General case $\omega = f(x_1, \dots, x_n)$ & $x_1(t_1, \dots t_m)$
$\vdots$
$x_n(t_1, \dots, t_m)$

$$\frac{\partial w}{\partial t_1} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \dots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_m}$$

$\vdots$

$$\frac{\partial w}{\partial t_m} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \dots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_m}.$$

matrix notation:

$$\frac{\partial w}{\partial t_i} = \sum_j \frac{\partial w}{\partial x_j}\frac{\partial x_j}{\partial t_i}$$

Example $f(x,y,z) = xyz$ $\quad x = s\cos t$, $y = s\sin t$, $z = t$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

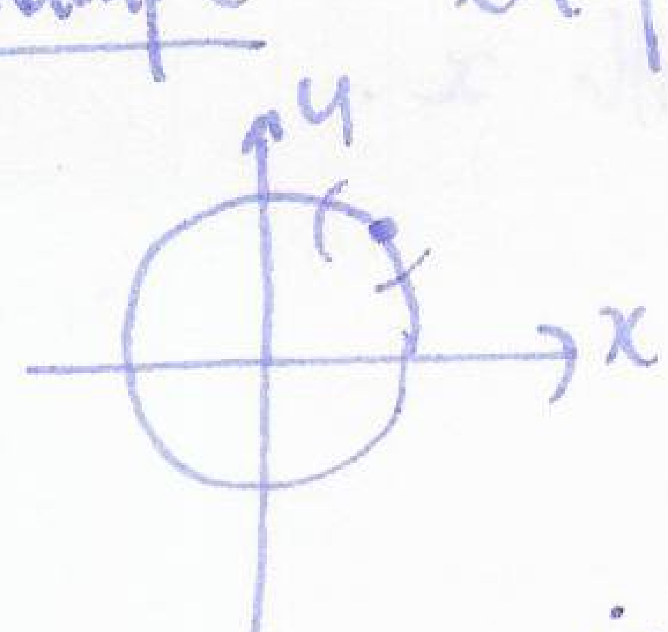
$$= yz\cos t + xz\sin t + yz.0 = st\sin t\cos t + st\cos t\sin t$$

$$= 2st\sin t\cos t.$$

Implicit partial differentiation

we know how to differentiate functions how can we differentiate equations?

Example $x^2+y^2=1$. ← equation, but implicitly defines a function locally near most points (x,y).



$w = F(x,y) = 0$ (where $F = x^2+y^2-1$ in this example).

imagine y is a function of x. $y = f(x)$.

$$w = F(x, f(x)) = 0$$

apply chain rule:

$$\underset{\overset{\shortparallel}{0!}}{\frac{dw}{dx}} = \frac{\partial F}{\partial x}\underset{\overset{\shortparallel}{1!}}{\underbrace{\frac{dx}{dx}}} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

so $\quad F_x + F_y \dfrac{dy}{dx} = 0 \quad$ i.e $\quad \dfrac{dy}{dx} = -\dfrac{F_x}{F_y} \quad (F_y \neq 0).$

<u>Example</u> $\quad F(x,y) = x^2+y^2-1.$

$$\frac{dy}{dx} = -\frac{2x}{2y} = -x/y.$$

<u>3 vars</u> $\quad F(x,y,z) = 0 \quad$ suppose this defines $z(x,y)$.

i.e. $\quad w = F(x,y,z(x,y))$

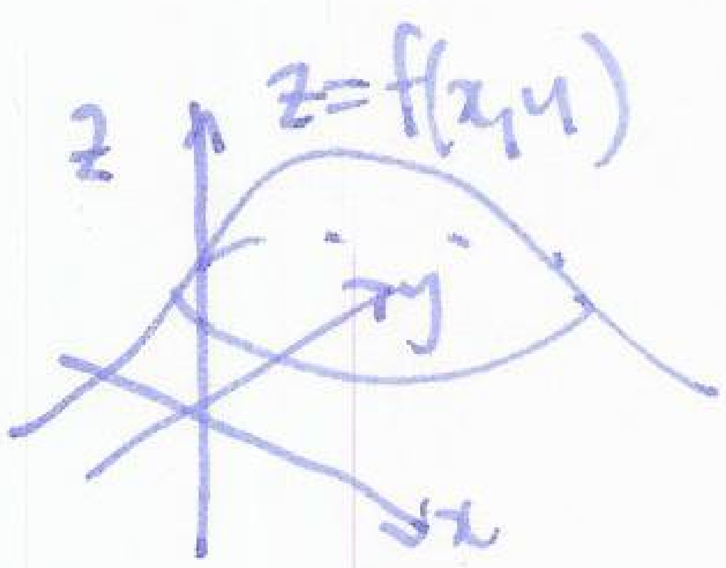
$$\left.\begin{aligned} \underset{\substack{\parallel \\ 0}}{\frac{\partial w}{\partial x}} &= F_x \underbrace{\frac{\partial x}{\partial x}}_{1} + F_y \underbrace{\frac{\partial y}{\partial x}}_{0} + F_z \frac{\partial z}{\partial x} \end{aligned}\right\} \text{ so } \quad \frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad (F_z \neq 0).$$

