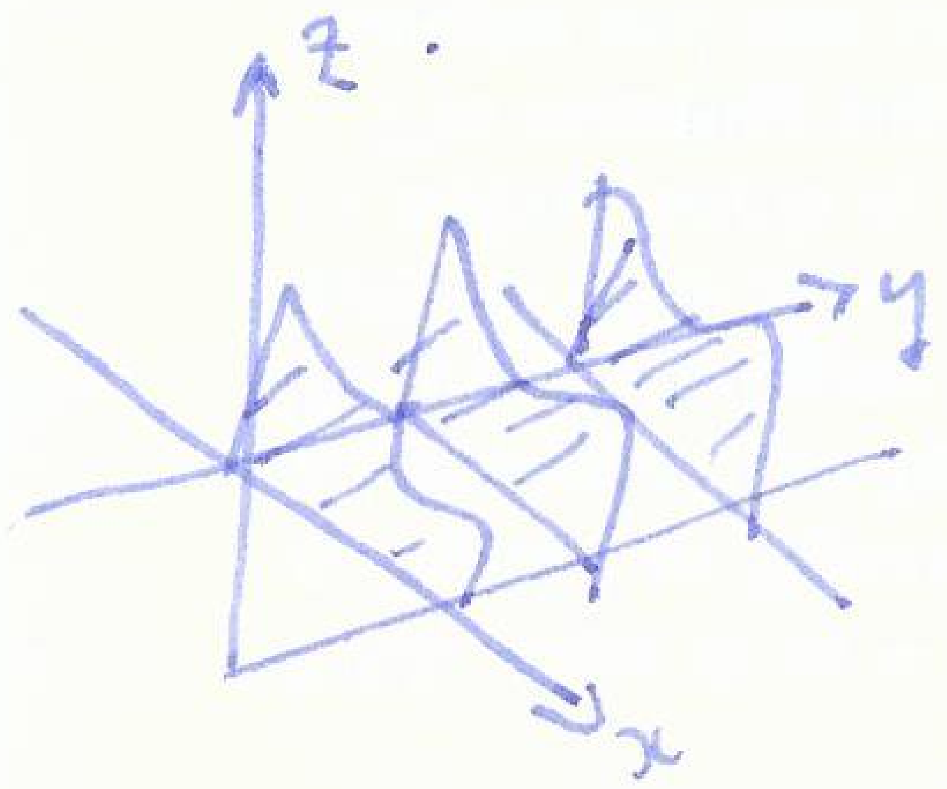
example $\int_0^1 \int_0^1 xy \, dx \, dy$ evaluate $\int_0^1 xy \, dx$ first.

$$\left[\tfrac{1}{2}x^2 y\right]_0^1 = \tfrac{1}{2}y$$

now evaluate $\int_0^1 \tfrac{1}{2}y \, dy = \left[\tfrac{1}{4}y^2\right]_0^1 = \tfrac{1}{4}$

what does this mean?



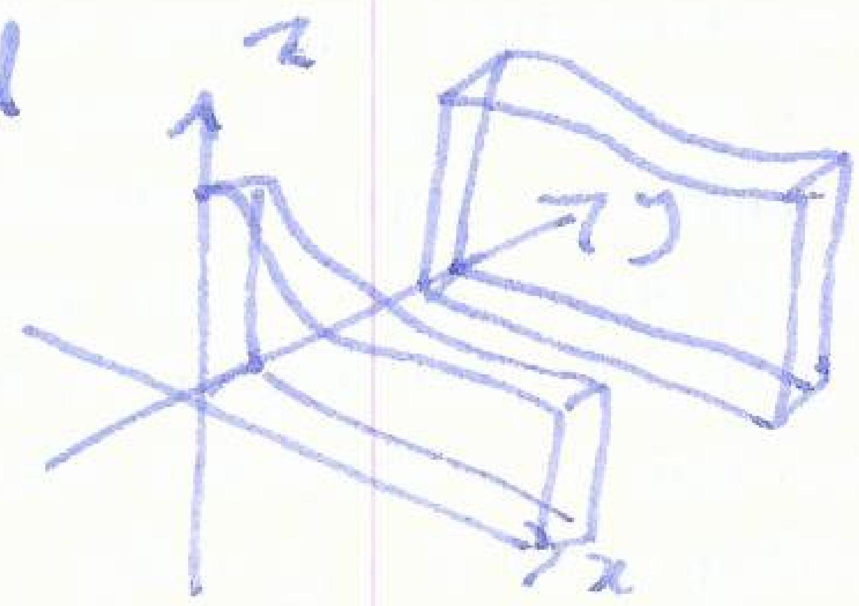
$\int_0^1 xy \, dx$ = area under curve y = const from $x=0$ to $x=1$

