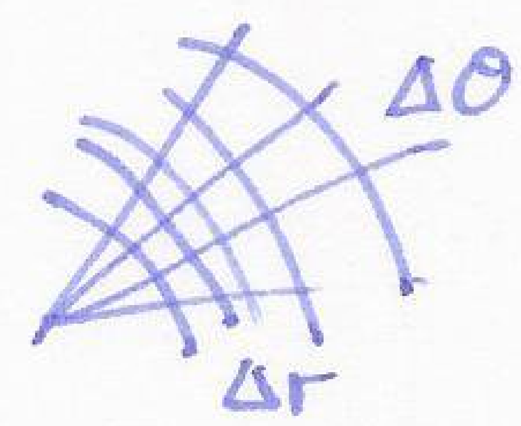
$\boxed{dA = r\,dr\,d\theta}$ do not forget the r.

__idea__ : cartesians:



each square of size $\Delta x \times \Delta y$

polars :



regions of different size

size approximately $r\Delta r\Delta\theta$.

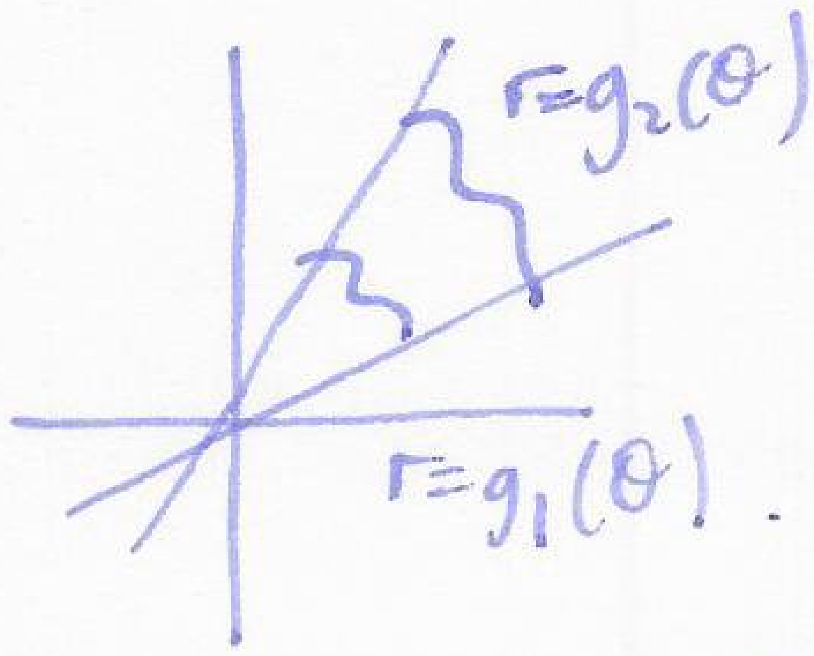
