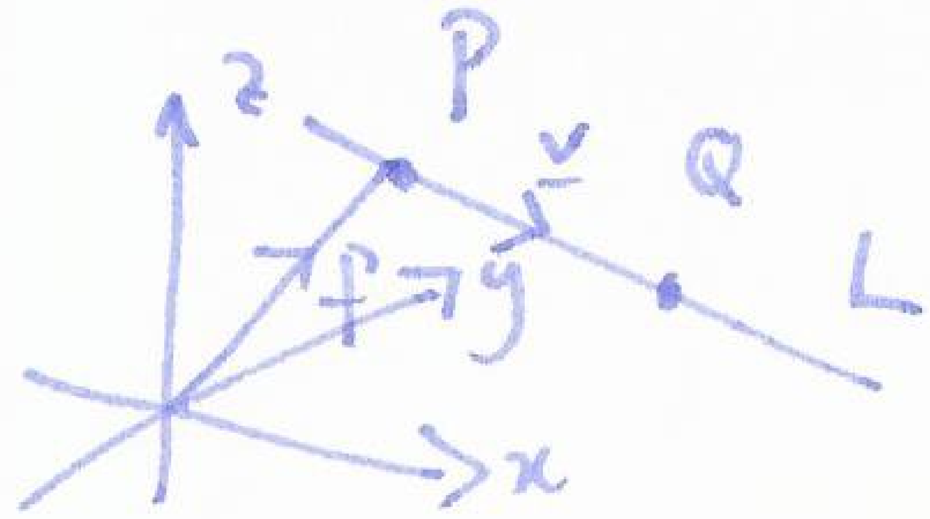
Lines and planes in 3d



how do we describe a line in $\mathbb{R}^3$?

let $\underline{v} = \overrightarrow{PQ}$

$\underline{p} = \overrightarrow{OP}$

then any point x on L can be written as

$\underline{p} + t\underline{v}$ for some $t \in \mathbb{R}$

this is a <u>parametric equation</u> for L $\quad \underline{x} = \underline{p} + t\underline{v}$.

equivalent to $\langle x, y, z\rangle = \langle x_1, y_1, z_1 \rangle + t\langle v_1, v_2, v_3\rangle$

$x = x_1 + tv_1, \; y = y_1 + tv_2, \; z = z_1 + tv_3$. (*)

if all of $v_1, v_2, v_3 \neq 0$ then can eliminate t in (*) to get

$$\frac{x - x_1}{v_1} = \frac{y - y_1}{v_2} = \frac{z - z_1}{v_3}$$

<u>symmetric equations</u> for L

Example ① L goes through $P = (1, -2, 4)$ parallel to $\underline{v} = \langle 2, 4, -4\rangle$

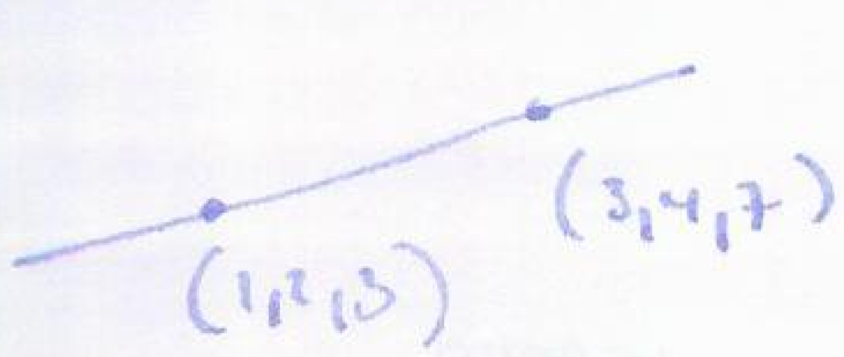
$\underline{x} \in L$ then $\underline{x} = \langle 1, -2, 4\rangle + t\langle 2, 4, -4\rangle$

$= \langle 1 + 2t, -2 + 4t, 4 - 4t\rangle$

or $x = 1 + 2t, \; y = -2 + 4t, \; z = 4 - 4t$.

