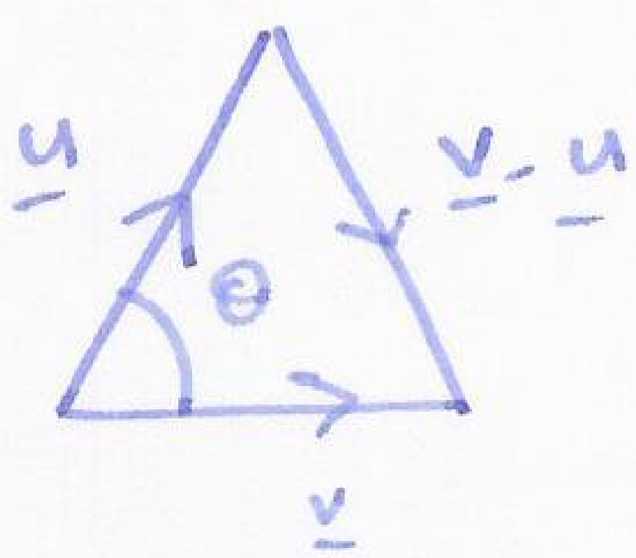
Key property if θ is the angle between two vectors $\underline{u}$ and $\underline{v}$

then $$\boxed{\underline{u}\cdot\underline{v} = \|\underline{u}\|\,\|\underline{v}\|\cos\theta}$$

$\underline{i}+\underline{k}$ 3d vect
$\underline{i}$? 2d or 3d?

Proof



recall: cosine law for triangles:

$$\|\underline{v}-\underline{u}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2 - 2\|\underline{u}\|\,\|\underline{v}\|\cos\theta.$$

(note: if $\theta = 90^\circ$ $\cos\theta = 0$, just Pythagoras).

$$\begin{aligned}
\|\underline{v}-\underline{u}\|^2 &= (\underline{v}-\underline{u})\cdot(\underline{v}-\underline{u}) \\
&= \underline{v}\cdot(\underline{v}-\underline{u}) - \underline{u}\cdot(\underline{v}-\underline{u}) \\
&= \underline{v}\cdot\underline{v} - \underline{v}\cdot\underline{u} - \underline{u}\cdot\underline{v} + \underline{u}\cdot\underline{u} \\
&= \|\underline{v}\|^2 + \|\underline{u}\|^2 - 2\underline{u}\cdot\underline{v}
\end{aligned}$$

$\therefore$ $\underline{u}\cdot\underline{v} = \|\underline{u}\|\,\|\underline{v}\|\cos\theta$ $\square$.

Corollary $$\boxed{\cos\theta = \frac{\underline{u}\cdot\underline{v}}{\|\underline{u}\|\,\|\underline{v}\|}}$$ θ angle between $\underline{u}$ and $\underline{v}$.

Defn. we say two vectors are <u>perpendicular</u> or <u>orthogonal</u>

if the angle between them is $\frac{\pi}{2}$, $\perp$

$\underline{u}, \underline{v}$ orthogonal iff $\underline{u}\cdot\underline{v} = 0$

Convention: $\underline{0}$ is orthogonal to all vectors.

Example find the angle between $\underset{\underline{u}}{\langle 2,1,-3\rangle}$ and $\underset{\underline{v}}{\langle 3,-3,1\rangle}$.

$$\cos\theta = \frac{\underline{u}\cdot\underline{v}}{\|\underline{u}\|\,\|\underline{v}\|}$$

