Example $\underline{r}(t) = t\underline{i} - t\underline{j}$ $\quad \underline{s}(t) = \underline{k}$

$$\frac{d}{dt}\left(\underline{r}(t) \times \underline{s}(t)\right) = \underline{r}'(t) \times \underline{s}(t) + \underline{r}(t) \times \underline{s}'(t)$$

$\underline{r}'(t) = \underline{i} - \underline{j}$ $\quad \underline{s}'(t) = \underline{0}$ $\qquad = (\underline{i} - \underline{j}) \times \underline{k} + \underline{0} = \underline{j} - \underline{i}$.

Integration

recall: scalar functions of 1-variable: definite integrals $\int_a^b f(t)\,dt$.

indefinite integrals $\int f(t)\,dt$

integrate vector valued functions componentwise

$\underline{r}(t) = f(t)\underline{i} + g(t)\underline{j}$

$$\int \underline{r}(t)\,dt = \int f(t)\,dt\ \underline{i} + \int g(t)\,dt\ \underline{j}$$

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$\underline{r}(t) = f(t)\underline{i} + g(t)\underline{j} + h(t)\underline{k}$

then $\int \underline{r}(t)\,dt = \int f(t)\,dt\ \underline{i} + \int g(t)\,dt\ \underline{j} + \int h(t)\,dt\ \underline{k}$.

indefinite integration: $\int f'(t)\,dt = f(t) + c$, if $\underline{r}(t) = \langle f'(t), g'(t) \rangle$

so $\int \underline{r}(t)\,dt = \langle (f(t) + c_1), g(t) + c_2 \rangle$

$= f(t)\underline{i} + g(t)\underline{j} + \underline{c}$. $\quad \underline{c} = \langle c_1, c_2 \rangle$.

example so if you know the velocity vector $\underline{r}'(t)$, and the starting point $\underline{r}(0)$, you can recover the path $\underline{r}(t)$.

$\underline{r}'(t) = \langle \frac{1}{1+t}, \frac{1}{1+t^2}, \sin^2 t \rangle$ $\quad \frac{1}{2} - \frac{1}{2}\cos 2\theta$ $\quad \cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$.

$\underline{r}(t) = \langle \ln|1+t|, \tan^{-1}(t), \frac{1}{2}t - \frac{1}{4}\sin 2t \rangle + \underline{c}$

so if $\underline{r}(0) = \langle 1, 0, 0 \rangle$ then $\underline{c} = \langle 0, 0, 0 \rangle$.

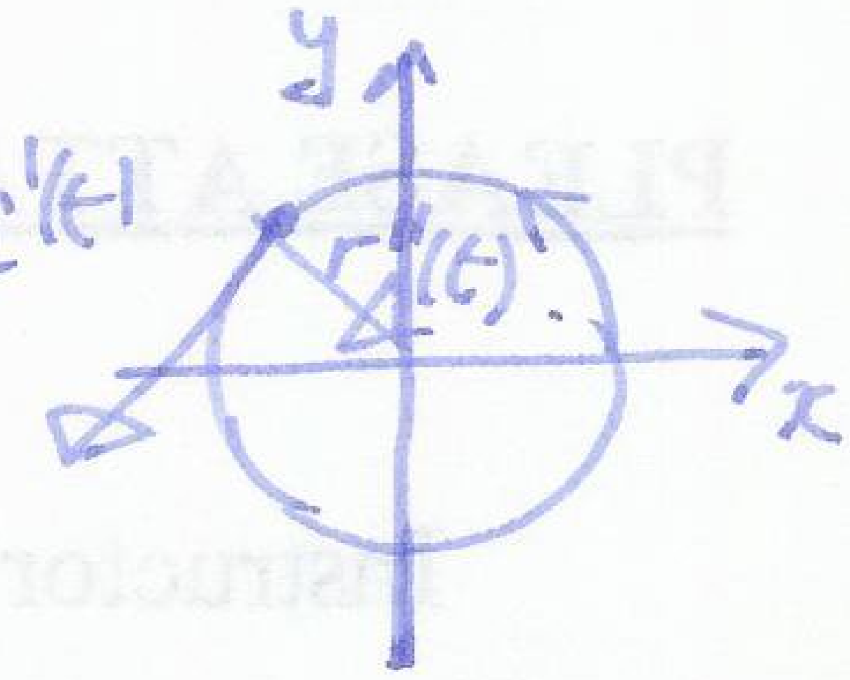
§12.3 Velocity and acceleration

suppose $\underline{r}(t)$ describes the movement of an object in space.

then $\underline{r}'(t)$ is the velocity $\underline{v}(t)$ speed $\|\underline{r}'(t)\| = \|\underline{v}(t)\|$.

$\underline{r}''(t)$ is the acceleration $\underline{a}(t)$.

