§11.4 Cross products

Webworks due W midnight

Matlab ① due F midnight

Important: cross product only works for 3d vectors!

Def^n $\underline{u} = \langle u_1, u_2, u_3 \rangle \qquad \underline{v} = \langle v_1, v_2, v_3 \rangle$

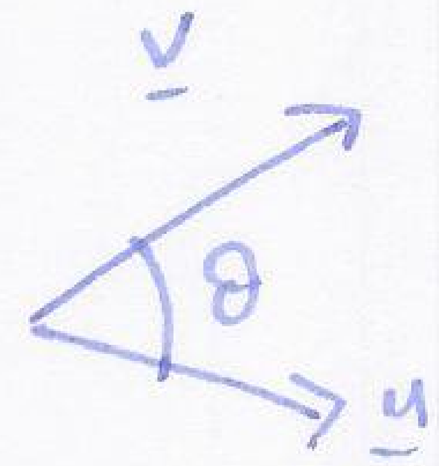
$$\underline{u} \times \underline{v} = \langle u_2 v_3 - u_3 v_2 ,\ -u_1 v_3 + u_3 v_1 ,\ u_1 v_2 - u_2 v_1 \rangle$$

mnemonic: 3x3 determinant $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

$$= \underline{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \underline{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \underline{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= \langle u_2 v_3 - v_2 u_3 ,\ -u_1 v_3 + u_3 v_1 ,\ u_1 v_2 - u_2 v_1 \rangle .$$

Q: what does it mean?



Geometric properties:

- $\underline{u} \times \underline{v}$ is orthogonal to both $\underline{u}$ and $\underline{v}$ (right hand rule!)
- $\|\underline{u} \times \underline{v}\| = \|\underline{u}\| \, \|\underline{v}\| \sin\theta$
- $\underline{u} \times \underline{v} = \underline{0}$ iff $\underline{u}, \underline{v}$ are parallel.
- $\|\underline{u} \times \underline{v}\| =$ area of parallelogram with sides parallel to $\underline{u}$ and $\underline{v}$.

Algebraic properties $\underline{u}, \underline{v}, \underline{w}$ vectors c scalar

- $\underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$
- $\underline{u} \times (\underline{v} + \underline{w}) = \underline{u} \times \underline{v} + \underline{u} \times \underline{w}$
- $c(\underline{u} \times \underline{v}) = c\underline{u} \times \underline{v} = \underline{u} \times c\underline{v}$
- $\underline{u} \times \underline{0} = \underline{0} = \underline{0} \times \underline{u}$
- $\underline{u} \times \underline{u} = \underline{0}$
- $\underline{u} \cdot (\underline{v} \times \underline{w}) = (\underline{u} \times \underline{v}) \cdot \underline{w}$.

Example ① Find a unit vector orthogonal to both $\underline{u} = \langle 1,2,3 \rangle$ and $\langle 1,0,-1 \rangle = \underline{v}$

$$\underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 1 & 0 & -1 \end{vmatrix} = \underline{i} \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} - \underline{j} \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + \underline{k} \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix}$$

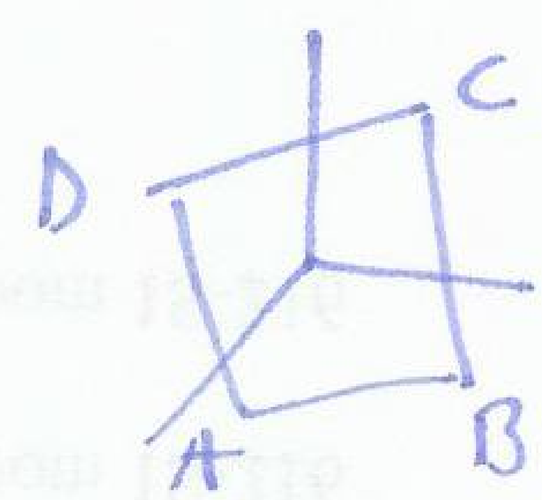
$$= \langle -2, 4, -2 \rangle$$

parallel to $\langle 1, -2, 1 \rangle$

unit vector : $\left\langle \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$

② find the area of the parallelogram with vertices $\underset{A}{(5,2,0)}$ $\underset{B}{(2,6,1)}$ $\underset{C}{(2,4,7)}$ $\underset{D}{(5,0,6)}$

check sides are parallel:



$$\vec{AB} = \langle -3, 4, 1 \rangle$$

$$\vec{AD} = \langle 0, 2, -6 \rangle$$

$$\vec{CD} = \langle 3, -4, -1 \rangle$$

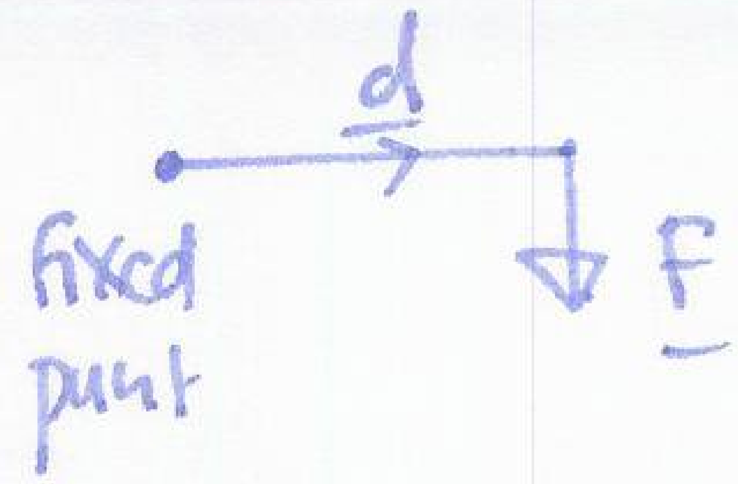
$$\vec{CB} = \langle 0, +2, -6 \rangle$$

$$\text{area} = \| \vec{AB} \times \vec{AD} \| = \left\| \begin{matrix} \underline{i} & \underline{j} & \underline{k} \\ -3 & 4 & 1 \\ 0 & 2 & -6 \end{matrix} \right\| = \underline{i} \begin{vmatrix} 4 & 1 \\ 2 & -6 \end{vmatrix} - \underline{j} \begin{vmatrix} -3 & 1 \\ 0 & 6 \end{vmatrix} + \underline{k} \begin{vmatrix} -3 & 4 \\ 0 & 2 \end{vmatrix}$$

$$= \langle -26, 18, -6 \rangle$$

$$\sqrt{26^2 + 18^2 + 36^2} \approx 32.19.$$

Application torque (moments).



$$\boxed{\underline{M} = \underline{d} \times \underline{F}}$$

turning force
torque.

Triple scalar product

$\underline{u} = \langle u_1, u_2, u_3 \rangle$

$\underline{v} = \langle v_1, v_2, v_3 \rangle$

$\underline{w} = \langle w_1, w_2, w_3 \rangle$

$$\underline{u}\cdot(\underline{v}\times\underline{w}) = \langle u_1, u_2, u_3 \rangle \cdot \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

useful property

$$\underline{u}\cdot(\underline{v}\times\underline{w}) = \underline{v}\cdot(\underline{w}\times\underline{u}) = \underline{w}\cdot(\underline{u}\times\underline{v}) = -\{\underline{u}\cdot(\underline{w}\times\underline{v})\} \cdots$$

Geometric meaning : $\underline{u}\cdot(\underline{v}\times\underline{w})$ = volume of parallelepiped spanned by $\underline{u}, \underline{v}, \underline{w}$.