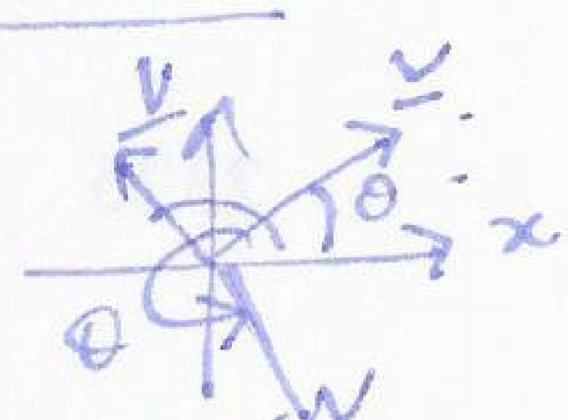
$$\|\underline{u}\| = \sqrt{4+1+9} = \sqrt{14}$$
$$\|\underline{v}\| = \sqrt{9+9+1} = \sqrt{19}.$$

$$\underline{u}\cdot\underline{v} = \langle 2,1,-3\rangle\cdot\langle 3,-3,1\rangle$$
$$= 2.3 + 1.-3 + -3.1$$
$$= 6 - 3 - 3 = 0.$$

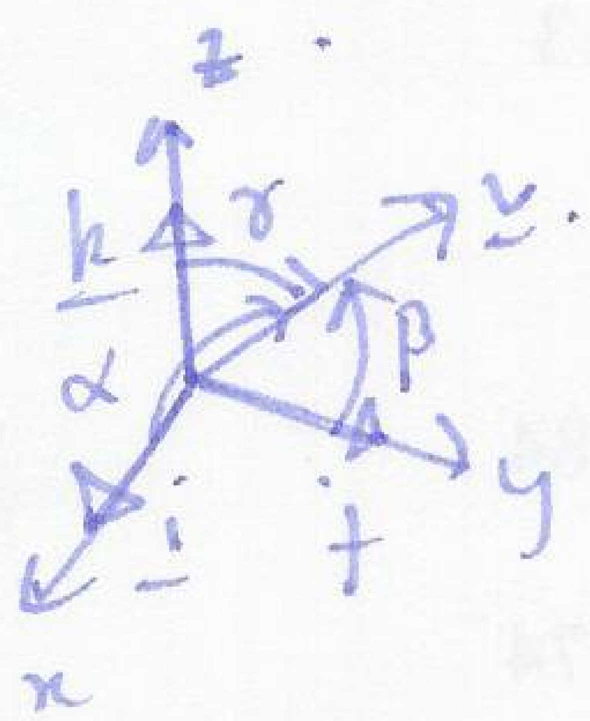
$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ so $\underline{u},\underline{v}$ are perpendicular

Direction cosines

in 2d:



in 3d more convenient to measure angle between $\underline{v}$ and the $\underline{i},\underline{j},\underline{k}$ standard unit vectors.



$$\cos\alpha = \frac{\underline{v}\cdot\underline{i}}{\|\underline{v}\|\,\|\underline{i}\|}$$

$\underline{v} = \langle v_1,v_2,v_3\rangle$

Note $\langle v_1,v_2,v_3\rangle\cdot\langle 1,0,0\rangle$
$= v_1.$ and $\|\underline{i}\| = 1$.

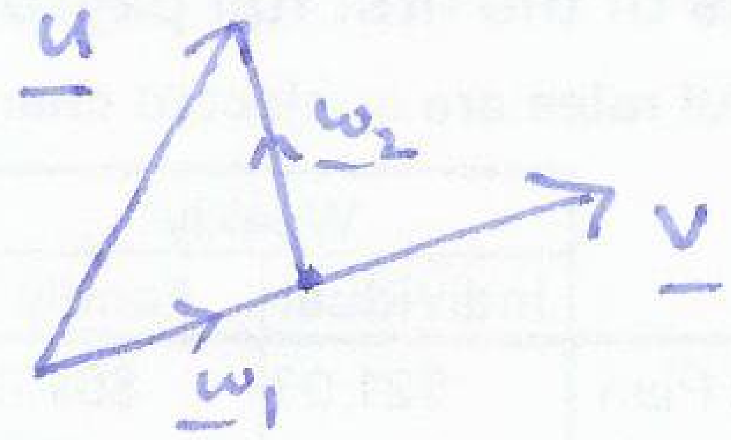
so $$\cos\alpha = \frac{v_1}{\|\underline{v}\|}$$

similarly $$\cos\beta = \frac{v_2}{\|\underline{v}\|}$$

$$\cos\gamma = \frac{v_3}{\|\underline{v}\|}.$$

Projections and components

Let $\underline{u}$ and $\underline{v}$ be vectors



we can write $\underline{u}$ as $\underline{u} = \underline{w}_1 + \underline{w}_2$ where $\underline{w}_1$ is parallel to $\underline{v}$

