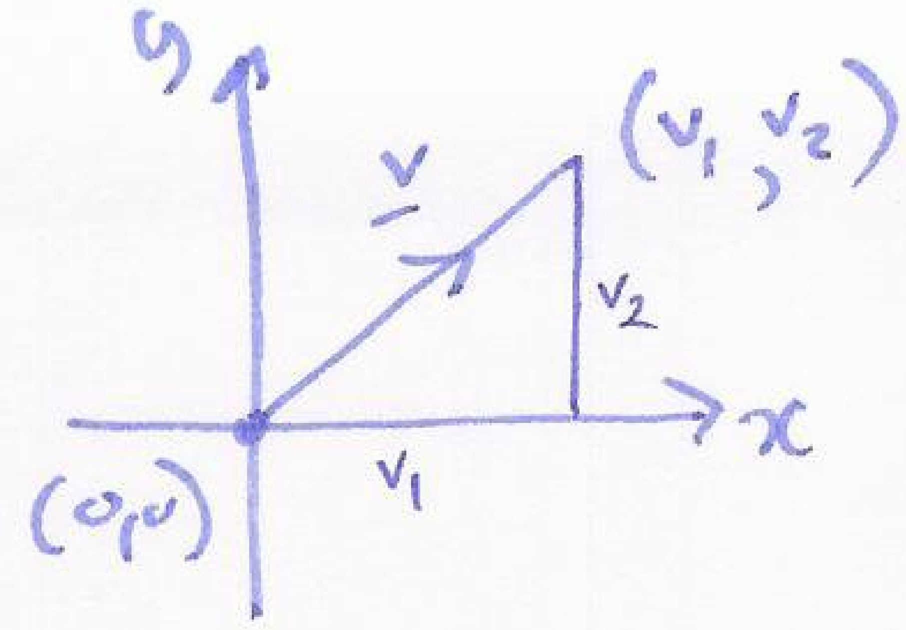
MATLAB.
1 9/16
2 10/5
3 11/9
4 11/25
Webworks.
Wed midnight.

<u>lengths of vectors</u>



$\underline{v} = \langle v_1, v_2 \rangle$

$\|\underline{v}\| = \sqrt{v_1^2 + v_2^2}$

<u>length of a scalar multiple</u> $\|c\underline{v}\| = |c| \, \|\underline{v}\|$

note: $\cancel{\|c\underline{v}\| = \|\langle cv_1, cv_2 \rangle\|}$ $c\underline{v} = c\langle v_1, v_2 \rangle$

$= c(v_1 \underline{i} + v_2 \underline{j})$

$= cv_1 \underline{i} + cv_2 \underline{j} = \langle cv_1, cv_2 \rangle$

<u>useful fact</u>: $\boxed{c\langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle}$

so $\|c\underline{v}\| = \|\langle cv_1, cv_2 \rangle\| = \sqrt{(cv_1)^2 + (cv_2)^2} = \sqrt{c^2(v_1^2 + v_2^2)}$

$= |c|\sqrt{v_1^2 + v_2^2}$

$= |c| \, \|\underline{v}\|$.

<u>unit vectors</u>: a vector of length 1 is called a <u>unit vector</u>.

e.g. $\langle 1, 0 \rangle, \langle 0, 1 \rangle, \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ are unit vectors.

given a vector $\underline{v}$, the vector $\frac{1}{\|\underline{v}\|}\underline{v}$ is a unit vector in the direction of $\underline{v}$.

check $\left\| \frac{1}{\|\underline{v}\|} \underline{v} \right\| = \frac{1}{\|\underline{v}\|} \|\underline{v}\| = 1$.