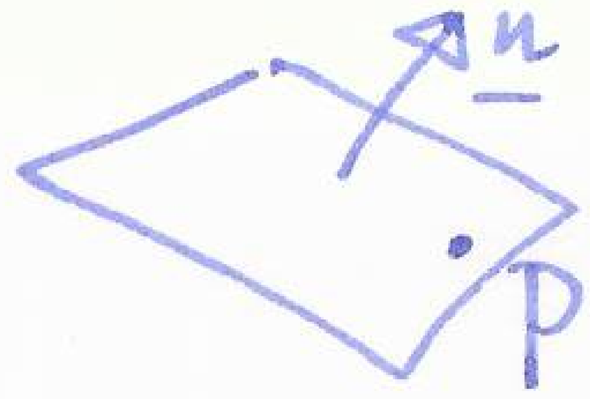
or $\dfrac{x-1}{2} = \dfrac{y+2}{4} = \dfrac{z-4}{-4}$

Th17 WW2
F18 M1
M21
W23 Midterm 1
R24 WW3,4.
} sample exam



? find $v = \langle 2, 2, 4\rangle$.

Planes



$\underline{n}$ normal vector.

Let Q be a point in the plane. Then $\overrightarrow{PQ}$ is perpendicular to $\underline{n}$

i.e. $\underline{n} \cdot \overrightarrow{PQ} = 0$.

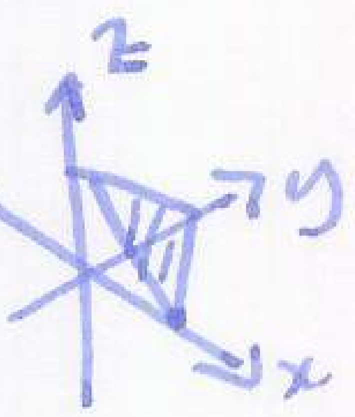
$\underline{n} = \langle a, b, c \rangle \qquad P = \langle x_1, y_1, z_1 \rangle \qquad Q = \langle x, y, z \rangle$

then $\underline{n} \cdot \overrightarrow{PQ} = \langle a, b, c \rangle \cdot \langle x - x_1, y - y_1, z - z_1 \rangle = 0$

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

or $\quad ax + by + cz + d = 0$.

Example find an equation for the plane containing $\overset{P}{(1,0,0)}, \overset{Q}{(0,1,0)}, \overset{R}{(0,0,1)}$



$\overrightarrow{PQ} = \langle -1, 1, 0 \rangle \qquad \underline{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$

$\overrightarrow{PR} = \langle -1, 0, 1 \rangle$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \underline{i} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \underline{j} \begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} + \underline{k} \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix}$$

$$= \langle 1, 1, 1 \rangle$$

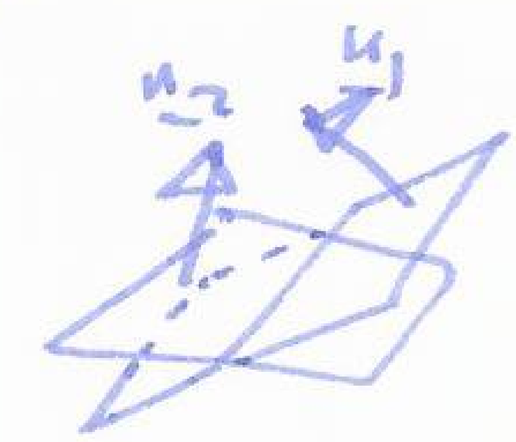
$\langle 1, 1, 1 \rangle \cdot \left(\langle 1, 0, 0 \rangle - \langle x, y, z \rangle \right) = 0$

$-x + 1 \mp y \mp z = 0 \qquad$ or. $\quad x + y + z + 1 = 0$.

angles between planes.

any two planes are either parallel

or intersect in a line



parallel $\Leftrightarrow$ normal vectors are parallel.

intersect in a line: direction of line is $\underline{n}_1 \times \underline{n}_2$!

angle between planes = angle between normal vectors !

$$\cos\theta = \frac{\underline{n}_1 \cdot \underline{n}_2}{\|\underline{n}_1\| \, \|\underline{n}_2\|}.$$