recall

1d

sum of rectangles $\approx$ ~~volume~~ area

2d



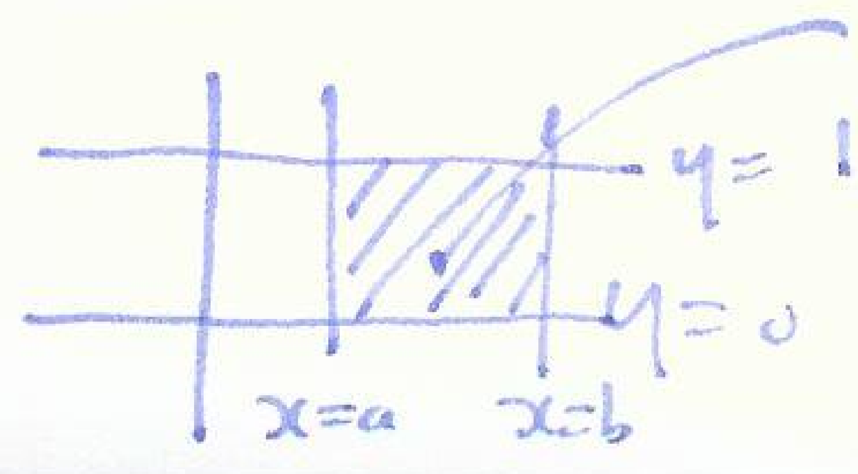
sum of slabs $\approx$ volume.

so $\int_0^1 \int_0^1 f(x,y) \, dx \, dy$ = volume under graph, over unit square $[0,1] \times [0,1]$.

~~important point: order doesn't matter~~:

~~volume under surface~~ $\cancel{= \int_0^1 \int_0^1 f(x,y) \, dx \, dy = \int_0^1 \int_0^1 f(x,y) \, dy \, dx}$.

