Example $\underline{r}(t) = \langle t^2-1, t \rangle$ $\quad \underline{T}(t) = \left\langle \frac{2t}{\sqrt{4t^2+1}}, \frac{1}{\sqrt{4t^2+1}} \right\rangle$

$$\cancel{\underline{N}(t)}\ \underline{T}'(t) = \left\langle \frac{\sqrt{4t^2+1}\cdot 2 - 2t\,\frac{1}{2}(4t^2+1)^{-1/2}\cdot 8t}{4t^2+1}, -\frac{1}{2}(4t^2+1)^{-3/2}\cdot 8t \right\rangle.$$

$$= \left\langle \frac{2(4t^2+1) - 8t^2}{(4t^2+1)^{3/2}}, \frac{-4t}{(4t^2+1)^{3/2}} \right\rangle$$

parallel to $\langle 2, -4t \rangle$

$$\underline{N}(t) = \left\langle \frac{2}{\sqrt{4+16t^2}}, \frac{-4t}{\sqrt{4+16t^2}} \right\rangle.$$

$\boxed{\text{WW9 \#4}}$

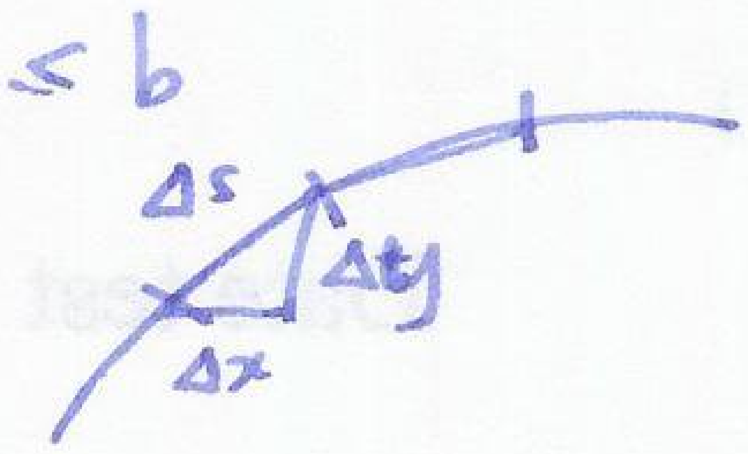
~~Useful fact if $\underline{r}(t)$ moves with unit speed, i.e. $\|\underline{r}(t)\| = 1$,~~

~~then~~

§12.5 Arc length and curvature

recall: length of plane curves $(x(t), y(t))$ $a \le t \le b$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$



$\sum \Delta s = \sum \sqrt{\Delta x^2 + \Delta y^2}$

vector notation: $\underline{r}(t) = \langle x(t), y(t) \rangle$

then $\quad s = \int_a^b \|\underline{r}'(t)\|\, dt$

3d: $\underline{r}(t) = \langle x(t), y(t), z(t) \rangle$ $\quad a \le t \le b$

then arc length $s = \int_a^b \|\underline{r}'(t)\|\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$.

last time paramaterized curves / vector valued functions of 1-variable.

$\underline{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\underline{r}'(t)$ "velocity" $\quad \underline{T}(t) = \dfrac{\underline{r}'(t)}{\|\underline{r}'(t)\|}.$

$\underline{r}''(t)$ "acceleration". $\quad N(t) = \dfrac{T'(t)}{\|T'(t)\|}$

THU 8 OCT	HW 8, 9, 10
MON 12	NO CLASS
TUE 13	MATLAB 2
WED 14	SMT 2
THU 15	HW 11, 12
MON 19	MT 2

§12.5 Arc length and curvature

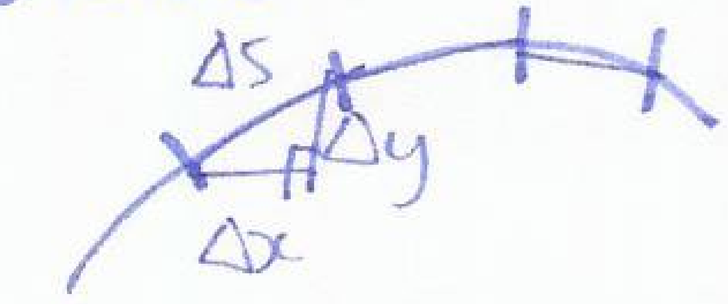
recall: length of plane curves $(x(t), y(t))$ $a \leq t \leq b$

then length $= \displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

vector notation: $\underline{r}(t) = \langle x(t), y(t) \rangle$, $\underline{r}'(t) = \langle x'(t), y'(t) \rangle$.