Example

find angle between planes $x + 2y + 3z + 4 = 0$

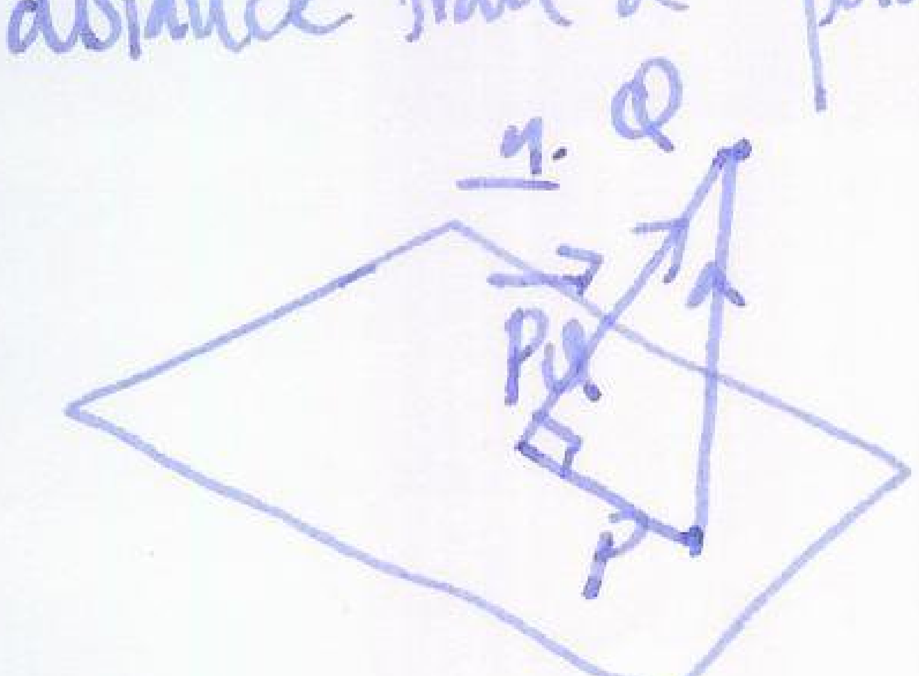
$2x - y - z - 1 = 0$

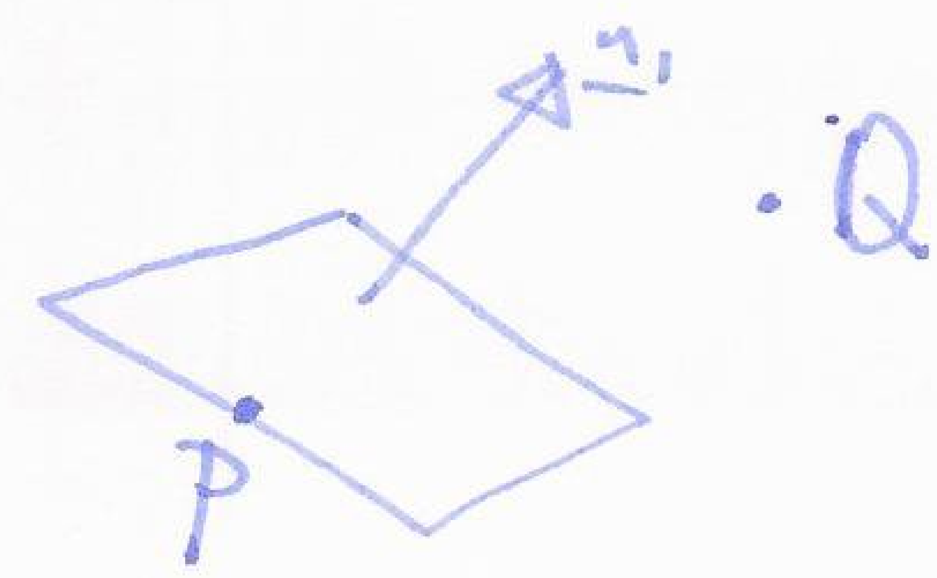
normal vectors: $\langle 1, 2, 3 \rangle$
$\langle 2, -1, -1 \rangle$