what about $\int_{y=0}^{y=1} \int_{x=a}^{x=b} f(x,y) \, dx \, dy$

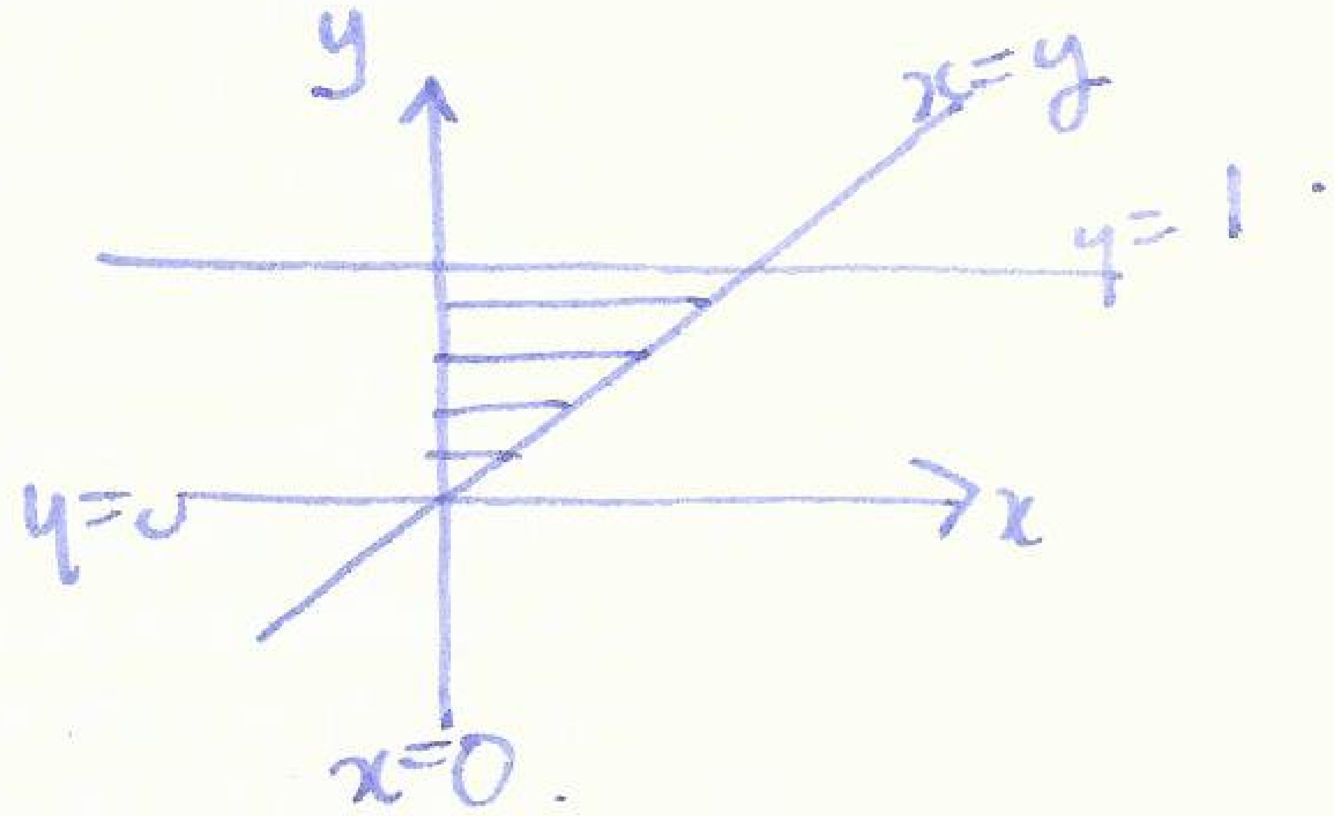


volume over this region.

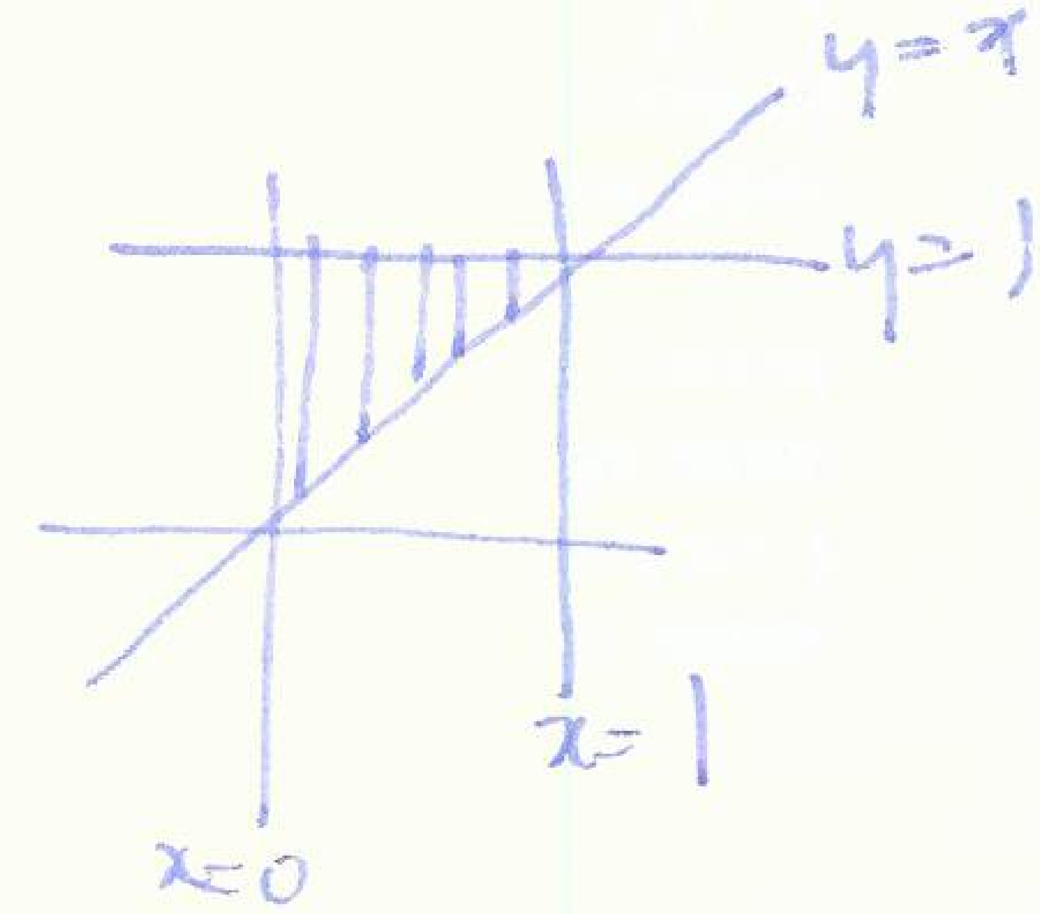
important point: limits describe the region over which you are integrating.

