Limits for vector valued functions:

Def$^{\underline{n}}$ $\lim_{t\to a} \underline{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t) \rangle$ if these limits exist

similarly $\lim_{t\to a} \underline{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$ if these limits exist

(otherwise limit doesn't exist)

Continuity for vector valued functions

$\underline{r}(t)$ is cts at $t=a$ if $\lim_{t\to a} \underline{r}(t)$ exists and is equal to $\underline{r}(a)$

$\underline{r}(t)$ is cts on an interval I if its cts at every point of I.

Example is $\underline{r}(t) = \langle \frac{\sin t}{t}, \frac{1}{t} \rangle$ cts at $t=0$?

§12.2 Differentiation / integration of vector valued functions

Differentiation

recall : 1d $f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ if this limit exists.

vector valued functions :

Def$^{\underline{n}}$ $\underline{r}'(t) = \lim_{\Delta t\to 0} \frac{\underline{r}(t+\Delta t) - \underline{r}(t)}{\Delta t}$ if this limit exists

other notations : $\frac{d\underline{r}}{dt}$, $\frac{d}{dt}(\underline{r}(t))$. $\underline{r}'(t)$ is tangent vector or velocity vector.

useful fact : differentiation works componentwise

$\underline{r}(t) = \langle f(t), g(t), h(t) \rangle$ then $\underline{r}'(t) = \langle f'(t), g'(t), f'(t) \rangle$.

original version of differentiation

check : $\underline{r}(t) = \langle f(t), g(t) \rangle$

$\underline{r}(t+\Delta t) = \langle f(t+\Delta t), g(t+\Delta t) \rangle$

$$\lim_{\Delta t \to 0} \frac{\underline{r}(t+\Delta t) - \underline{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\langle f(t+\Delta t), g(t+\Delta t)\rangle - \langle f(t), g(t)\rangle}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \langle f(t+\Delta t) - f(t), g(t+\Delta t) - g(t) \rangle$$

$$= \lim_{\Delta t \to 0} \left\langle \frac{f(t+\Delta t) - f(t)}{\Delta t}, \frac{g(t+\Delta t) - g(t)}{\Delta t} \right\rangle \quad \square .$$

Example $\underline{r}(t) = \langle e^{2t}, \ln t^2 \rangle$ $\quad \underline{r}'(t) = \langle 2e^{2t}, \frac{2}{t} \rangle$.

Example $\underline{r}(t) = \langle \cos t, \sin t, t \rangle$ $\quad$ real/scalar valued function!

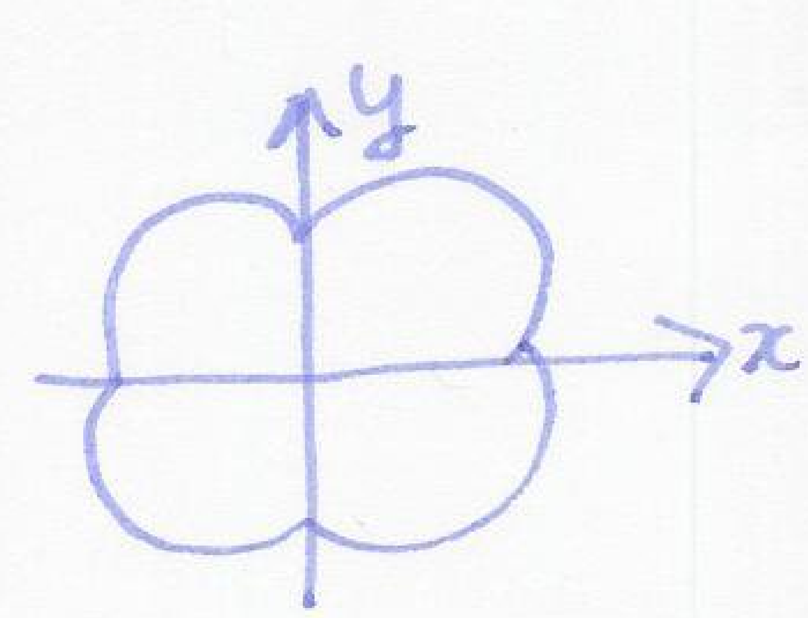
$\underline{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$ $\quad \underline{r}'(t) \cdot \underline{r}''(t) = \langle -\sin t, \cos t, 1 \rangle \cdot \langle -\cos t, -\sin t, 0 \rangle$

$\underline{r}''(t) = \langle -\cos t, -\sin t, 0 \rangle$ $\quad = \sin t \cos t - \cos t \sin t + 0 = 0$.