<u>Example</u> $\underline{v} = \langle 2, -3 \rangle$ find unit vector in direction of $\underline{v}$:

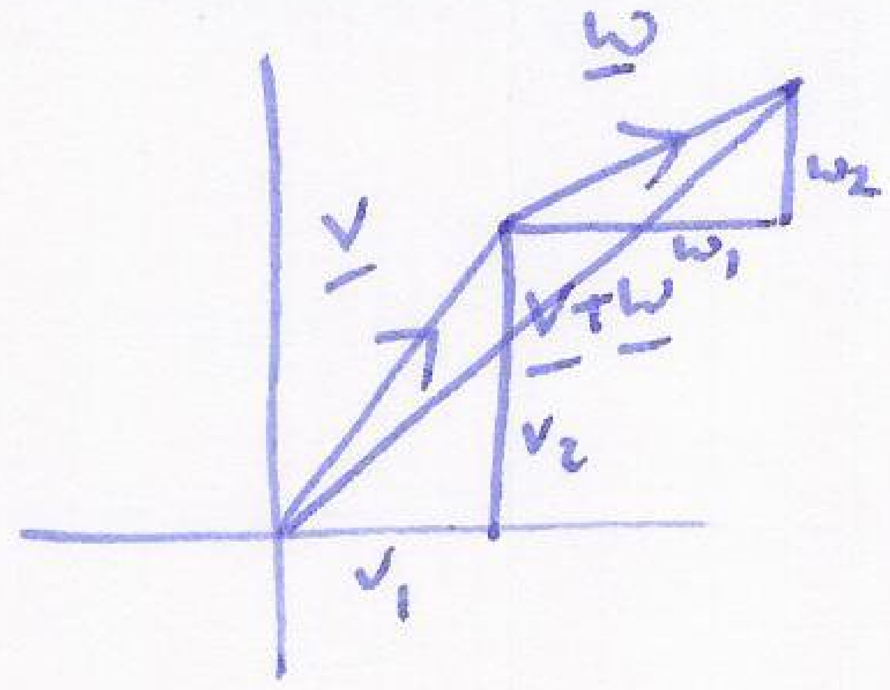
$\|\underline{v}\| = \sqrt{2^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$

so unit vector is $\frac{1}{\sqrt{13}}\langle 2, -3 \rangle = \langle \frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}} \rangle$.

adding vectors in coordinates

$\underline{v} = \langle v_1, v_2 \rangle$

$\underline{w} = \langle w_1, w_2 \rangle$

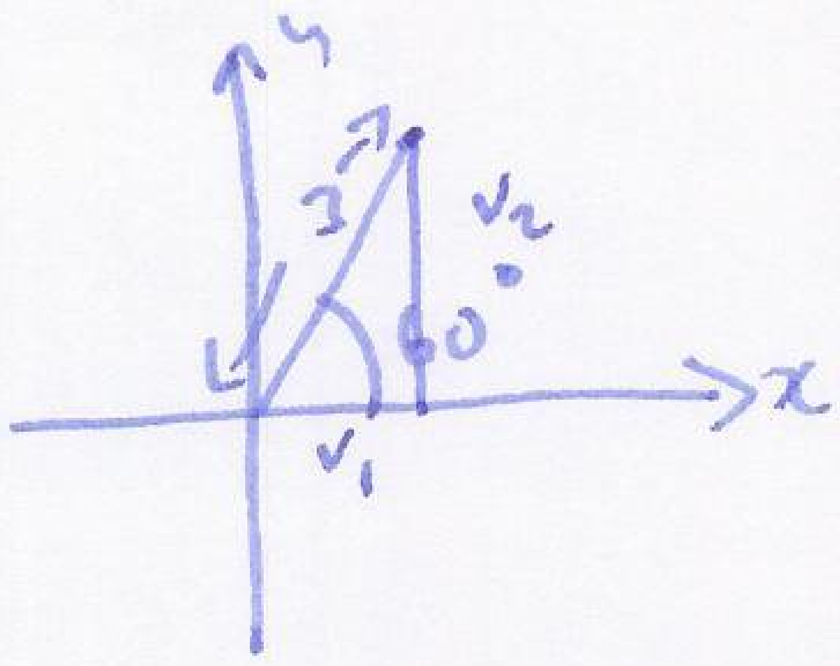


so $\underline{v} + \underline{w} = \langle v_1 + w_1, v_2 + w_2 \rangle$

"addition is componentwise".