1d: $\int_{x=a}^{x=b} f(x)\,dx$ 2d: $\iint_D f(x,y)\,dA$ integrate over region D

Example $\int_{y=0}^{y=1}\int_{x=0}^{x=y} f(x,y)\,dx\,dy$

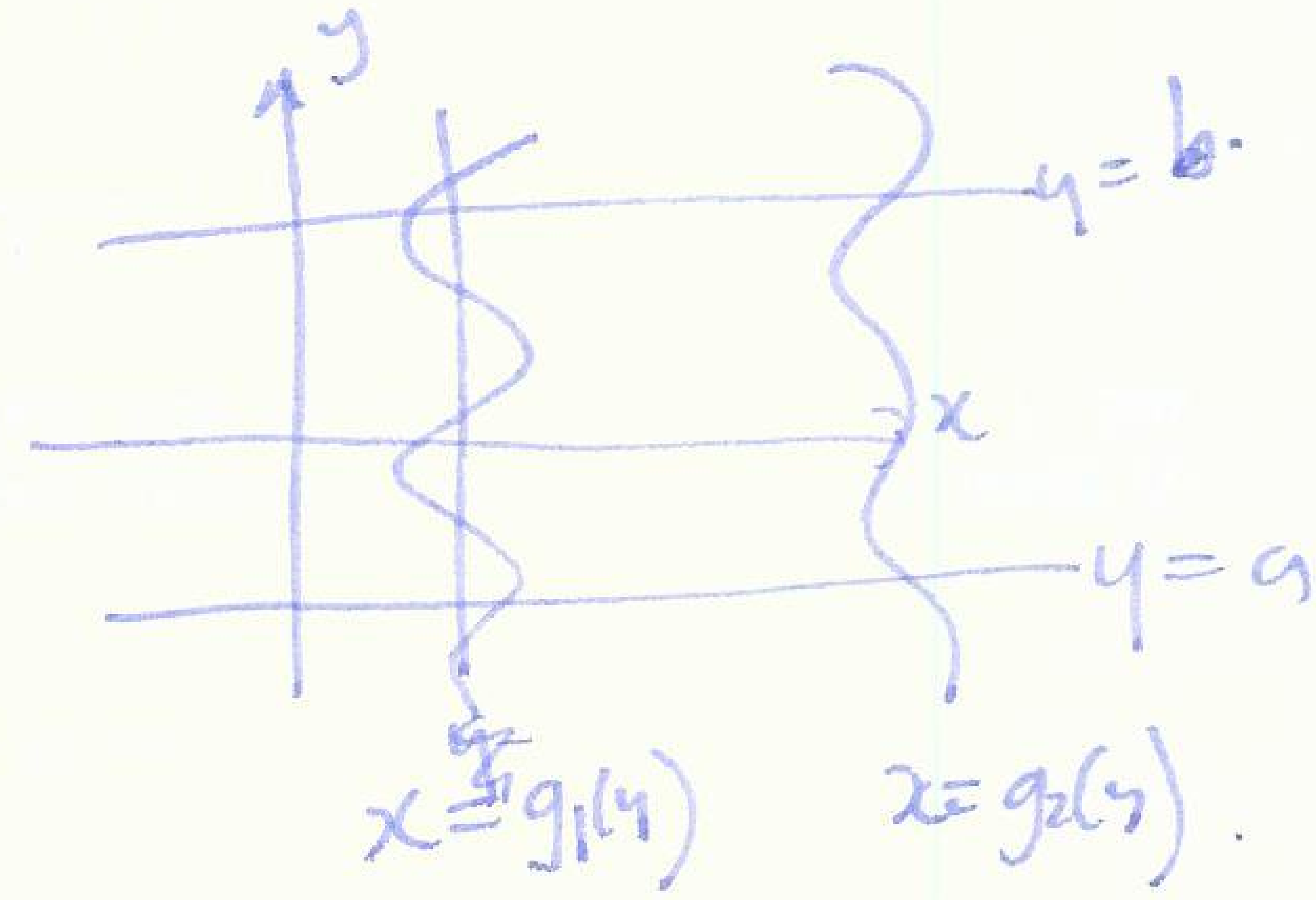