similarly $\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z}.$

<u>Example</u> $\quad x^2+y^2+z^2 = 1$

$$\frac{\partial z}{\partial x} = -\frac{2x}{2z} \qquad \frac{\partial z}{\partial y} = \frac{-2y}{2z}$$

<u>§13.6 Gradients / directional derivatives</u>



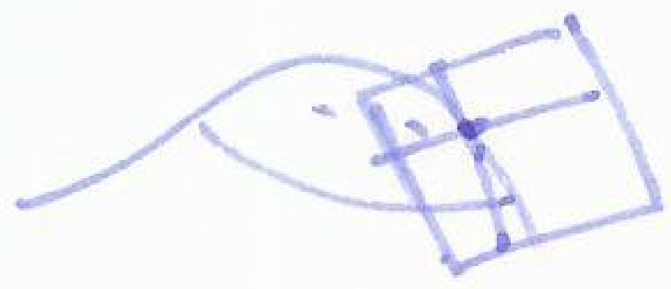
$\dfrac{\partial f}{\partial x}$ = rate of change in x-direction.

$\dfrac{\partial f}{\partial y}$ = rate of change in y-direction.

<u>Q</u>: what about rate of change in some other direction?

specify direction with unit vector $\underline{v} = \langle \cos\theta, \sin\theta \rangle$.

assume $f(x,y)$ differentiable, so tangent plane is good approximation.



$\frac{\partial f}{\partial x}$ = slope of tangent plane in x-direction

$\frac{\partial f}{\partial y}$ = slope of tangent plane in y-direction

so. equation for tangent plane is $z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0)$
$\qquad + f_y(x_0,y_0)(y-y_0)$.

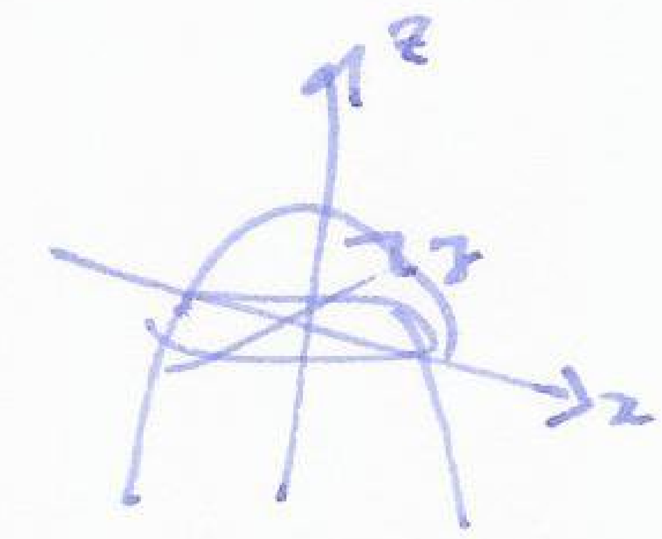
so what's the slope in direction $\underline{v}$? plug in $x = x_0 + \cos\theta$
$y = y_0 + \sin\theta$.

get: $\Delta z = f_x(x_0,y_0)\cos\theta + f_y(x_0,y_0)\sin\theta$.

Def^{n}: The directional derivative is

$$D_{\underline{v}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \quad (*)$$

Example $f(x,y) = 1 - x^2 - y^2$.



direction $\theta = \frac{\pi}{4}$. $\langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4} \rangle$.

$D_{\underline{v}} f(x,y) = -2x\cos\frac{\pi}{4} - 2y\sin\frac{\pi}{4}$.

Observation $(*)$ looks like a dot product.

$$D_{\underline{v}} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle . \underbrace{\langle \cos\theta, \sin\theta \rangle}_{\underline{v}}.$$

Def^{n}: The gradient vector is $\nabla f = \langle f_x, f_y \rangle$.

$$D_{\underline{v}} f = \nabla f \cdot \underline{v}$$

Important properties

- the gradient vector points in the direction of greatest increase.
- the length of the gradient vector is the rate of change in the direction of greatest increase.