area $r\Delta\theta$

$r\Delta\theta$

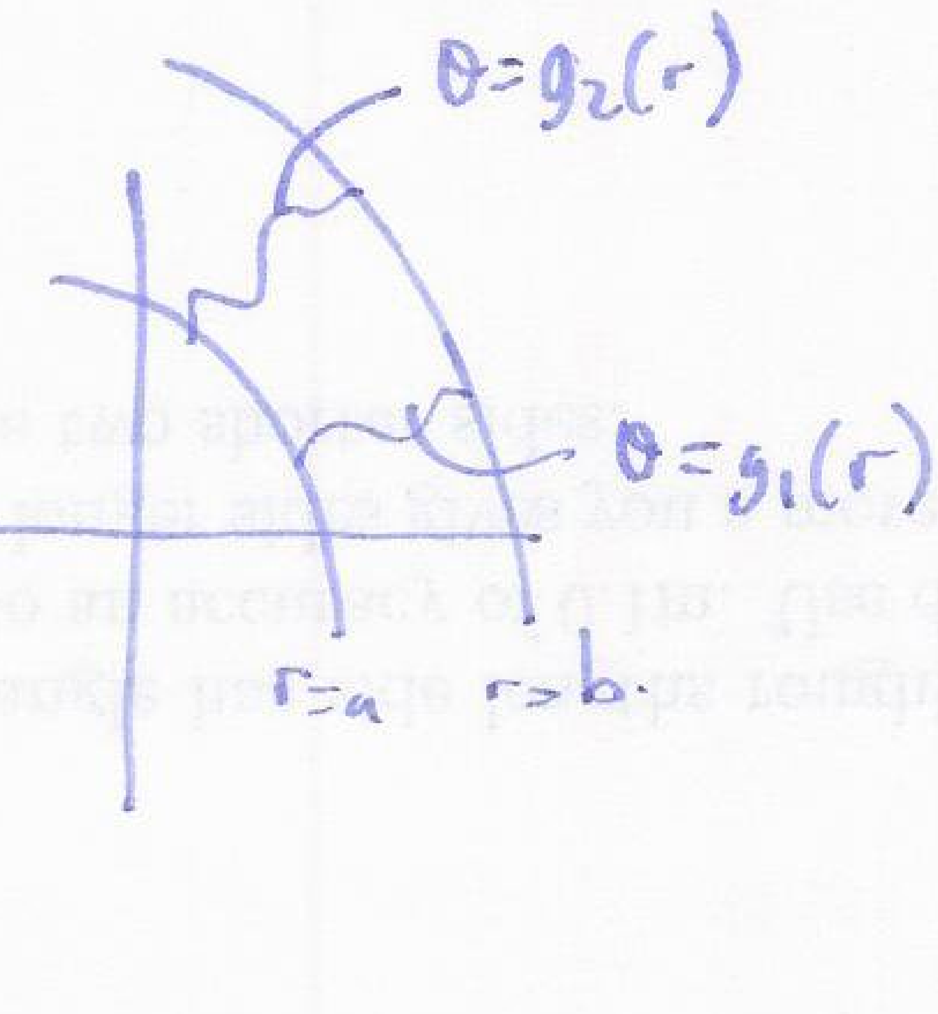
$(r+\Delta r)\Delta\theta - r\Delta\theta$

$= r\Delta r\Delta\theta$.

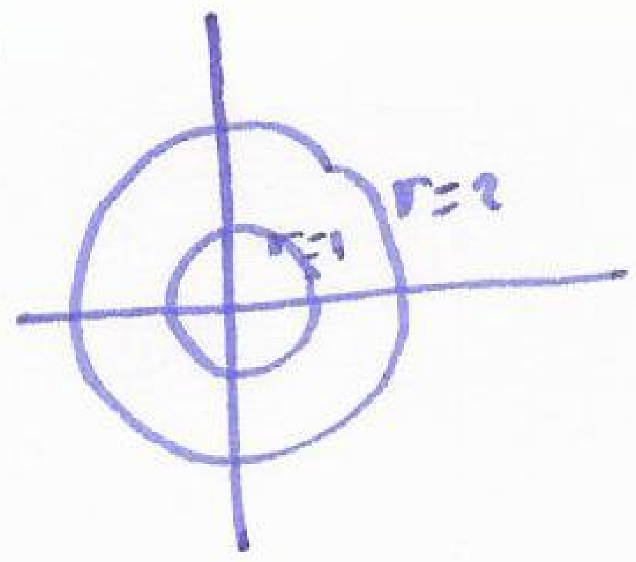
$$\int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} r\,dr\,d\theta$$