$\int_{x=0}^{x=1}\int_{y=x}^{y=1} f(x,y)\,dy\,dx$ ← same region described in the opposite order

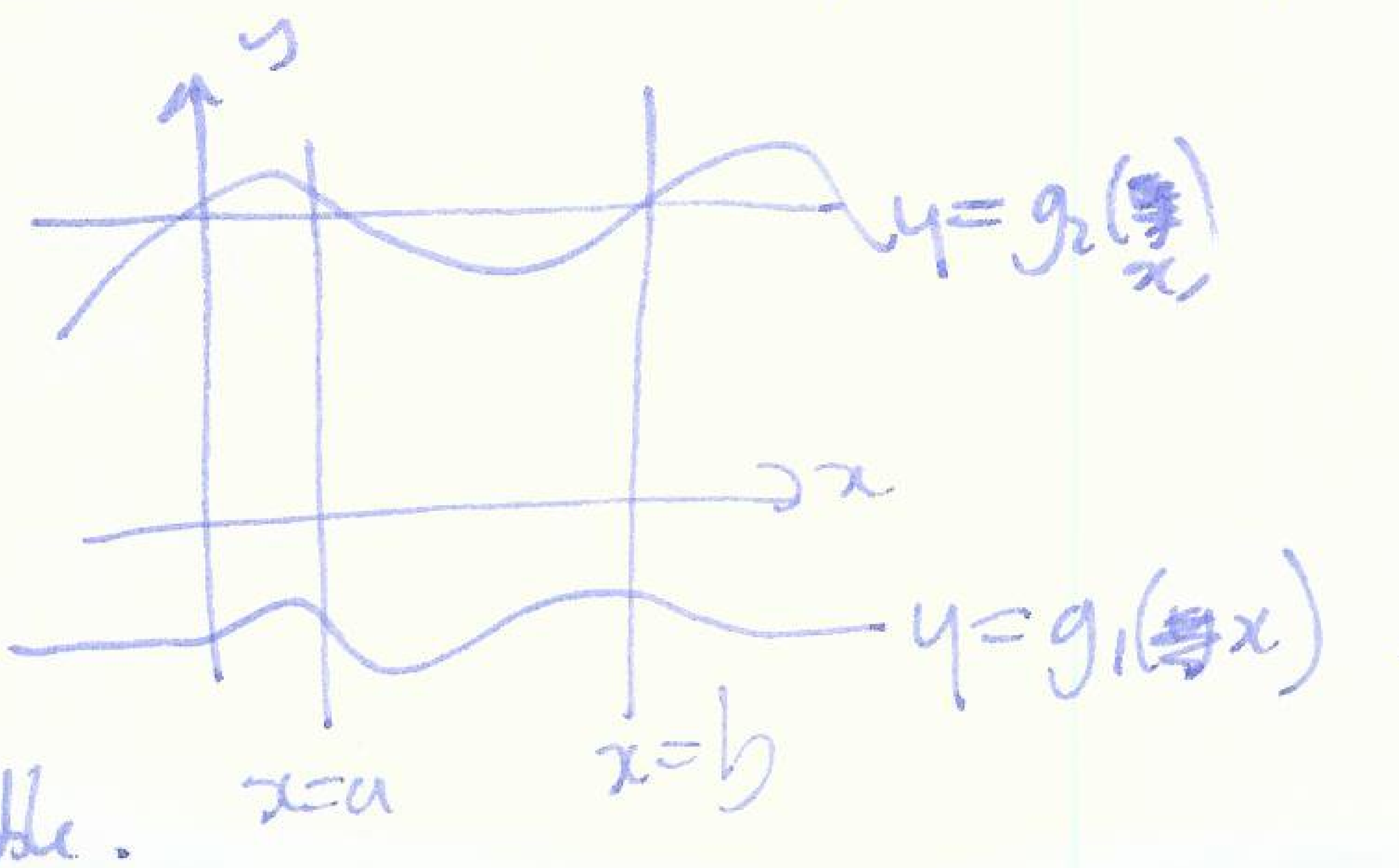


important point: changing order of integration gives same answer, but limits might look different!