$\underline{w}_2$ is orthogonal to $\underline{v}$.

$\underline{w}_1$ is called the projection of $\underline{u}$ onto $\underline{v}$ notation: $\text{proj}_{\underline{v}}(\underline{u})$.

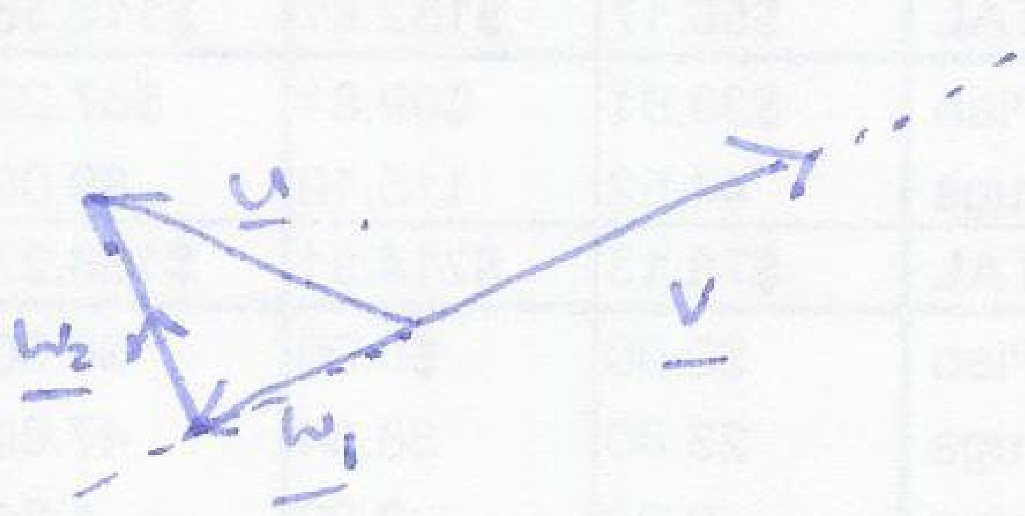
"shadow cast by $\underline{u}$ on $\underline{v}$"

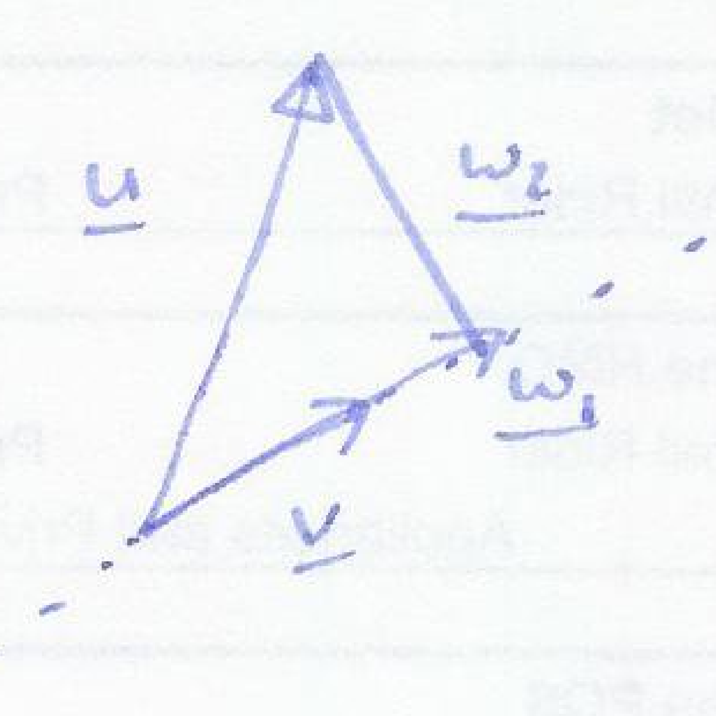
"amount of $\underline{u}$ in the $\underline{v}$ direction".