Proof

$$D_{\underline{v}} f = \underline{\nabla f} \cdot \underline{v} = \|\nabla f\| \underbrace{\|\underline{v}\|}_{=1} \cos\theta = \|\nabla f\| \cos\theta .$$

max value of $\cos\theta = 0$ when $\theta = 0$.

then $\|D_v f\| = \|\nabla f\|$ in that special direction. $\square$.

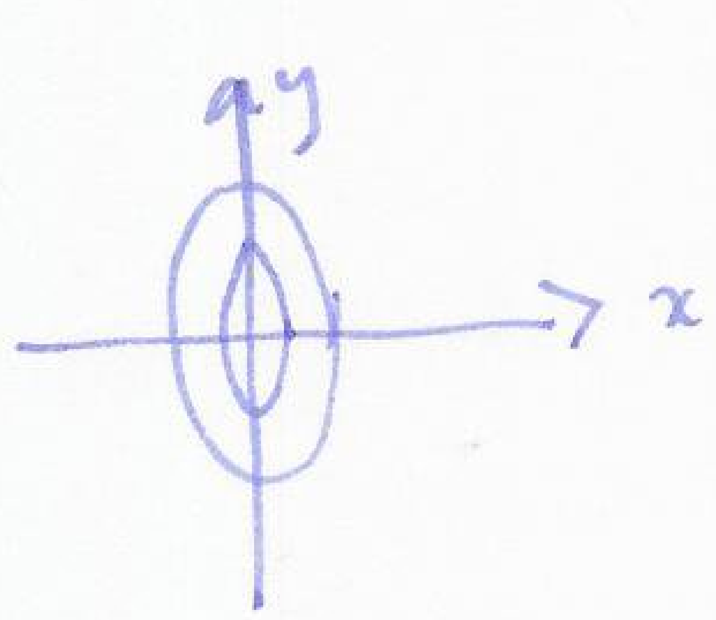
Observation · $-\underline{\nabla f}$ gives direction of greatest decrease.

· if $\underline{\nabla f} = \underline{0}$ then $D_{\underline{v}} f = 0$ for all $\underline{v}$.

Example on the ellipse: $z = 4 - 4x^2 - y^2$ what's the fastest way down?

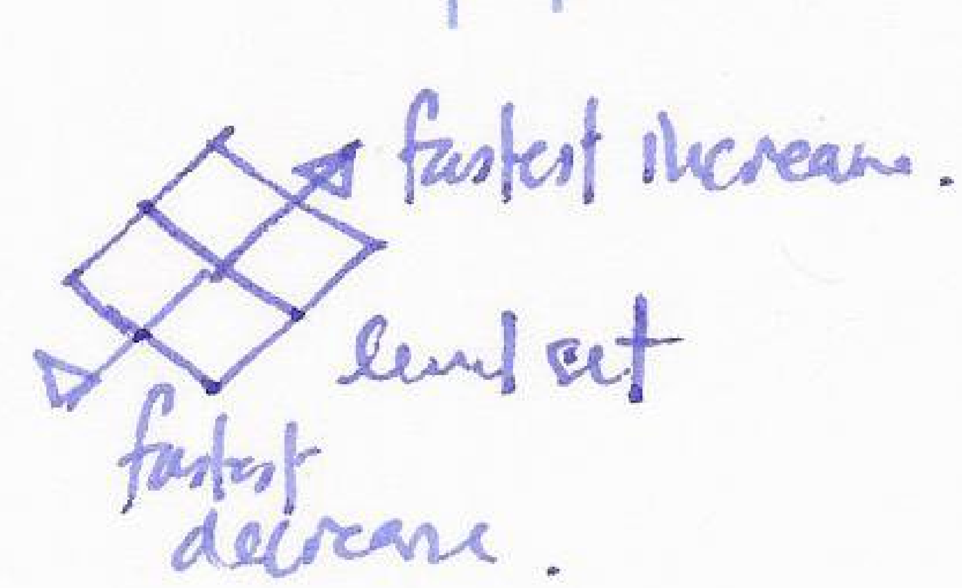
$$\nabla f = \langle -8x, -2y \rangle .$$

Gradient vector and contour lines



The gradient vector is perpendicular to the contour lines/level sets.

linear approx:



tangent plane: $z = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$ normal vector $\langle f_x, f_y, -1 \rangle$