Examples: $\int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$



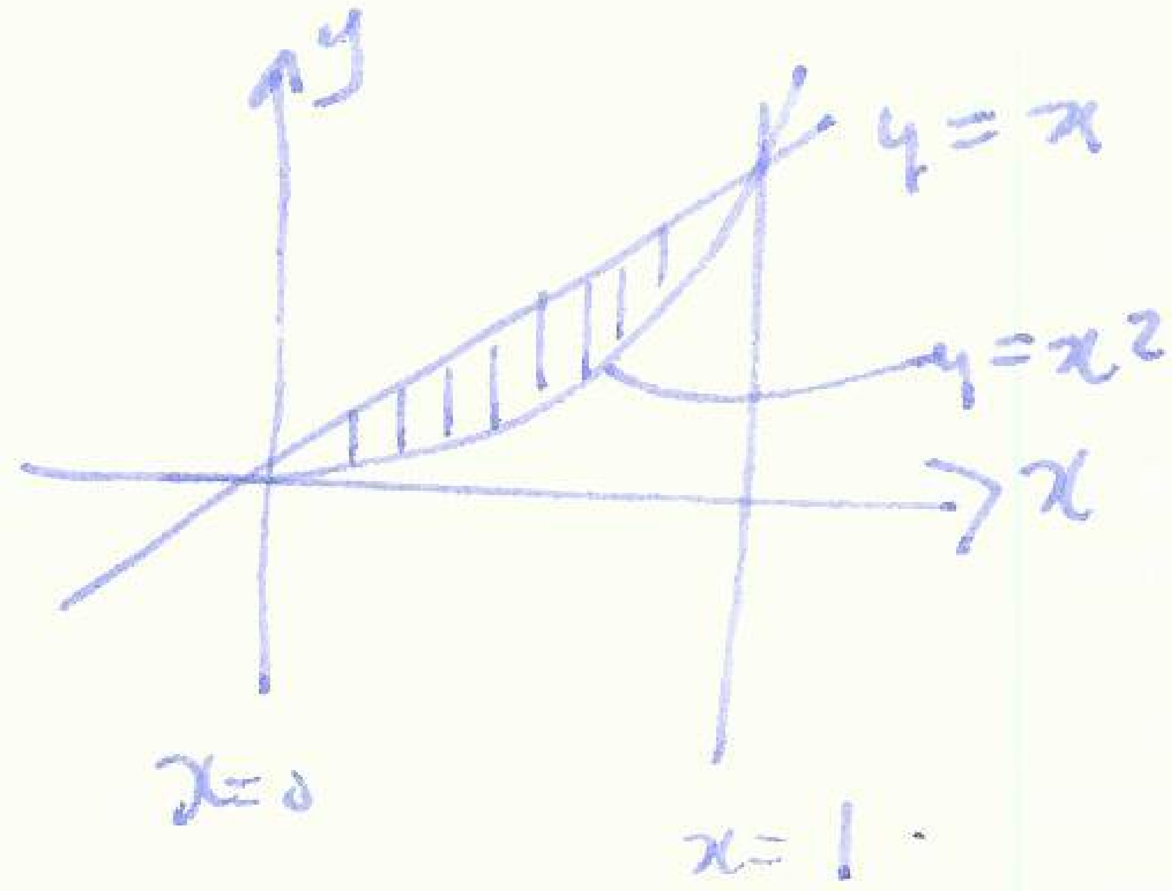
or $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$