$\underline{w}_2 = \underline{u} - \underline{w}_1$ is called the vector component of $\underline{u}$ orthogonal to $\underline{v}$

Remark $\underline{w}_1$ is always parallel to $\underline{v}$, but may point in opposite direction.

$\text{proj}_{\underline{v}}(\underline{u})$





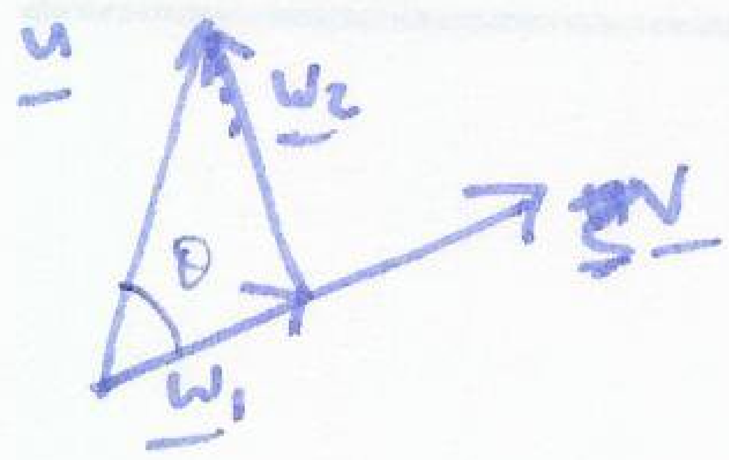
$\underline{w}_1$ may also be larger than $\underline{v}$.

Notation component vs vector component

$\text{proj}_{\underline{v}}(\underline{u})$ is the vector component

$\|\text{proj}_{\underline{v}}(\underline{u})\|$ is the component

Finding vector components



$$\underline{w}_1 = \boxed{\mathrm{proj}_{\underline{v}}(\underline{u}) = \left(\frac{\underline{u}\cdot\underline{v}}{\|\underline{v}\|^2}\right)\underline{v}}\,.$$

why does this work?

~~recall~~ : $\underline{u}\cdot\underline{v} = \|\underline{u}\|\|\underline{v}\|\cos\theta$

~~also~~ projection $\|\underline{w}_1\| = \|\mathrm{proj}_{\underline{v}}(\underline{u})\| = \|\underline{u}\|\cos\theta$

<u>recall</u> $\underline{u}\cdot\underline{v} = \|\underline{u}\|\,\|\underline{v}\|\cos\theta$

so $\|\underline{w}_1\| = \dfrac{\underline{u}\cdot\underline{v}}{\|\underline{v}\|}$

so $\underline{w}_1 = \|\underline{w}_1\|\,\hat{\underline{v}}$ where $\hat{\underline{v}}$ = unit vector in $\underline{v}$ direction

$= \dfrac{1}{\|\underline{v}\|}\underline{v}$

so $\underline{w}_1 = \dfrac{\underline{u}\cdot\underline{v}}{\|\underline{v}\|}\cdot\dfrac{1}{\|\underline{v}\|}\underline{v} = \dfrac{\underline{u}\cdot\underline{v}}{\|\underline{v}\|^2}\underline{v}$ as required $\square$.