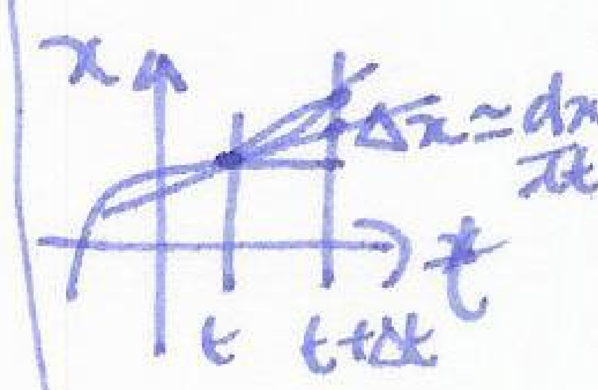
length of $\underline{r}(t)$, $a \leq t \leq b$ $= \displaystyle\int_a^b \|\underline{r}'(t)\|\, dt$.

mnemonic



$\sum \Delta s \qquad \Delta s^2 = \Delta x^2 + \Delta y^2$

$\sum \sqrt{\Delta x^2 + \Delta y^2}$.



$\Delta x \simeq \dfrac{dx}{dt}\Delta t$. $\quad \Delta y = \dfrac{dy}{dt}\Delta t$

3d version: $\underline{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\underline{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$

length of $\underline{r}(t)$, $a \leq t \leq b$ $= \displaystyle\int_a^b \|\underline{r}'(t)\|\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$.

Example find arc length of helix $\underline{r}(t) = \langle \cos t, \sin t, t \rangle$ $0 \leq t \leq 2\pi$

$\underline{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$.

length $= \displaystyle\int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\, dt = \int_0^{2\pi} \sqrt{2}\, dt = \left[\sqrt{2}\,t\right]_0^{2\pi} = 2\pi\sqrt{2}.$

Example: find arc length of $\underline{r}(t) = t\,\underline{i} + \frac{4}{3}t^{3/2}\,\underline{j} + \frac{1}{2}t^2\,\underline{k}$

$0 \le t \le 1$

$x(t) = t\sqrt{3} \quad y(t) = \frac{4}{3}t^{3/2} \quad z(t) = \frac{1}{2}t^2$

$\frac{dx}{dt} = \sqrt{3} \quad \frac{dy}{dt} = 2t^{1/2} \quad \frac{dz}{dt} = t$

arc length $= \int_0^1 \sqrt{\sqrt{3}^2 + (2t^{1/2})^2 + t^2}\,dt$

$= \int_0^1 \sqrt{3 + 4t + t^2}\,dt$ $\qquad t^2 + 4t + 1$

$\qquad (t+2)^2 - 4 + 1$

$= \int_0^1 \sqrt{(t+2)^2 - 3}\,dt$ $\qquad$ mnemonic: $\sqrt{x^2 - 1}$.

↑ want perfect square.

substitute $t + 2 = \sec x$ $\qquad \cos^2 x + \sin^2 x = 1$

$\frac{dt}{dx} = \sec x \tan x.$ $\qquad 1 + \tan^2 x = \sec^2 x$

$\qquad \tan^2 x = \sec^2 x - 1$

$\int_\phi^\theta \sqrt{\sec^2 x - 1}\,\sec x \tan x\,dx = \int_\phi^\theta \sec x \tan^2 x\,dx$

$= \int_\phi^\theta \sec x \tan x \,.\, \tan x\,dx = \left[\sec x \tan x\right] - \int_\phi^\theta \sec x \sec^2 x\,dx$

$= \sec x \tan x - \int_\phi^\theta \sec x + \sec x \tan^2 x\,dx.$

$= \sec x \tan x - \ln|\sec x + \tan x| - \int_0^1 \sec x \tan^2 x\,dx$

so $\int_\phi^\theta \sec^3 x \tan^2 x\,dx = \frac{1}{2}\left[\sec x \tan x - \ln|\sec x + \tan x|\right]_{\sec^{-1}(2)}^{\sec^{-1}(3)}.$