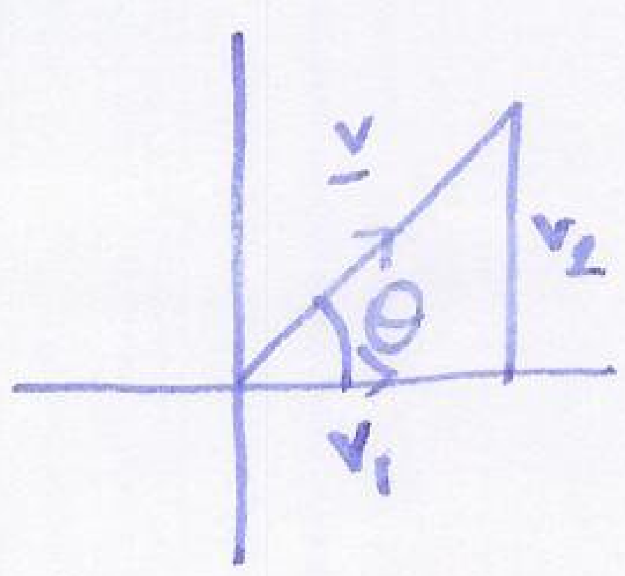
converting between descriptions:

"$\underline{v}$ is vector of length 3 making angle of $+60^\circ$ anticlockwise with x-axis".



$\frac{v_1}{3} = \cos 60^\circ \qquad \frac{v_2}{3} = \sin 60^\circ$

so $\underline{v} = \langle v_1, v_2 \rangle = \langle 3\cos 60^\circ, 3\sin 60^\circ \rangle$.



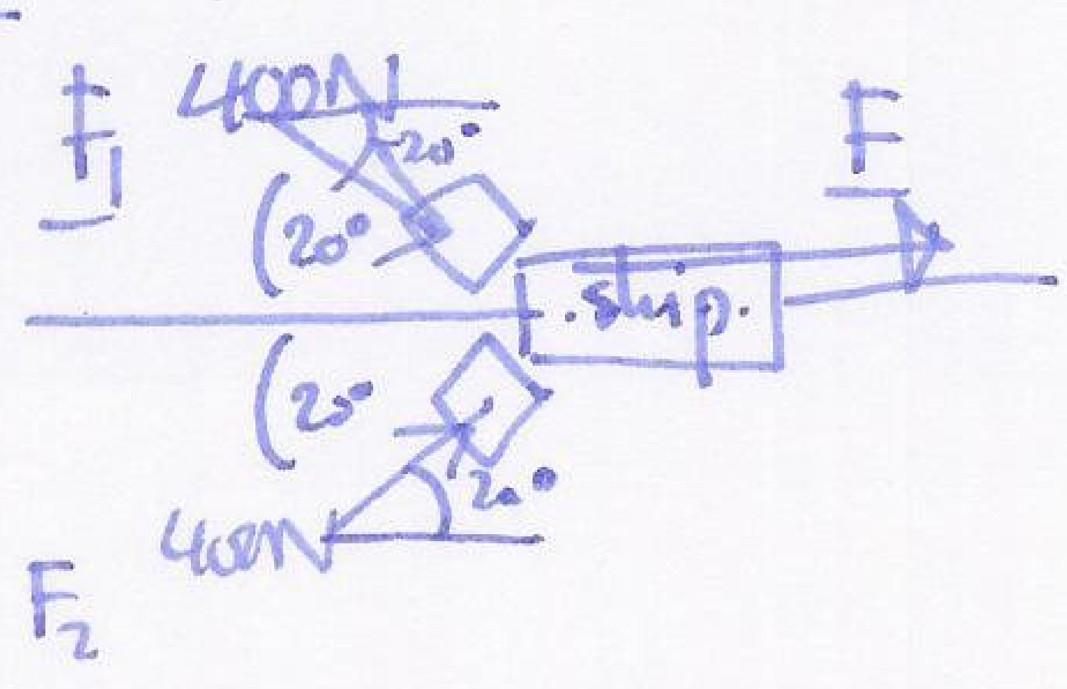
$\|\underline{v}\| = \sqrt{v_1^2 + v_2^2}$

