Heegaard splittings and virtual fibers

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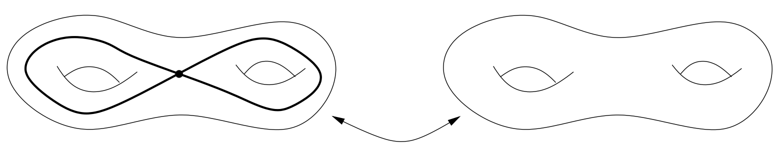
Thm: Let M be a closed hyperbolic 3-manifold, with a sequence of finite covers of bounded Heegaard genus. Then M is virtually fibered.

• hyperbolic: $M = \mathbb{H}^3/\Gamma$, $G < \text{Isom}(\mathbb{H}^3)$ discrete cocompact

• Heegaard splitting: $M = H_1 \cup H_2$, $H_i =$ handlebody:

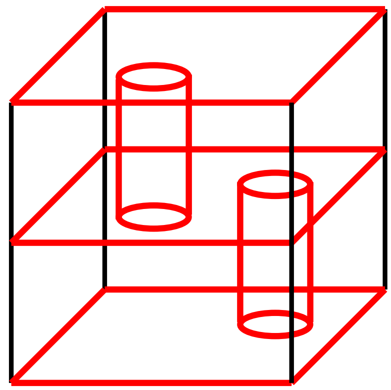


• fibered: $M_\phi = S \times I/\sim$, $(x,1) \sim (\phi(x),0)$

• virtually fibered: some finite cover is fibered

Thm: [Lackenby] as above for regular covers

Cyclic covers of fibered manifolds have bounded Heegaard genus.



Thm: M closed hyperbolic 3-manifold with a sequence of finite covers with

- bounded Scharlemann-Thompson width

- Heegaard gradient $\chi_i/d_i \to 0$

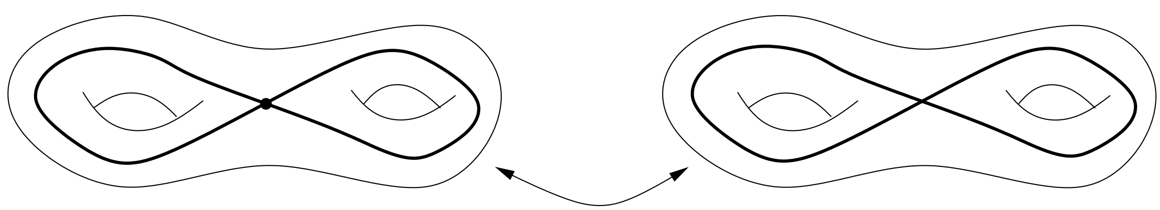
then all but finitely many M_i are fibered over S^1 or I^*

Independently announced by Agol

Lackenby, using [Lubotzky, Sarnack] showed Heegaard genus grows linearly in congruence covers $\Gamma \to PSL(2, \mathbb{F}_q)$ of arithmetic manifolds

[Lubotzky] subgroup growth exponential, proportion of congruence covers $\to 0$ as index $\to \infty$

Proof: Sweepouts: $f : S \times I \to M$,
$f_* : H_3(S \times I, \partial) \to H_3(M, \Gamma)$