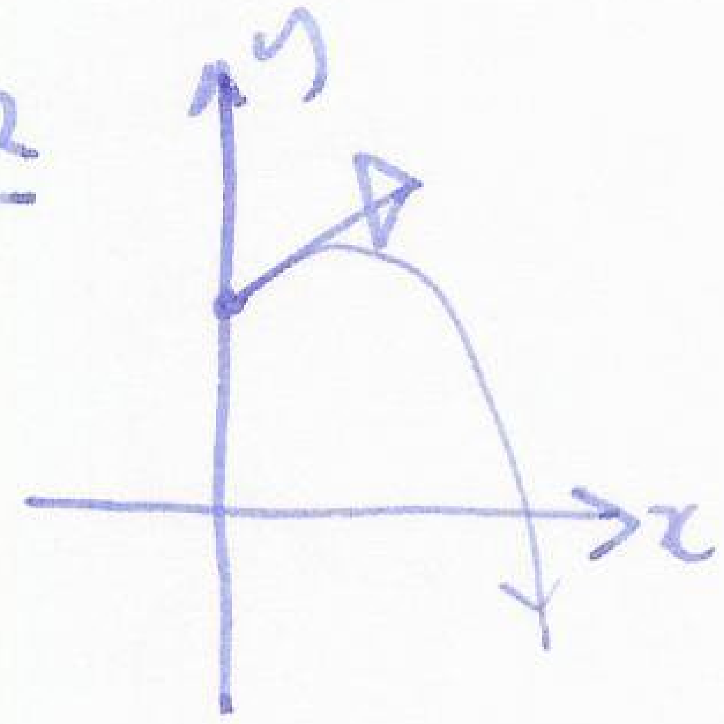
Example $\underline{r}(t) = \langle \cos t, \sin t \rangle$



$\underline{r}'(t) = \langle -\sin t, \cos t \rangle$

$\underline{r}''(t) = \langle -\cos t, -\sin t \rangle$

Example



at time $t=0$ an object of mass 1 is thrown from $(0,1)$ with a velocity of $\langle 1,1 \rangle$, and falls under gravity $\underline{F} = -mg\underline{j}$.

find the path $\underline{r}(t)$ of the object.

$\underline{r}(t)$ path $\quad \underline{r}(0) = \langle 0,1 \rangle$

$\underline{r}'(t)$ velocity $\quad \underline{r}'(0) = \langle 1,1 \rangle$

$\underline{r}''(t)$ acceleration. $\quad \underline{r}''(t) = \langle 0, -mg \rangle$

$\S$ $\underline{r}'(t) = \int \underline{r}''(t)\,dt = \langle c_1, -gt + c_2 \rangle$

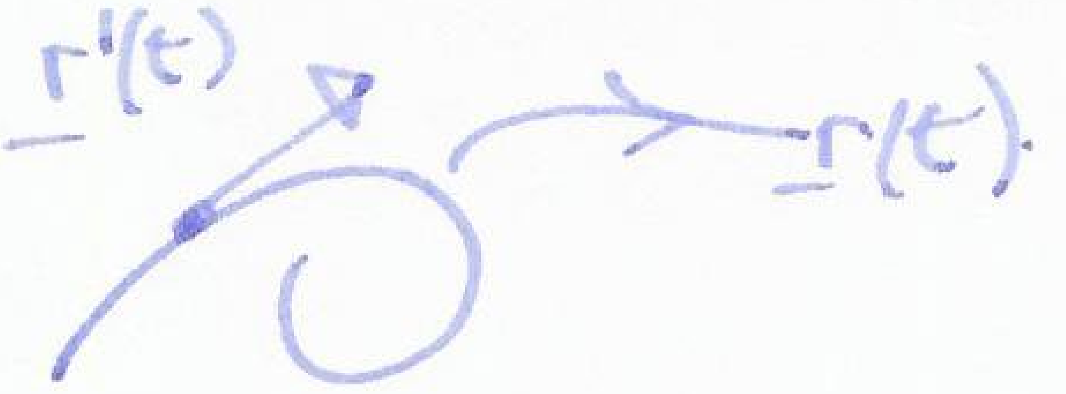
$\underline{r}'(0) = \langle 1,1 \rangle = \langle c_1, c_2 \rangle$, so $\underline{r}'(t) = \langle 1, 1-gt \rangle$

$\underline{r}(t) = \int \underline{r}'(t)\,dt = \langle t + c_1, t - \frac{1}{2}gt^2 + c_2 \rangle$

$\underline{r}(0) = \langle 0,1 \rangle = \langle c_1, c_2 \rangle \quad c_1 = 0,\ c_2 = 1.$

so $\underline{r}(t) = \langle t, 1 + t - \frac{1}{2}gt^2 \rangle$.