$\underline{r}'(t) \times \underline{r}''(t)$ .... vector valued function!

Defⁿ $\underline{r}(t) = \langle f'(t), g'(t), h'(t) \rangle$ is smooth on an interval I, if f', g', h' are cts on I and $\underline{r}'(t) \neq \underline{0}$ for any $t \in I$.

Example $\underline{r}(t) = (5\cos t - \cos 5t)\underline{i} + (5\sin t - \sin 5t)\underline{j}$



$\underline{r}'(t) = \langle -5\sin t + 5\sin 5t, \ 5\cos t - 5\cos 5t \rangle$

solve $\underline{r}'(t) = \underline{0} = \langle 0, 0 \rangle$. $\quad$ ~~$\sin t = \sin 5t$ and $\cos t = 5\cos 5t$~~

$-\sin t + \sin 5t = 0$ $\quad$ recall : $\sin 2\theta = 2\sin\theta\cos\theta$ $\quad \sin(A+B) = \sin A \cos B + \sin B \cos A$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\quad \cos(A+B) = \cos A \cos B - \sin A \sin B$

$B - A = t \quad A + B = t$

$B = 3t$, $A = 2t$

$\sin(A+B) + \sin(A-B) = 2\sin A \cos B$.

$2\sin 2t \cos 3t = 0$. $\quad t = 0, \pi/2, \pi, \frac{3\pi}{2}$ etc....

also need $\cos t - \cos 5t = 0$

$2\sin 3t \sin 3t = 0$.

$\cos(A-B) - \cos(A+B) = \cos A \cos B + \sin A \sin B$ (3)
$- \cos A \cos B + \sin A \sin B$.
$= 2\sin A \sin B$.

Useful properties of differentiation

$\underline{r}(t), \underline{s}(t)$ smooth, vector valued. $f(t)$ scalar valued
c constant.

- $\frac{d}{dt}\left(c\underline{r}(t)\right) = c\underline{r}'(t)$
- $\frac{d}{dt}\left(\underline{r}(t)+\underline{s}(t)\right) = \underline{r}'(t)+\underline{s}'(t)$
- $\frac{d}{dt}\left(f(t)\underline{r}(t)\right) = f'(t)\underline{r}(t) + f(t)\underline{r}'(t)$.
- $\frac{d}{dt}\left(\underline{r}(t)\cdot\underline{s}(t)\right) = \underline{r}(t)\cdot\underline{s}'(t) + \underline{r}'(t)\cdot\underline{s}(t)$ ⊕
- $\frac{d}{dt}\left(\underline{r}(t)\times\underline{s}(t)\right) = \underline{r}(t)\times\underline{s}'(t) + \underline{r}'(t)\times\underline{s}(t)$
- $\frac{d}{dt}\left(\underline{r}(f(t))\right) = \underline{r}'(f(t))\, f'(t)$
- if $\underline{r}(t)\cdot\underline{r}(t) = c$ then $\underline{r}(t)\cdot\underline{r}'(t) = 0$.

check ⊕ $\underline{r}(t) = \langle x_1(t), y_1(t)\rangle$
$\underline{s}(t) = \langle x_2(t), y_2(t)\rangle$

$\underline{r}(t)\cdot\underline{s}(t) = x_1(t)x_2(t) + y_1(t)y_2(t)$

$\frac{d}{dt}\left(\underline{r}(t)\cdot\underline{s}(t)\right) = (x_1x_2 + y_1y_2)' = x_1'x_2 + x_1x_2' + y_1'y_2 + y_1y_2'$

$= x_1'x_2 + y_1'y_2 + x_1x_2' + y_1y_2'$

$= \underline{r}'(t)\cdot\underline{s}(t) + \underline{r}(t)\cdot\underline{s}'(t)$.