$\tan\theta = \frac{v_2}{v_1}$

Applications

force is a vector (size + direction)

Example tugboats



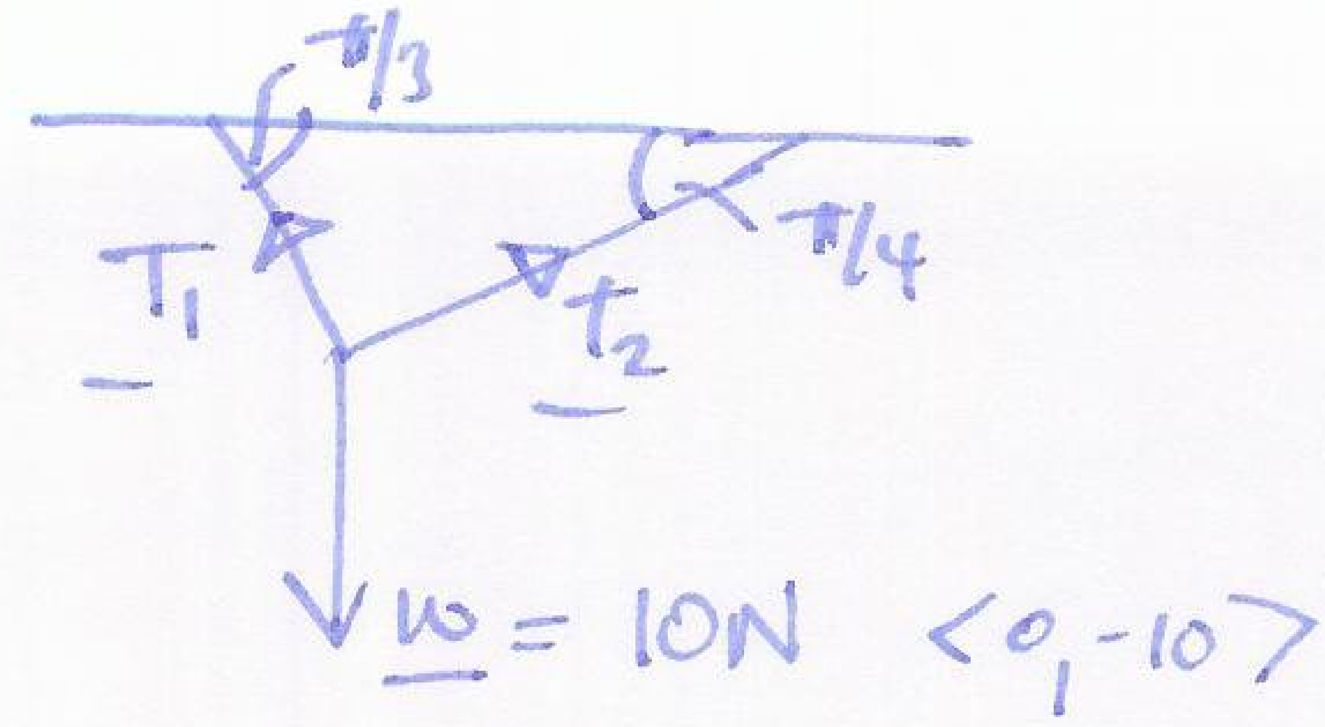
Q: what is resulting force on the ship?

$\underline{F_1} = 400 \langle \cos(-20^\circ), \sin(20^\circ) \rangle$.

$\underline{F_2} = 400 \langle \cos(20^\circ), \sin(20^\circ) \rangle$

$\underline{F} = \underline{F_1} + \underline{F_2} = \langle 800\cos(20^\circ), 0 \rangle \approx \langle 752, 0 \rangle = 752\,\underline{i}$.