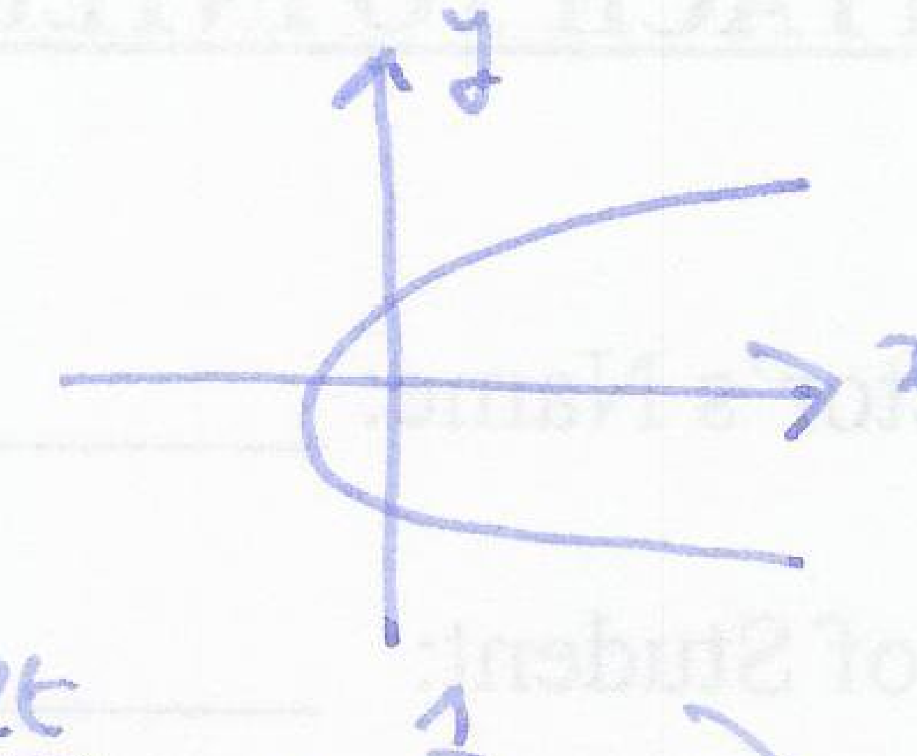
§12.4 Tangent vectors and normal vectors.



the unit tangent vector $\underline{T}(t)$ is

$$\underline{T}(t) = \frac{\underline{r}'(t)}{\|\underline{r}'(t)\|}, \quad \underline{r}'(t) \neq \underline{0}.$$

Example $\underline{r}(t) = \langle t^2-1, t \rangle$



$\underline{r}'(t) = \langle 2t, 1 \rangle$.

$$\underline{T}(t) = \frac{1}{\sqrt{4t^2+1}} \langle 2t, 1 \rangle = \left\langle \frac{2t}{\sqrt{4t^2+1}}, \frac{1}{\sqrt{4t^2+1}} \right\rangle.$$

note $\underline{T}(t)$ is a vector valued function of t (in fact takes values in $S^1 \subset \mathbb{R}^2$, $S^2 \subset \mathbb{R}^3$).

Defn principal unit normal vector $\quad N(t) = \dfrac{\underline{T}'(t)}{\|\underline{T}'(t)\|}$.

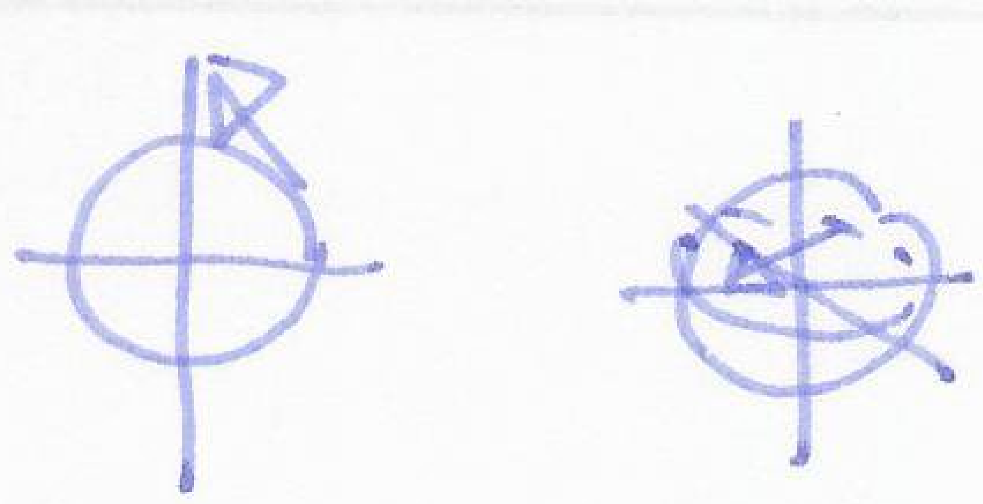
useful fact: $\underline{T}'(t)$ is orthogonal to $T(t)$.

Proof $\quad \underline{T}(t) \cdot \underline{T}(t) = \|T(t)\|^2 = 1$.

$$\frac{d}{dt}\left(T(t) \cdot T(t)\right) = \frac{d}{dt}(1) = 0$$

$$= \underline{T}'(t) \cdot \underline{T}(t) + \underline{T}(t) \cdot T'(t) = 2\underline{T}(t) \cdot \underline{T}'(t) = 0. \qquad \square$$

why?



tangent vectors to $S^1 \subset \mathbb{R}^2$, $S^2 \subset \mathbb{R}^3$ always $\perp$ to S^1, S^2.

useful fact if $\underline{r}(t)$ moves with unit speed, ~~then~~ i.e. $\|\underline{r}'(t)\| = 1$, then $\underline{r}'(t)$ and $\underline{r}''(t)$ perpendicular.