gradient for functions of 3 vars

$f(x,y,z)$ $\quad \underline{\nabla} f = \langle f_x, f_y, f_z \rangle$

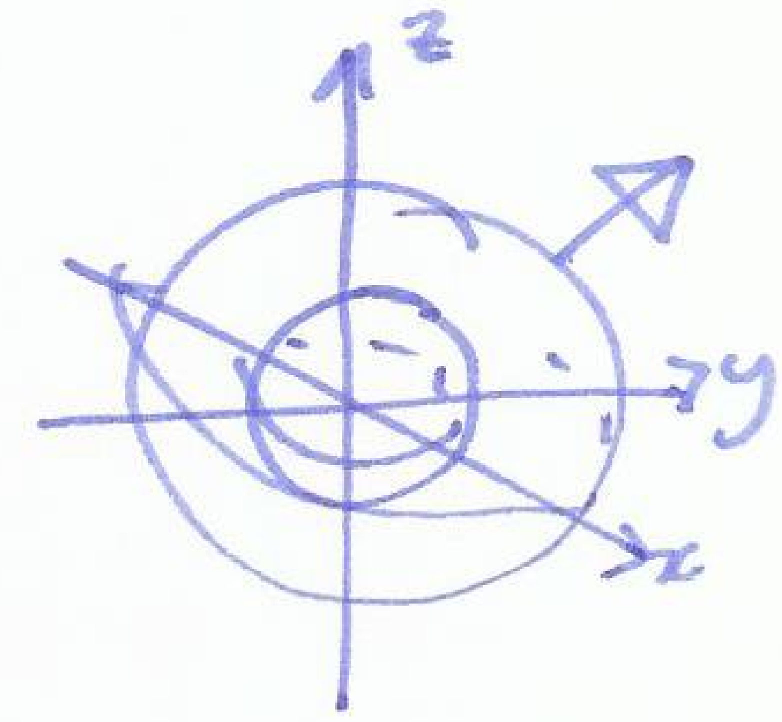
means • direction of fastest increase

• $\|\nabla f\|$ fastest rate of change

directional derivative: $\underline{v}$ unit vector $\quad D_{\underline{v}} f = \underline{v} \cdot \underline{\nabla} f$.

Example $f(x,y,z) = x^2+y^2+z^2$

$\nabla f = \langle 2x, 2y, 2z \rangle$



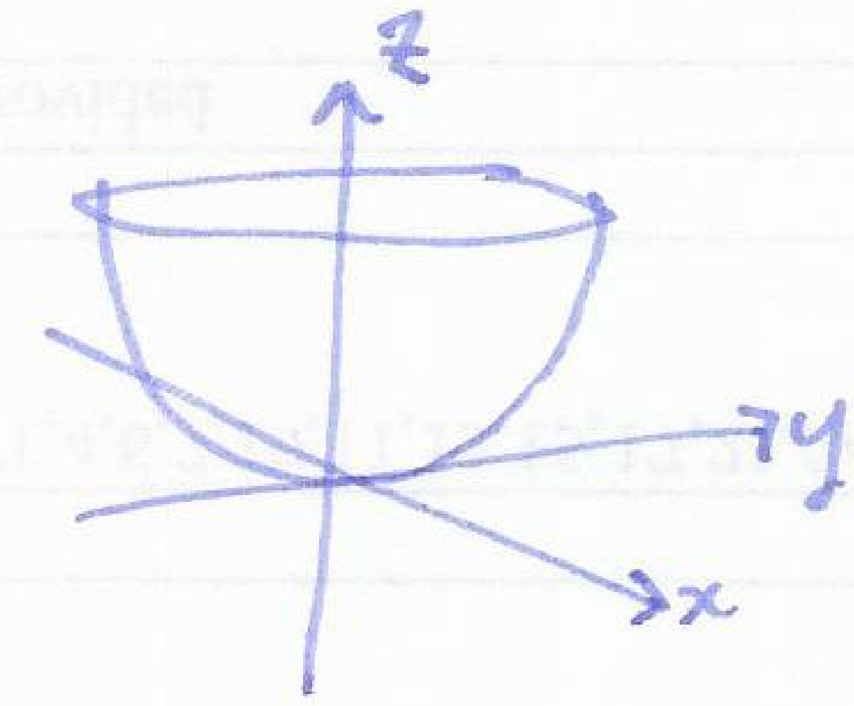
gradient vector is perpendicular to level sets:

so normal vector to tangent plane for $f(x,y,z) = x^2+y^2+z^2 = c$

is given by $\nabla f = \langle 2x, 2y, 2z \rangle$.