$$\int_a^b \int_{g_1(r)}^{g_2(r)} r\,d\theta\,dr$$



Example

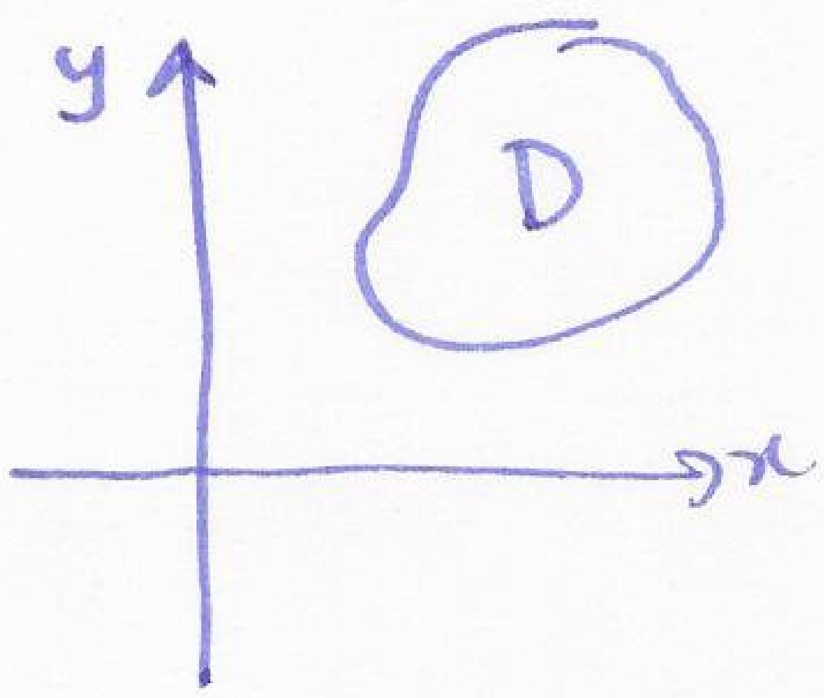


$$\iint_D x^2+y^2 \, dA .$$

$$\int_0^{2\pi}\int_1^2 r^2 \, r \, dr \, d\theta \qquad \left[\tfrac{1}{4}r^4\right]_1^2 = \tfrac{1}{4}(16-1)$$

$$\int_0^{2\pi} \frac{15}{4}\, d\theta = \frac{15\pi}{2} .$$

§14.4 Centers of mass

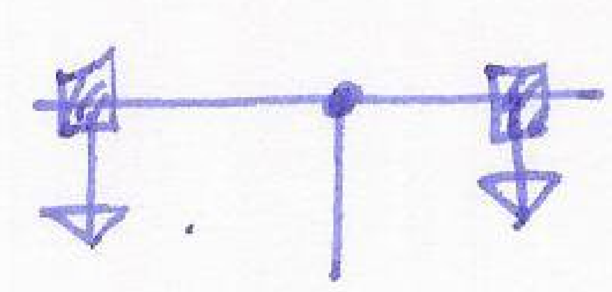


$\rho(x,y) =$ density of region D.

$$\text{mass} \quad m = \iint_D \rho(x,y)\, dA .$$

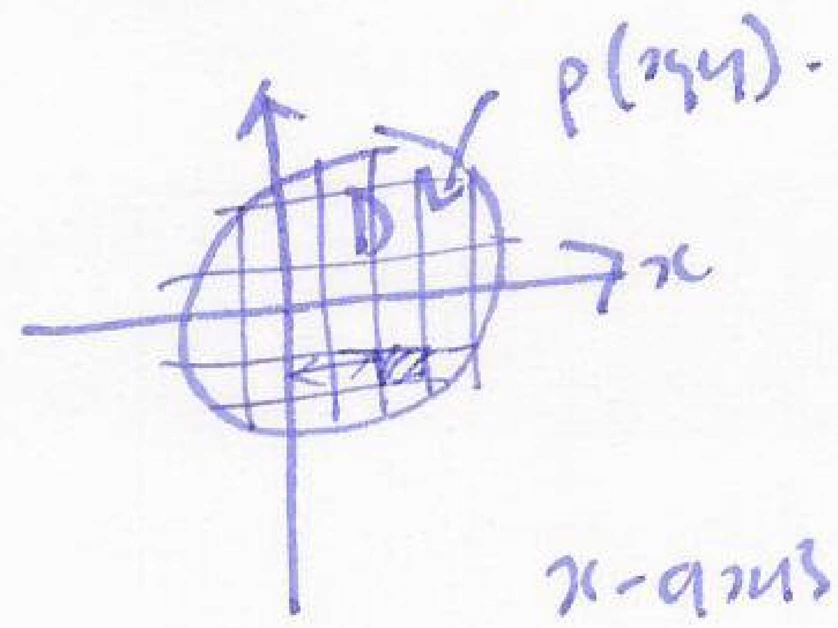
moments, turning force

1d:



turning force = force × distance

2d



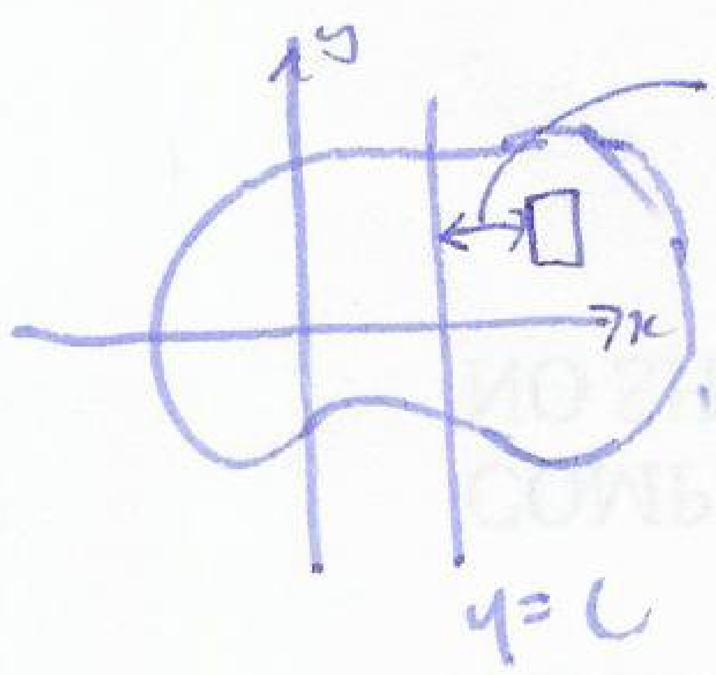
turning force / moment about x-axis

x-axis: $M_x = \displaystyle\iint_D y\,\rho(x,y)\, dA$

y-axis: $M_y = \displaystyle\iint_D x\,\rho(x,y)\, dA$.

center of mass : $(\bar{x},\bar{y}) = \left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right)$ $\quad m =$ total mass.

moment about x-axis $\Leftrightarrow$ $y=0$. what about moment about $y=c$?



$y-c$ instead of y

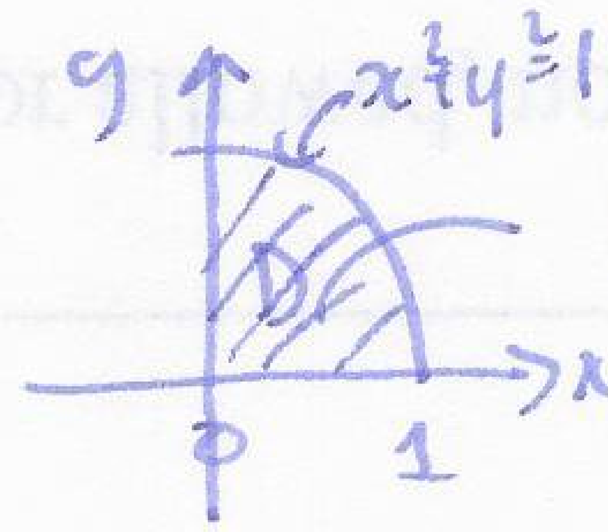
$$\iint_D (y-c)\rho\, dA = \iint_D y\rho\, dA - \iint_D c\rho\, dA.$$

$$= M_x - cm = 0$$

↑ balanced.

$$\Rightarrow \quad c = \frac{M_x}{m}.$$

Example find center of mass of



$\rho(x,y) = x^2+y^2$.

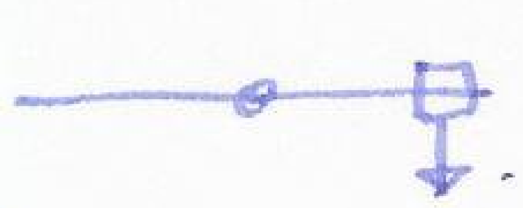
$$m = \iint_D (x^2+y^2)\, dA \qquad M_x = \iint_D y(x^2+y^2)\, dA \qquad M_y = \iint_D x(x^2+y^2)\, dA.$$

$$m = \int_0^1 \int_0^{\sqrt{1-x^2}} x^2+y^2\, dy\, dx = \int_0^{\pi/2}\int_0^1 r^2\, r\, dr\, d\theta = \frac{\pi}{2}\left[\frac{1}{4}r^4\right]_0^1 = \frac{\pi}{8}.$$

$$M_x = \int_0^{\pi/2}\int_0^1 r\sin\theta\, r^2\, r\, dr\, d\theta = \int_0^1 r^4\, dr = \left[\frac{1}{5}r^5\right]_0^1 = \frac{1}{5}.$$

$$\frac{1}{5}\int_0^{\pi/2} \sin\theta\, d\theta = \frac{1}{5}\left[-\cos\theta\right]_0^{\pi/2} = \frac{1}{5}(-0+(+1)) = \frac{1}{5}. \quad \left(\frac{8}{5\pi}, \frac{8}{5\pi}\right).$$

Moment of intertia / angular momentum.

1d  angular moment = d × force.

$$I_y = \iint x^2\rho\, dA \qquad I_x = \iint y^2 \rho\, dA.$$