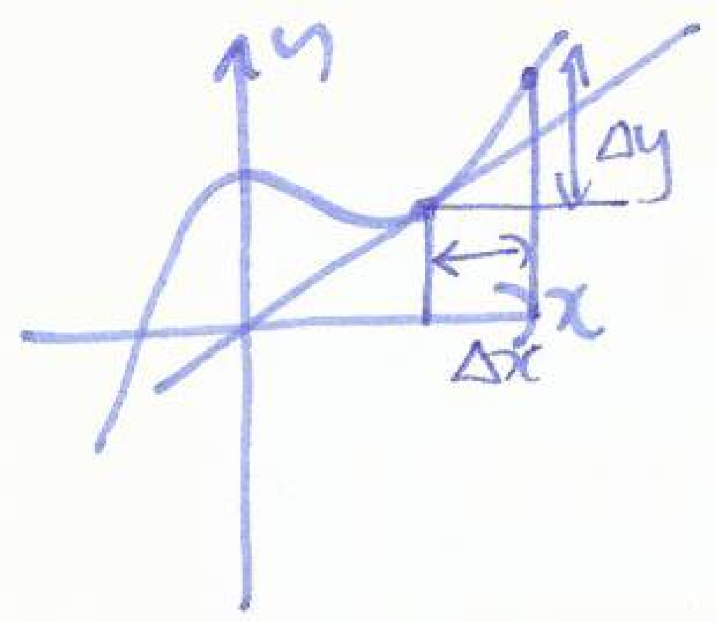
§13.4 Differentials

1 var



$y=f(x)$ the <u>differential</u> of y is $dy = f'(x)\,dx$

$$\Delta y = f(x+\Delta x) - f(x)$$

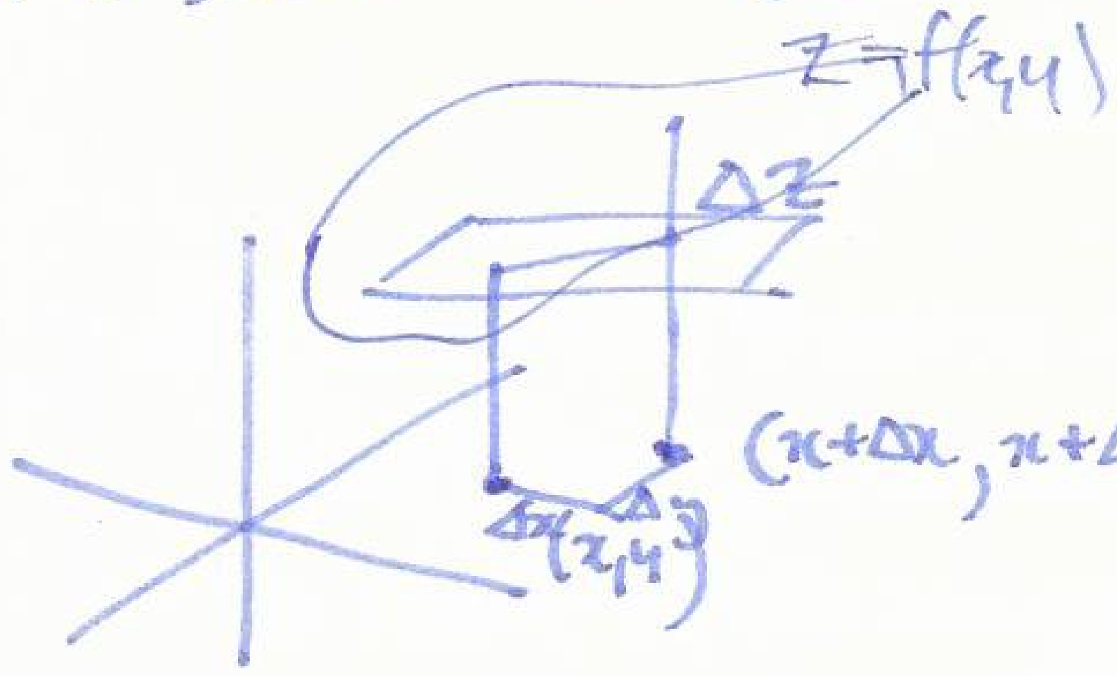
$$\Delta y \approx dy = f'(x)\,dx = f'(x)\Delta x .$$

Δx : small change in x.

Δy : corresponding change in $f(x)$.

$dy = f'(x)\,dx$ a (linear) equation approximating Δy for small Δx.