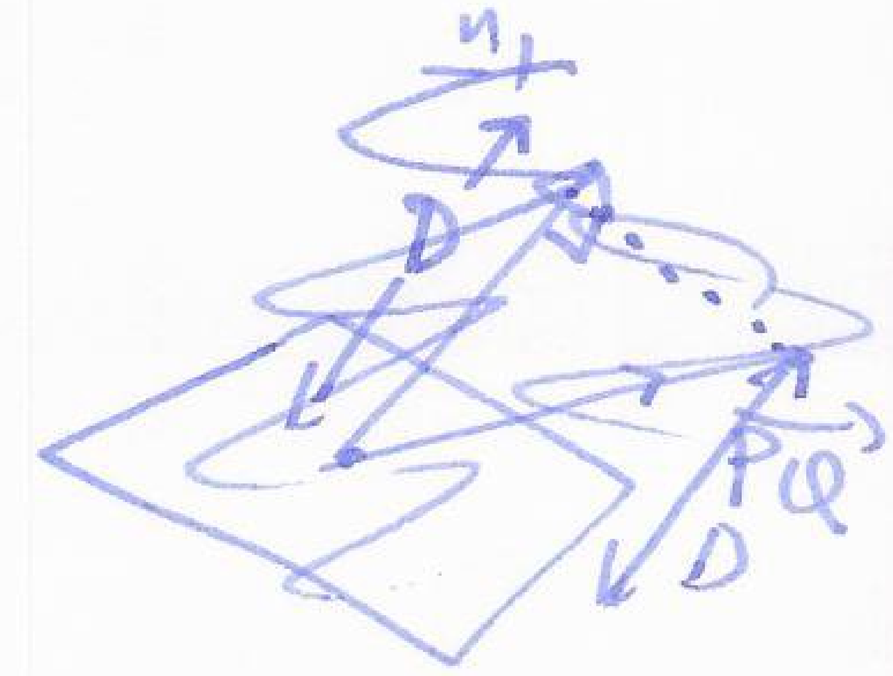
$$\cos\theta = \frac{2 - 2 - 3}{\sqrt{14}\,\sqrt{6}} = \frac{-3}{\sqrt{14}\sqrt{6}}.$$

Distances between points, planes, lines

- distance from a point to a plane:



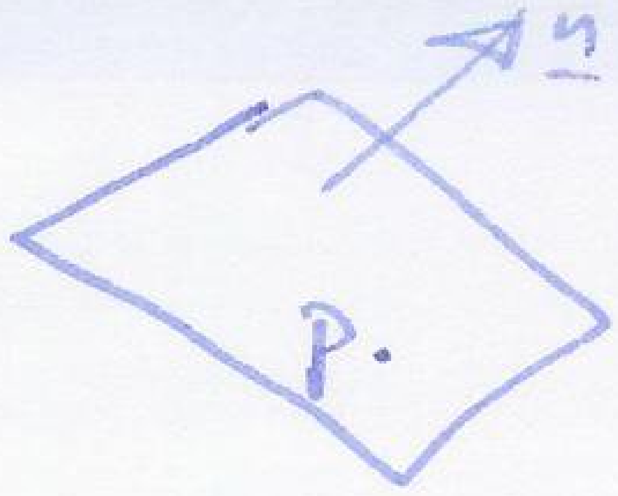




$$\text{distance from } Q \text{ to plane} = \| \mathrm{proj}_{\underline{n}}(\overrightarrow{PQ}) \|.$$

$$= \frac{\overrightarrow{PQ} \cdot \underline{n}}{\|\underline{n}\|}$$

Example



$\cdot Q = (4,3,2)$

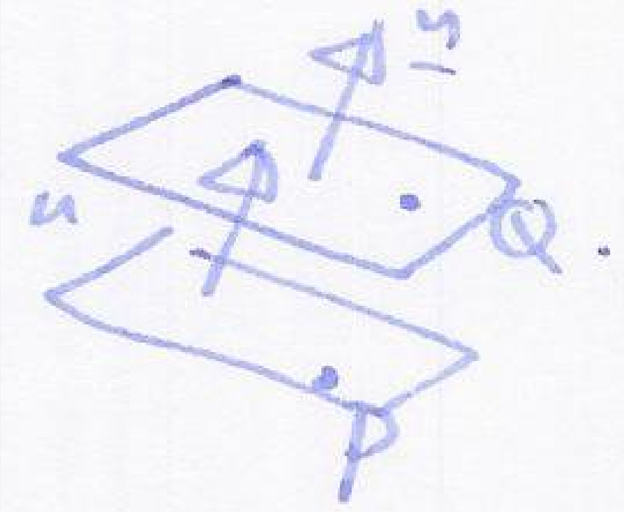
plane: $x + 2y - 3z + 1 = 0$.

normal vector: $\langle 1, 2, -3 \rangle$

point on plane? $\langle -1, 0, 0 \rangle$ $\quad \overrightarrow{PQ} = \langle 5, 3, 2 \rangle$

$$\text{proj}_{\underline{n}} \overrightarrow{PQ} = \frac{\langle 5,3,2\rangle \cdot \langle 1,2,-3\rangle}{\|\langle 1,2,-3\rangle\|} = \frac{5+6-6}{\sqrt{14}} = \frac{5}{\sqrt{14}}.$$