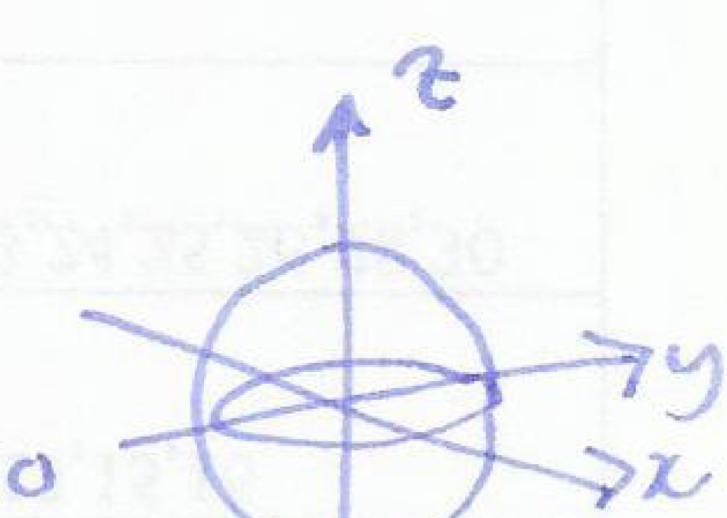
§13.7 Tangent planes and normal lines

Describing surfaces:



① graphs $z = f(x,y)$ $\quad$ example $z = x^2+y^2$

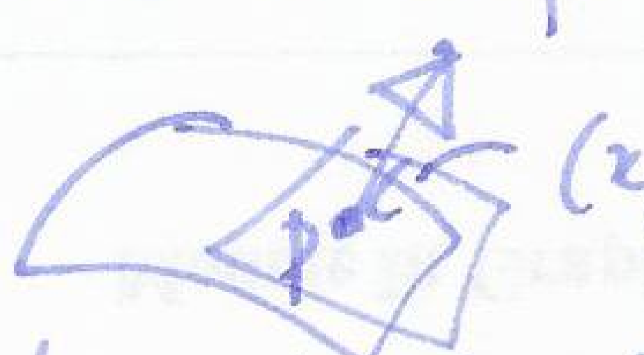
② equations/ level sets $f(x,y,z) = 0$ $\quad$ example $x^2+y^2+z^2-1=0$



can turn a graph into an equation: $F(x,y,z) = f(x,y) - z = 0$

(can't necessarily go other way: vertical line test!).

Suppose $F(x,y,z)=0$ describes a surface and $p=(x_0,y_0,z_0)$ is a point in the surface, and $\underline{\nabla F} \neq \underline{0}$.



$\underline{\nabla F}(x_0,y_0,z_0)$ is the normal vector to the tangent plane at P.

$\underline{\nabla F}(x_0,y_0,z_0)$ is the direction of the normal line at P.

we can write down equations: $\nabla F = \langle F_x, F_y, F_z \rangle$

tangent plane: $F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0.$

normal line: $(x_0,y_0,z_0) + t\,\nabla F(x_0,y_0,z_0)$.

Example $z = f(x,y) = x^2+y^2$ find tangent plane at $(1,1,2)$.

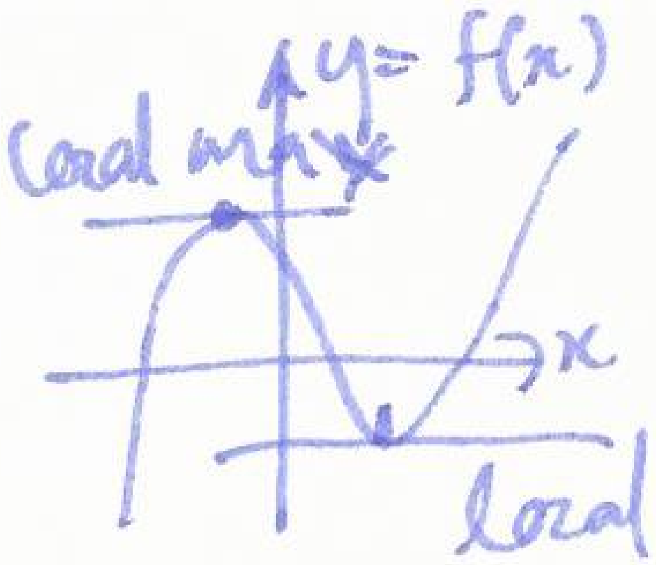
write as $F(x,y,z) = x^2+y^2-z = 0$

$\nabla f = \langle 2x, 2y, -1 \rangle \qquad \nabla f(1,1,2) = \langle 2,2,-1 \rangle.$

$2(x-1) + 2(y-1) + -(z-2) = 0$ tangent plane.

normal line: $t\langle 2,2,-1\rangle + \langle 1,1,2 \rangle$.

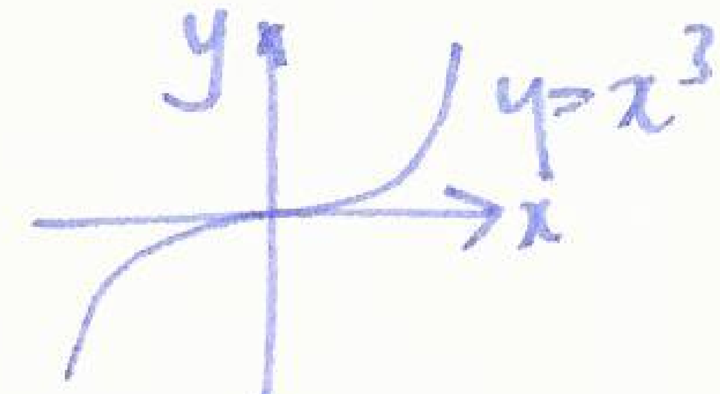
§13.8 Extreme values of functions of two vars.