Example (forces in balance)



a weight hang in balance / equilibrium suspended by wires making angles of $\frac{\pi}{3}$ and $\frac{\pi}{4}$. Find the force/tension in each wire.

equilibrium : $\underline{T_1} + \underline{T_2} = \underline{w}$ (two equations!)

$$\underline{T_1} = \left\langle -\|\underline{T_1}\| \overset{1/2}{\cos\tfrac{\pi}{3}}, \|\underline{T_1}\| \overset{\sqrt{3}/2}{\sin\tfrac{\pi}{3}} \right\rangle$$

$$\underline{T_2} = \left\langle \|\underline{T_2}\| \underset{\sqrt{2}/2}{\cos\tfrac{\pi}{4}}, \|\underline{T_2}\| \underset{\sqrt{2}/2}{\sin\tfrac{\pi}{4}} \right\rangle$$

$$\left\langle -\|\underline{T_1}\|\cdot\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\|\underline{T_1}\| \right\rangle + \left\langle \tfrac{\sqrt{2}}{2}\|\underline{T_2}\|, \tfrac{\sqrt{2}}{2}\|\underline{T_2}\| \right\rangle = \langle 0, +10\rangle$$

$$\left\langle -\tfrac{1}{2}\|\underline{T_1}\| + \tfrac{\sqrt{2}}{2}\|\underline{T_2}\|, \tfrac{\sqrt{3}}{2}\|\underline{T_1}\| + \tfrac{\sqrt{2}}{2}\|\underline{T_2}\| \right\rangle = \langle 0, 10\rangle$$

$$-\tfrac{1}{2}\|\underline{T_1}\| + \tfrac{\sqrt{2}}{2}\|\underline{T_2}\| = 0 \quad (1)$$

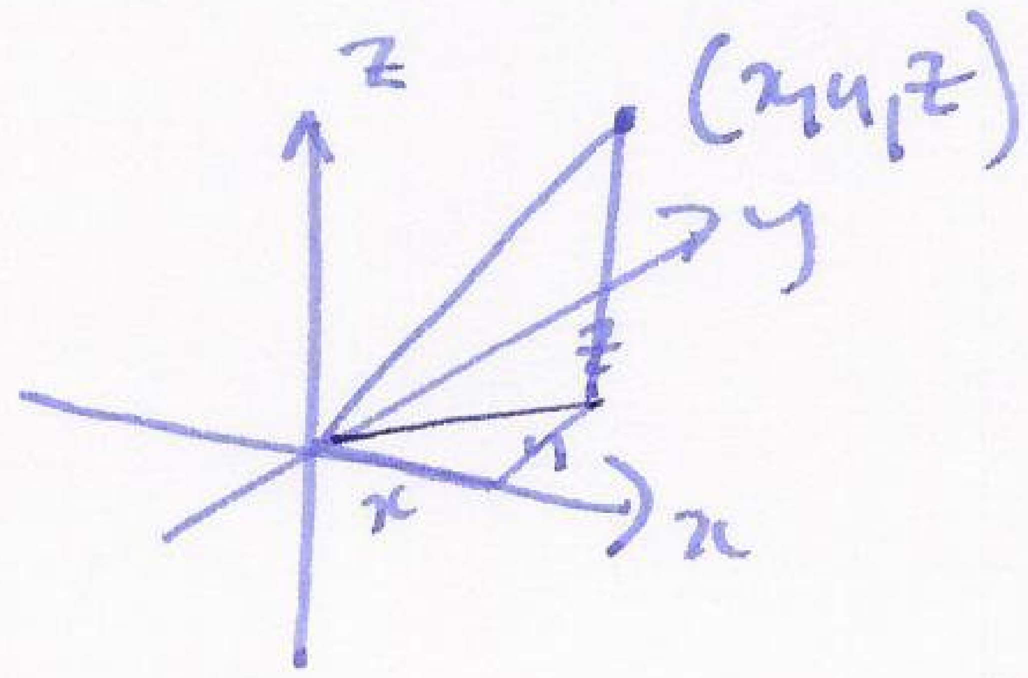
$$\tfrac{\sqrt{3}}{2}\|\underline{T_1}\| + \tfrac{\sqrt{2}}{2}\|\underline{T_2}\| = 10 \quad (2)$$