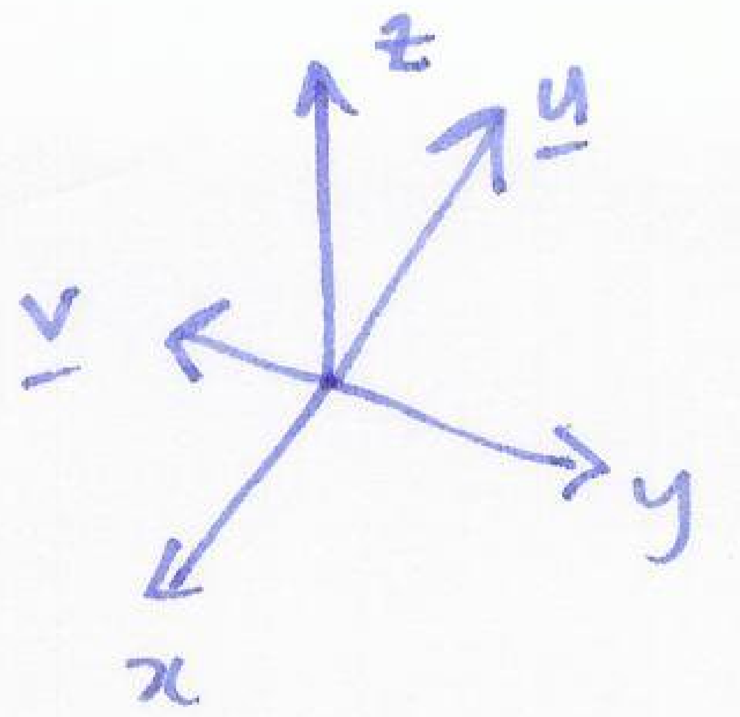
now $\underline{w}_2 = \underline{u} - \underline{w}_1 = \underline{u} - \dfrac{\underline{u}\cdot\underline{v}}{\|\underline{v}\|^2}\underline{v}$.

<u>check</u> $\underline{w}_2$ is perpendicular to $\underline{v}$:

$$\left(\underline{u} - \frac{\underline{u}\cdot\underline{v}}{\|\underline{v}\|^2}\underline{v}\right)\cdot\underline{v} = \underline{u}\cdot\underline{v} - \frac{\underline{u}\cdot\underline{v}}{\|\underline{v}\|^2}\underline{v}\cdot\underline{v}$$

$$= \underline{u}\cdot\underline{v} - \frac{\underline{u}\cdot\underline{v}}{\|\underline{v}\|^2}\|\underline{v}\|^2 = \underline{u}\cdot\underline{v} - \underline{u}\cdot\underline{v} = 0 \ \square$$

Example



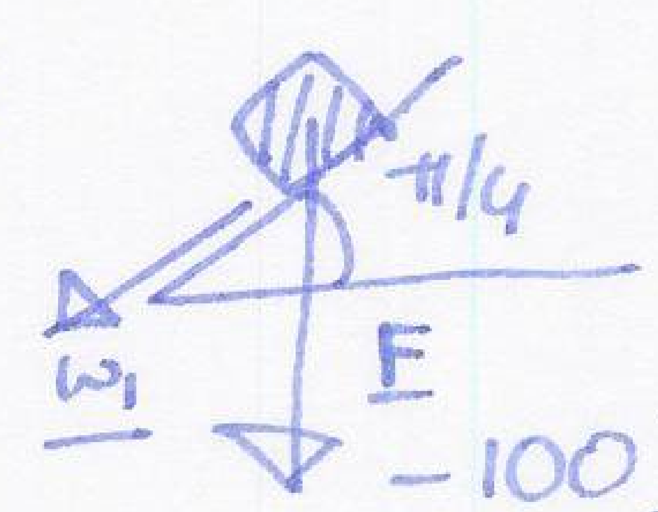
find the vector component of $\underline{u} = \langle 1,2,3 \rangle$ in the direction of $\underline{v} = \langle 1,-1,-1 \rangle$