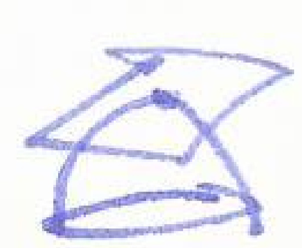
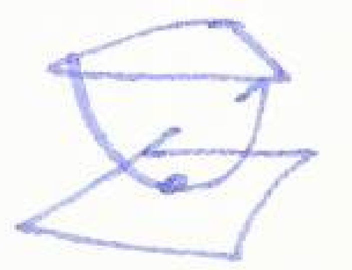
1var 

note $\frac{df}{dx} = 0$ at a max or min.

recall x is a critical point if $\frac{df}{dx}(x) = 0$.

note not all critical points are max or min!



2vars:  local max  local min.

Defn (x_0,y_0) is a critical point for $f(x,y)$ if

- $\frac{\partial f}{\partial x}(x_0,y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0,y_0) = 0$

or

- one of $\frac{\partial f}{\partial x}(x_0,y_0)$, $\frac{\partial f}{\partial y}(x_0,y_0)$ does not exist.

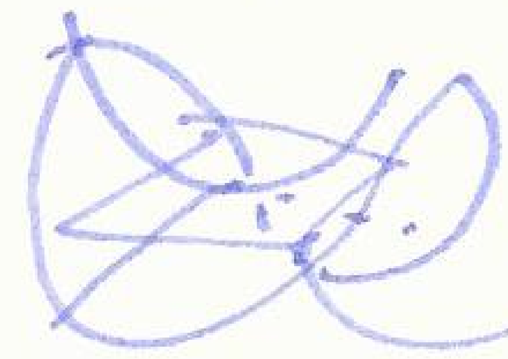
recall normal vector to $y = f(x,y)$ is ∇F where $F = f(x,y) - z$
$= \langle f_x, f_y, -1 \rangle$.

so $f_x(x_0,y_0) = 0 = f(x_0,y_0) \Rightarrow$ horizontal tangent plane.

note: critical point $\not\Rightarrow$ max/min!

Example saddlesurface: $\underset{f(x,y)}{y} = x^2 - y^2$ $\frac{\partial f}{\partial x} = 2x$ $\frac{\partial f}{\partial y} = -2y$

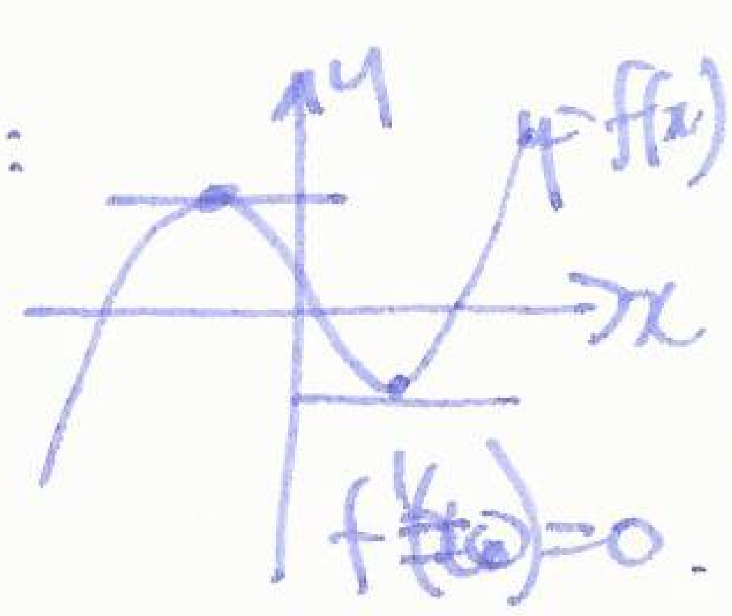
so $\frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial y}(0,0)$



horizontal tangent plane, but not/maximum.

Q: how do we tell if a critical point is a local max or min?

recall 1 var:



2nd derivative test: if $f'(x_0) = 0$

and $f''(x_0) > 0$ local min

$f''(x_0) < 0$ local max

$f''(x_0) = 0$ no information!

2 vars 2nd derivative test

$z = f(x,y)$ and (x_0,y_0) is a critical point (so $f_x(x_0,y_0) = 0 = f_y(x_0,y_0)$).

then set $D = f_{xx}(x_0,y_0) f_{yy}(x_0,y_0) - (f_{xy}(x_0,y_0))^2$

- $D > 0$ and $f_{xx}(x_0,y_0) > 0$ local min.
- $D > 0$ and $f_{xx}(x_0,y_0) < 0$ local max
- $D < 0$ saddle.
- $D = 0$ no information!

mnemonic: $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$.