$\sqrt{3}(1) + (2)$: $\frac{\sqrt{6}}{2}\|\underline{T_2}\| + \frac{\sqrt{2}}{2}\|\underline{T_2}\| = 10$ $\qquad \|\underline{T_2}\| = \frac{20}{\sqrt{6}+\sqrt{2}} \approx 5.2.$

$(2) - (1)$: $\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right)\|\underline{T_1}\| = 10$ $\qquad \|\underline{T_1}\| = \frac{20}{1+\sqrt{3}} \approx 7.3.$

§11.2 Vectors in 3d

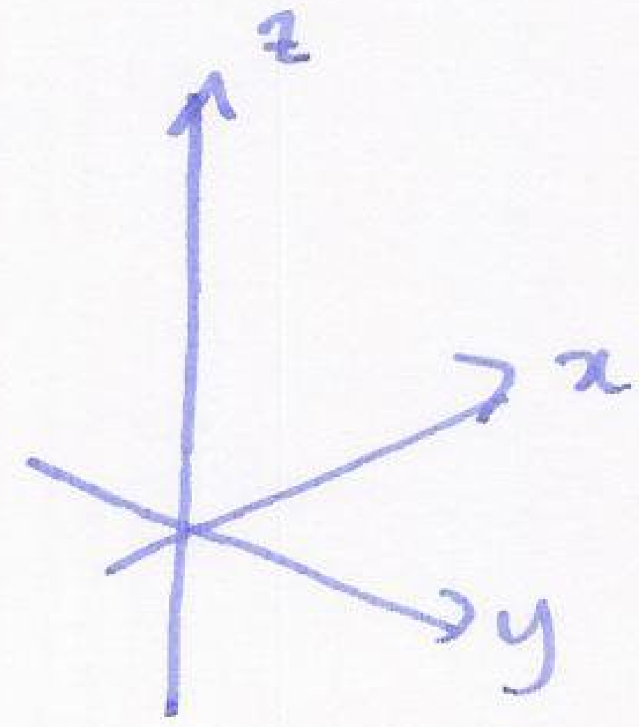
3d coordinates. ($\mathbb{R}^3$)



RIGHT HANDED!

USE THIS ONE

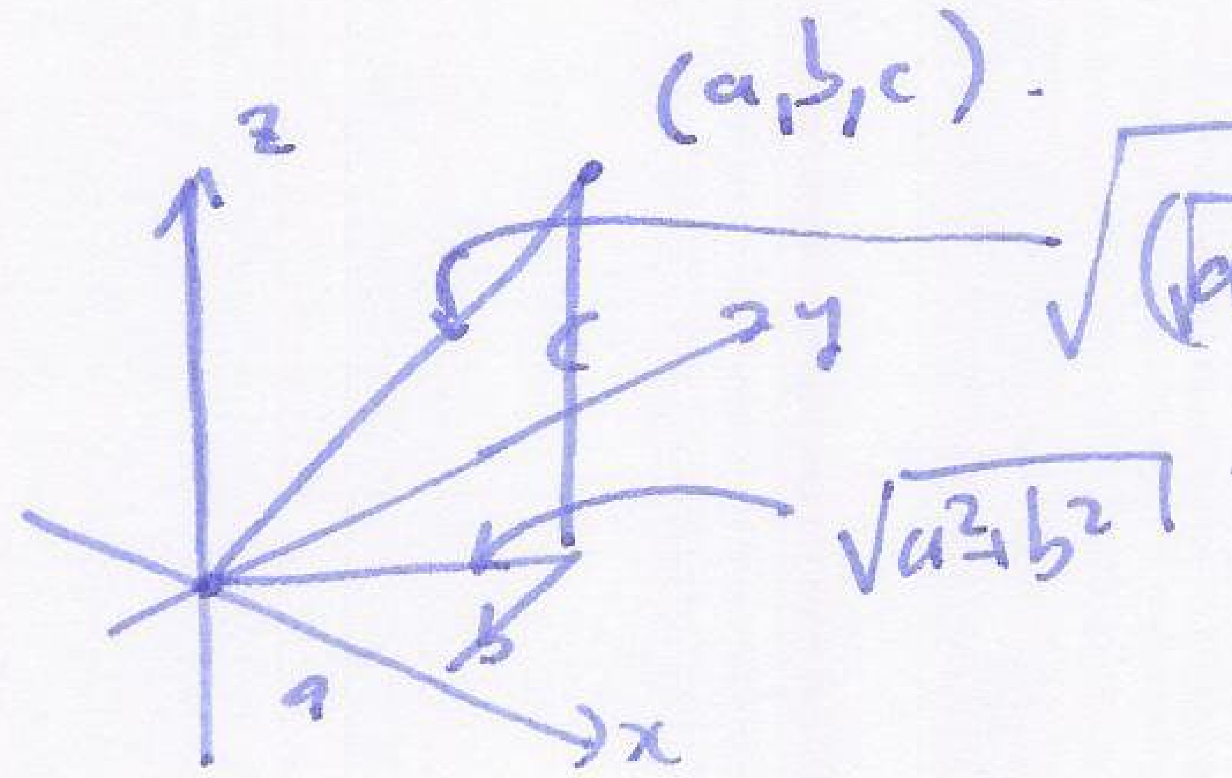
right hand.