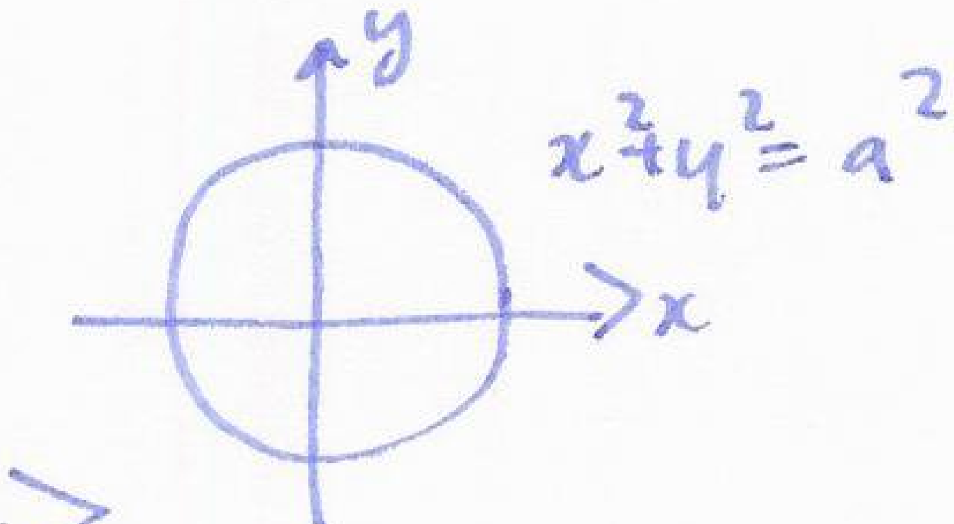
arc length paramaterization

suppose the length of $\underline{r}(t)$ between $t=0$ and $t=l$ is equal to l, then we say $\underline{r}(t)$ is an arc length parameterization of the curve.

useful fact : $\|\underline{r}'(t)\| = 1 \iff \underline{r}(t)$ is an arc length parameterization.
for all t

why?. recall $\int_0^l \|r'(t)\|\,dt$ if $\|\underline{r}'(t)\|=1$ then this is $\int_0^l 1\,dt$
$= [t]_0^l = l$.

Example circle of radius a :



parameterize : $\underline{r}(\theta) = \langle a\cos\theta, a\sin\theta \rangle$

$\underline{r}'(\theta) = \langle -a\sin\theta, a\cos\theta \rangle$ $\quad \|\underline{r}'(\theta)\| = \sqrt{(-a\sin\theta)^2 + (a\cos\theta)^2}$
$= a$. ($\neq 1$ in general!)

length: $\int_0^l a\,d\theta = [a\theta]_0^l = la$.

try new parameterization : $\underline{r}(\theta) = \langle a\cos\left(\frac{\theta}{a}\right), a\sin\left(\frac{\theta}{a}\right) \rangle$.

$\underline{r}'(\theta) = \langle -a.\cancel{\tfrac{1}{2}}\sin\left(\frac{\theta}{a}\right)\cdot\frac{1}{a}, a\cos\left(\frac{\theta}{a}\right)\cdot\frac{1}{a} \rangle$.

$= \langle -\sin\left(\frac{\theta}{a}\right), \cos\left(\frac{\theta}{a}\right) \rangle$.

$\|\underline{r}'(\theta)\| = 1$ as required.

so this is an arc-length paramaterization.

Curvature

Let $\underline{r}(t)$ be a curve parameterized by arc length then the curvature $\kappa(t) = \left\| \frac{dT(t)}{dt} \right\| = \|\underline{T}'(t)\|$ where $\underline{T}(t) = \frac{\underline{r}'(t)}{\|\underline{r}'(t)\|} = \underline{r}'(t)$!

Curvature

Let $\underline{r}(s)$ be a curve parameterized by arc length. Then we define the curvature $K(s) = \|\underline{T}'(s)\|$.

recall $\underline{T}(s) = \dfrac{\underline{r}'(s)}{\|\underline{r}'(s)\|} = \underline{r}'(s)$.