rules for limits

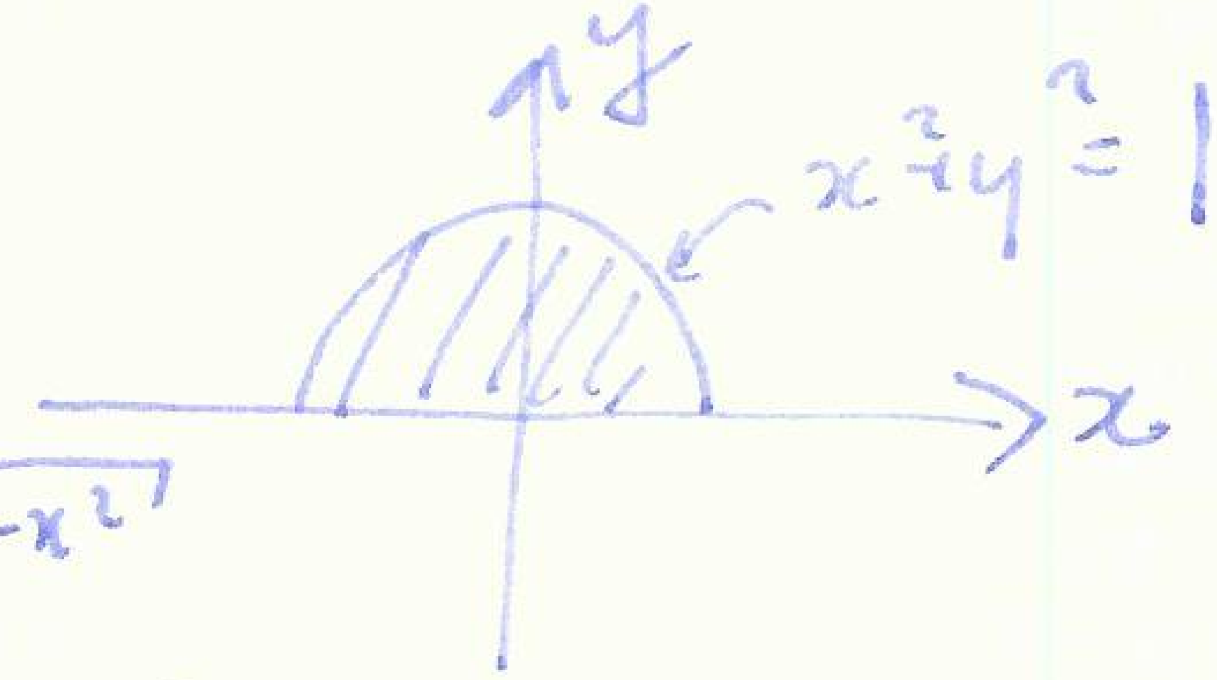
$\underbrace{\int}_{\text{numbers}}\underbrace{\int}_{\text{functions of other variable.}} f(x,y)\,dy\,dx$