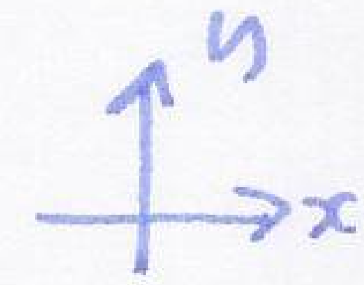
$$\text{proj}_{\underline{v}}(\underline{u}) = \frac{\underline{u}\cdot\underline{v}}{\|\underline{v}\|^2}\,\underline{v} = \frac{\langle 1,2,3\rangle . \langle 1,-1,-1\rangle}{\sqrt{1^2+1^2+1^2}} \langle 1,-1,-1\rangle$$

$$= \frac{1-2-3}{\sqrt{3}} \langle 1,-1,-1\rangle = \frac{-4}{\sqrt{3}} \langle 1,-1,-1\rangle$$

$$= \left\langle \frac{-4}{\sqrt{3}}, \frac{+4}{\sqrt{3}}, \frac{+4}{\sqrt{3}} \right\rangle .$$

Application ~~Example~~ an object sits on a ramp of angle $\frac{\pi}{4}$:





if the object weighs 100N, how large must the frictional force be to keep the object from slipping down?

answer: find component of $\underline{F}$ in direction of slope

unit vector $\underline{v}$ in direction of slope is $\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

so
$$\underline{w_1} = \frac{\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\rangle . \langle 0,-100\rangle}{1} . \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle .$$

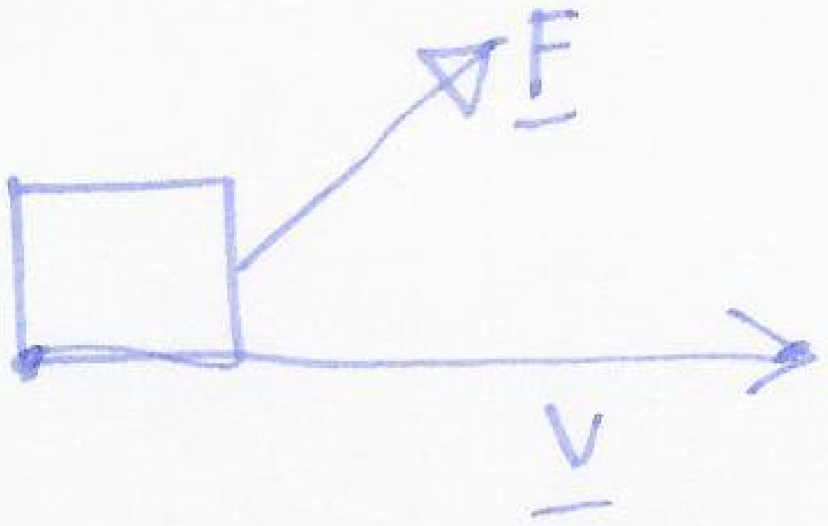
$$= 50\sqrt{2} \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

magnitude $50\sqrt{2} \simeq 70$.

Application : work

work done $W = (\text{magnitude of force})(\text{distance}) = \|\underline{F}\| \, \|\underline{v}\|$.

$\underline{v}$ parallel to $\underline{F}$.



$\underline{v}$ not parallel

$W = \|\mathrm{proj}_{\underline{v}} \underline{F}\| \, \|\underline{v}\| = \underline{F} \cdot \underline{v}$.

Def^{n} work done $W = \underline{F} \cdot \underline{v}$

$\underline{F}$ = force

$\underline{v}$ = distance moved / displacement

Example find the work done by a force $\underline{F} = 100N$ at an angle of 45° pulling the box horizontally a distance of 10m.

$$\underline{F} \cdot \underline{v} = \left\langle 100\tfrac{\sqrt{2}}{2}, 100\tfrac{\sqrt{2}}{2} \right\rangle \cdot \langle 10, 0 \rangle = 1000\tfrac{\sqrt{2}}{2} \approx 700.$$