2 var



$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) .$$

the <u>differential</u> of z is $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = f_x(x,y)\,dx + f_y(x,y)\,dy$

(a linear equation approximating Δz for small $\Delta x, \Delta y$).

$$\Delta z \approx f_x\,\Delta x + f_y\,\Delta y$$

3 vars $w = f(x,y,z)$ has differential $dw = f_x\,dx + f_y\,dy + f_z\,dz$

$\Delta w \approx f_x\,\Delta x + f_y\,\Delta y + f_z\,\Delta z$ for small $\Delta x, \Delta y, \Delta z$.

Example $z = x^2+y^2$ $\quad dz = 2x\,dx + 2y\,dy$.

Differentiability of functions of two vars.

$z=f(x,y)$ is <u>differentiable</u> at (x_0,y_0) if Δz can be written as

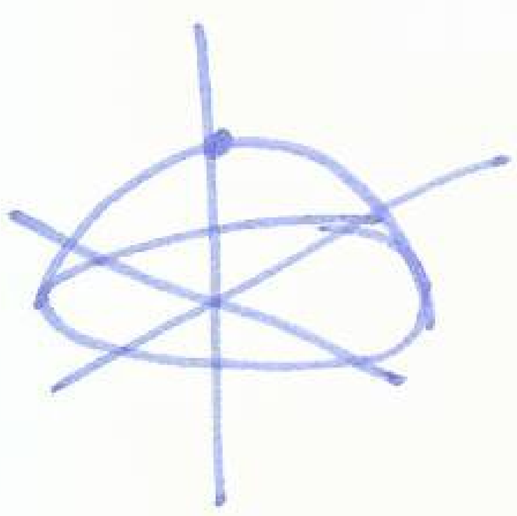
$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0) + \varepsilon(\Delta x, \Delta y) \quad \text{where } \varepsilon \to 0 \text{ as } (\Delta x,\Delta y)\to 0.$$

Useful fact f_x f_y cts at $(x_0,y_0) \Rightarrow f$ is differentiable.

Using differentials for approximations

$z = f(x,y)$: $\Delta z \approx f_x(x,y)\Delta x + f_y(x,y)\Delta y$

Example $z = \sqrt{6 - x^2 - y^2}$ $f(1,1) = 2$.



~~app~~ estimate $f(1.01, 1.01)$

$f_x(x,y) = \frac{1}{2}(6 - x^2 - y^2)^{-1/2} \cdot -2x$

$f_y(x,y) = \frac{1}{2}(6 - x^2 - y^2)^{-1/2} \cdot -2y$

$$\Delta z \approx f_x(1,1)(0.01) + f_y(1,1)(0.01)$$

$$= \frac{-2}{2.2}(0.01) + \frac{-2}{2.2}(0.01) \approx -0.01.$$

so $f(1.01, 1.01) \approx 1.99$.

Example A tin can is 10cm high and 5cm in radius, and the top and bottom have thickness 0.2 cm, and the sides have thickness 0.1cm.
Estimate the amount of metal in the can.

area $\approx$ change in volume. $V = 2\pi r h$

$$\Delta V \approx \frac{\partial V}{\partial r}\Delta r + \frac{\partial V}{\partial h}\Delta h = 2\pi h \Delta r + 2\pi r \Delta h$$

$h = 10$ $\Delta h = 0.4$
$r = 5$ $\Delta r = 0.1$

$$\Delta V \approx 2\pi . 10 . 0.1 + 2\pi 5 . 0.4.$$

$$\approx 6\pi.$$

§13.5 Chain rule

recall $\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)).g'(x).$

$f(y),\ y=g(x) \quad \frac{df}{dx} = \frac{df}{dy}\frac{dy}{dx}.$

Chain rule for many variables "differentiate wrt to all possible variables".

~~Example~~ Special case $w = f(x,y)$ where $x = g(t)$, $y = h(t)$ $\quad (w(t) = f(g(t), h(t)))$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

Example $w = x^2 y \quad x = t \quad y = \sin t$

$$\frac{dw}{dt} = 2xy\,1 + x^2\cos t = 2t\sin t + t^2\cos t.$$

Special case $w = f(x,y)$ $x = g(s,t)$, $y = h(s,t)$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Example $w = xy \quad x = s^2+t^2$, $y = s/t$.

$$\frac{\partial w}{\partial s} = y\,2s + x\,2t = \frac{2s^2}{t} + 2t(s^2+t^2).$$