LEFT HANDED.

useful facts

distances in 3d:



$$\sqrt{\left(\sqrt{a^2+b^2}\right)^2+c^2} = \sqrt{a^2+b^2+c^2}.$$

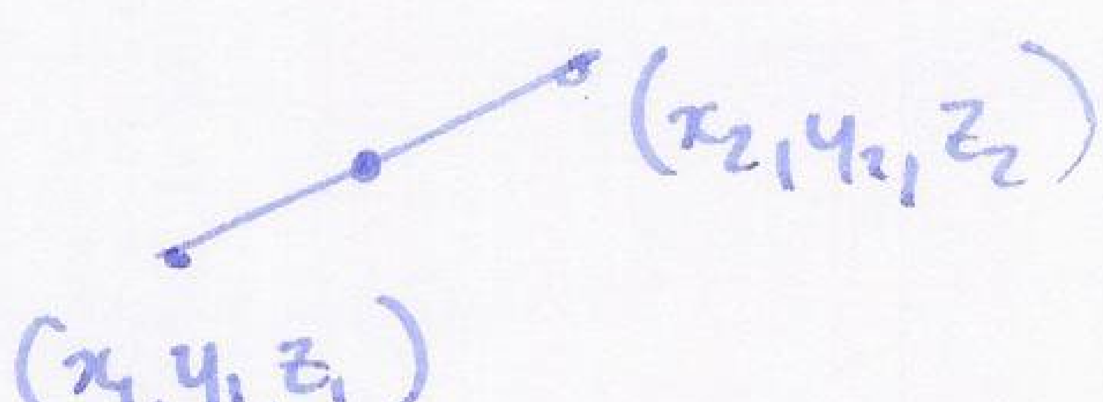
so distance between (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

a sphere in $\mathbb{R}^3$ is the collection of points a constant distance r from $(0,0,0)$ origin.