Example circle of radius a

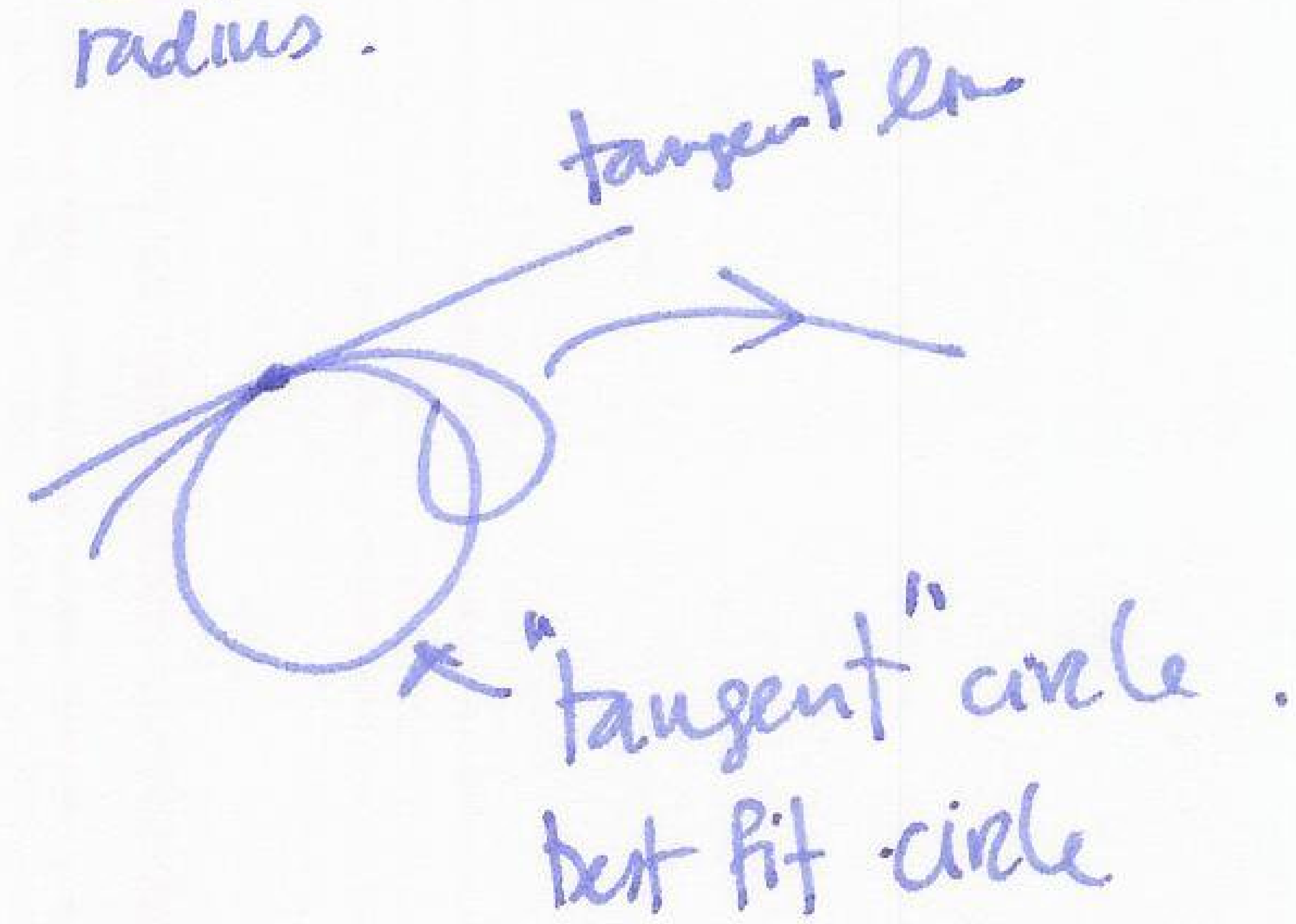
$$\underline{r}(\theta) = \left\langle a\cos\left(\tfrac{\theta}{a}\right), a\sin\left(\tfrac{\theta}{a}\right)\right\rangle$$

$$\underline{r}'(\theta) = \left\langle -\sin(\theta/a), \cos(\theta/a)\right\rangle = \underline{T}(\theta)$$

$$\underline{T}'(\theta) = \left\langle -\tfrac{1}{a}\cos(\theta/a), -\sin(\theta/a)\cdot\tfrac{1}{a}\right\rangle$$

$$K(\theta) = \|\underline{T}'(\theta)\| = \frac{1}{a} \cdot = \frac{1}{\text{radius}}.$$

Meaning of curvature K



$K(s) = \dfrac{1}{\text{radius}}$ of circle which best approximates curve at $\underline{r}(s)$.

What happens if $\underline{r}(t)$ not parameterized by arc length?

$$K(t) = \frac{\|\underline{T}'(t)\|}{\|\underline{r}'(t)\|}$$