Example $$\int_0^1 \int_{x^2}^{x} 1 \, dy \, dx$$



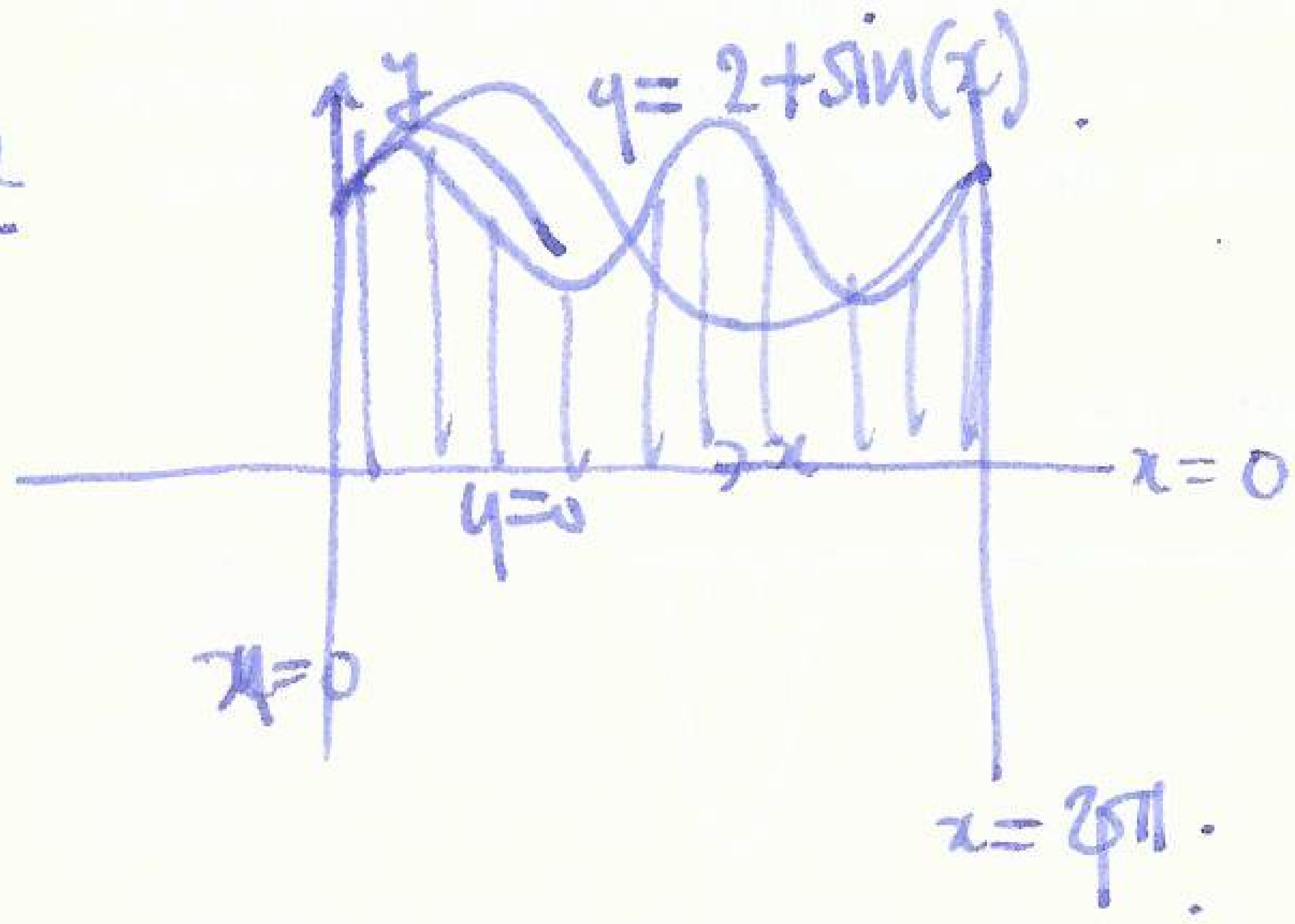
= area of region!

Example turn region into limits:



$$\int_{0}^{1} \int_{-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} f(x,y)\, dx\, dy \quad \text{or} \quad \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} f(x,y)\, dy\, dx .$$

Example

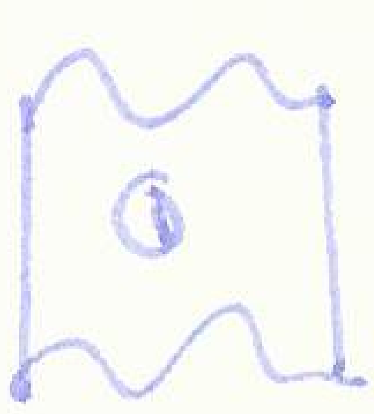


$$\int_0^{2\pi} \int_0^{2+\sin x} f(x,y)\, dy\, dx$$

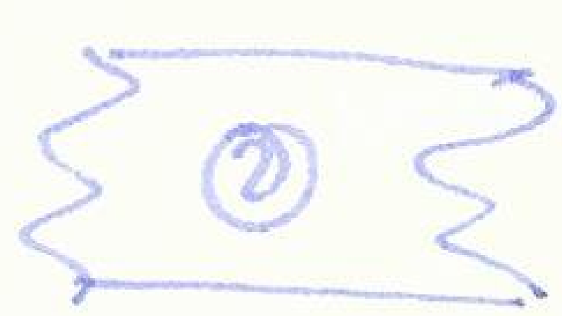
other way?

problem want region to have ~~straight~~ vertical or horizontal sides, with other boundaries given by functions.

eg.



or



bad region

solution: break region up into smaller chunks like 1 or 2.

observation limits determine region need to be able to go from

limits → region

limits ← region