• distance between two parallel planes:



just distance from any point Q on one plane to the other plane!

example $\quad 3x - y + 2z + 1 = 0 \qquad \underline{n} = \langle 3, -1, 2 \rangle$

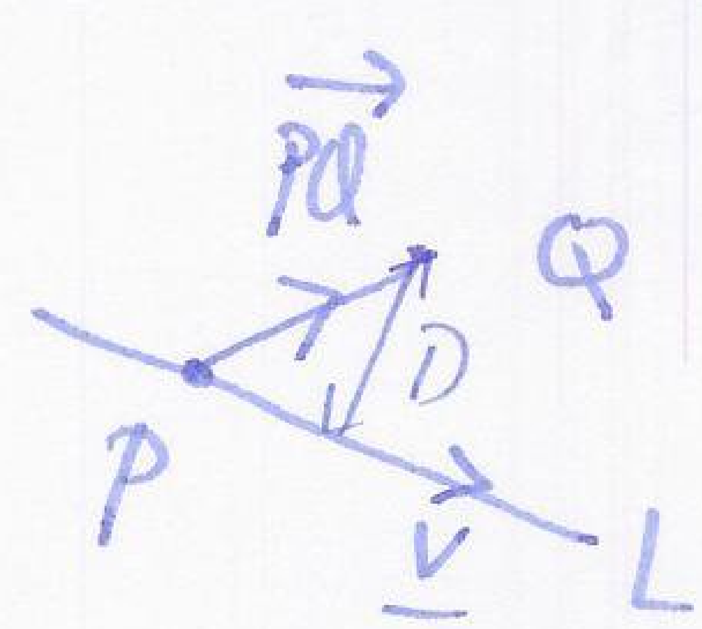
$\qquad 6x - 2y + 4z + 8 = 0 \qquad \underline{n} = \langle 6, -2, 4 \rangle$.

find $Q \in$ first plane: $\langle 0, -1, 0 \rangle$ $\qquad \overrightarrow{PQ} = \langle 0, 1, -2 \rangle$.

$P \in$ second plane: $\langle 0, 0, 2 \rangle$

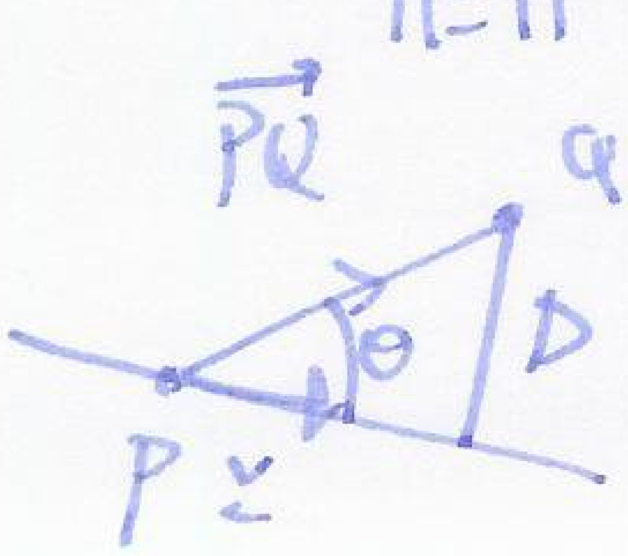
$$\text{distance: } \frac{\overrightarrow{PQ} \cdot \underline{n}}{\|\underline{n}\|} = \frac{\langle 0,1,-2\rangle \cdot \langle 3,-1,2\rangle}{\|\langle 3,-1,2\rangle\|} = \frac{-5}{\sqrt{14}}.$$

• distance between a point and a line.



$$D = \frac{\|\overrightarrow{PQ} \times \underline{v}\|}{\|\underline{v}\|}$$

why?



$$\frac{D}{\|\overrightarrow{PQ}\|} = \sin\theta \qquad D = \|\overrightarrow{PQ}\| \sin\theta$$

$$= \frac{\|\overrightarrow{PQ}\| \, \|\underline{v}\| \sin\theta}{\|\underline{v}\|} = \frac{\|\overrightarrow{PQ} \times \underline{v}\|}{\|\underline{v}\|}.$$