i.e. solution to equation $\sqrt{x^2+y^2+z^2} = r$

$$x^2+y^2+z^2 = r^2.$$

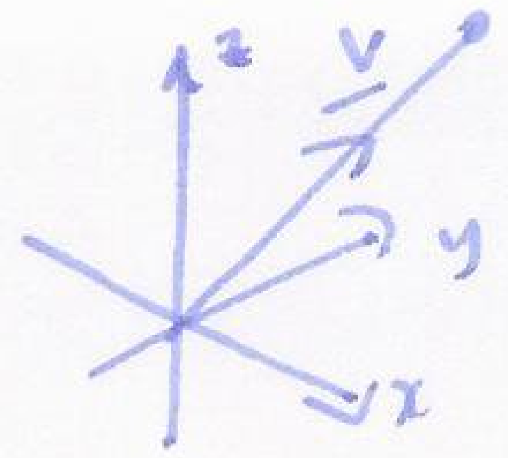
midpoints



$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right).$$

vectors in 3d

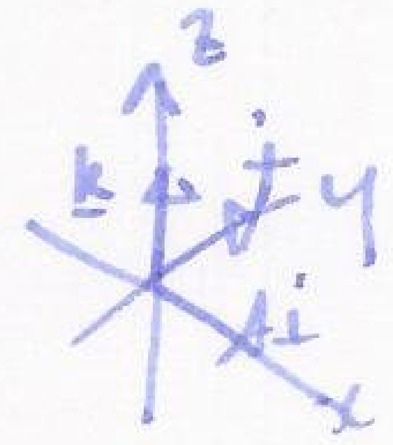
- pictures



- verbal description "go 1 unit in x-direction"

- coordinate descriptions.

- special unit vectors



$$\left.\begin{aligned} \underline{i} &= \langle 1,0,0\rangle \\ \underline{j} &= \langle 0,1,0\rangle \\ \underline{k} &= \langle 0,0,1\rangle \end{aligned}\right\}$$ standard unit vectors in $\mathbb{R}^3$

useful facts

let $\underline{u} = \langle u_1, u_2, u_3\rangle$ $\underline{v} = \langle v_1, v_2, v_3\rangle$ be vectors, c a scalar

- $\underline{u} = \underline{v}$ iff $u_1 = v_1$, $u_2 = v_2$, $u_3 = v_3$.

- if $\underline{v}$ has initial point (x_1, y_1, z_1) and final point (x_2, y_2, z_2) then $\underline{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$.

- length of $\underline{v}$, $\|\underline{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

- unit vector in direction of $\underline{v}$: $\frac{1}{\|\underline{v}\|}\underline{v}$. $(\underline{v} \neq \underline{0})$

- addition : $\underline{u} + \underline{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle$ (componentwise addition).

- scalar multiplication : $c\underline{v} = c\langle v_1, v_2, v_3\rangle = \langle cv_1, cv_2, cv_3\rangle$.

Remark : all this works in any dimension!

vectors in $\mathbb{R}^4$: $\langle a_1, a_2, a_3, a_4\rangle$ and so on...

§11.3 Dot products

We can take the dot product of two vectors and get a scalar.

Def^n $\underline{u} = \langle u_1, u_2 \rangle$ then $\underline{u}.\underline{v} = u_1v_1 + u_2v_2$

$\underline{v} = \langle v_1, v_2 \rangle$

similarly: $\underline{u} = \langle u_1, u_2, u_3 \rangle$ then $\underline{u}.\underline{v} = u_1v_1 + u_2v_2 + u_3v_3$.

$\underline{v} = \langle v_1, v_2, v_3 \rangle$

Important $\underbrace{\underline{u}}_{\text{vector}} . \underbrace{\underline{v}}_{\text{vector.}}$ is a scalar

Useful properties

$\underline{u}, \underline{v}, \underline{w}$ vectors, c scalar

- $\underline{u}.\underline{v} = \underline{v}.\underline{u}$ (commutative)
- $\underline{u}.(\underline{v}+\underline{w}) = \underline{u}.\underline{v} + \underline{u}.\underline{w}$ (distributive)
- $c(\underline{u}.\underline{v}) = (c\underline{u}).\underline{v} = \underline{u}.(c\underline{v})$.
- $\underline{0}.\underline{v} = 0$ (note: $\underline{0}.\underline{v} \neq \underline{0}$!).
- $\underline{v}.\underline{v} = \|\underline{v}\|^2$.

check $\underline{0}.\underline{v} = \langle 0,0,0 \rangle . \langle v_1, v_2, v_3 \rangle = 0$ (number not $\langle 0,0,0 \rangle$!)

check $\underline{v} = \langle v_1, v_2, v_3 \rangle$

$\underline{v}.\underline{v} = \langle v_1, v_2, v_3 \rangle . \langle v_1, v_2, v_3 \rangle = v_1^2 + v_2^2 + v_3^2 = \|\underline{v}\|^2$.