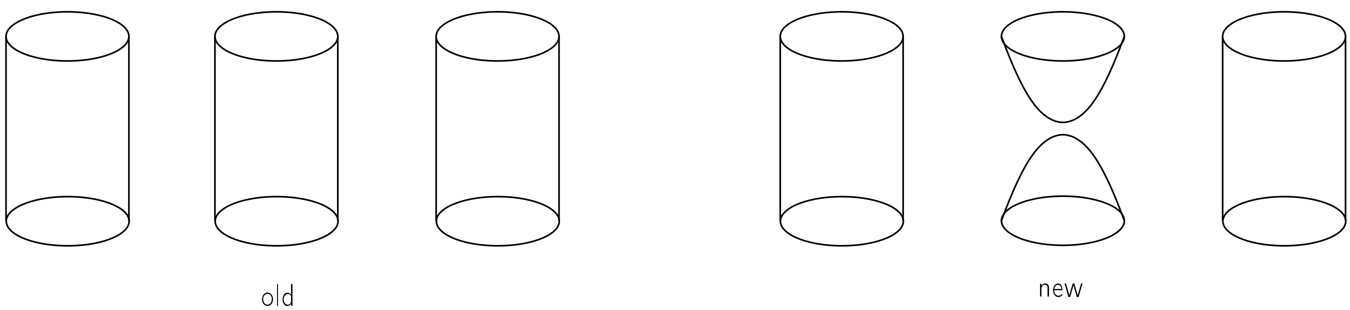
Simplicial sweepouts [Bachman, Cooper, White]:

continuous family of triangulations, with bounded number of triangles

straighten triangles in hyperbolic metric

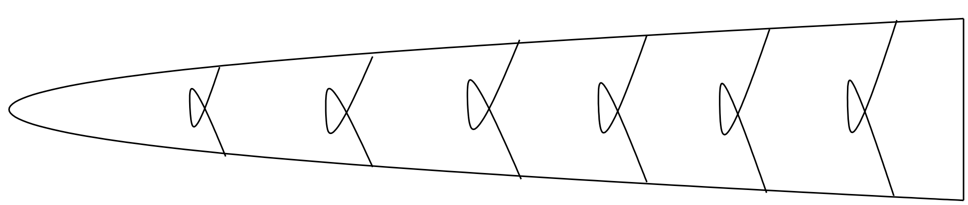
get immersed surfaces with area bound

Generalised sweepouts: $f : \Sigma \to M$ degree 1, $h : \Sigma \to \mathbb{R}$ Morse function



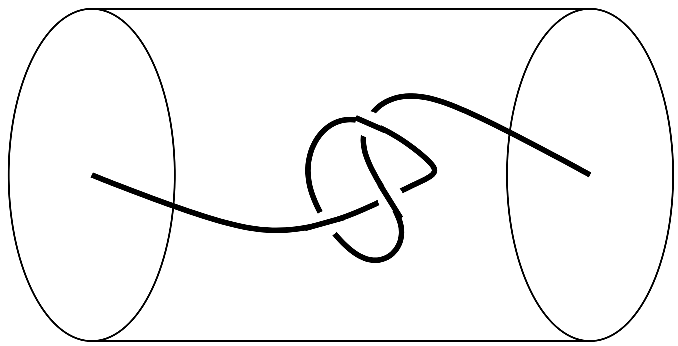
Get immersed surfaces with diameter bound

For M_i of large degree there is a handlebody in M_i with many disjoint nested sweepout surfaces.

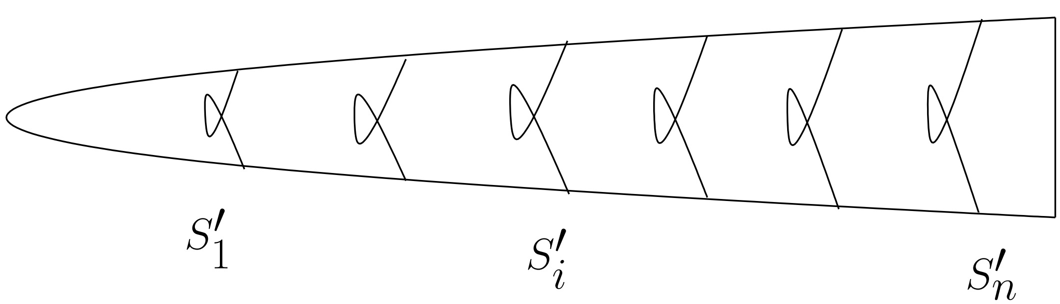


Assume surfaces have the same genus

Nested ⇒ homotopic, use compressing discs

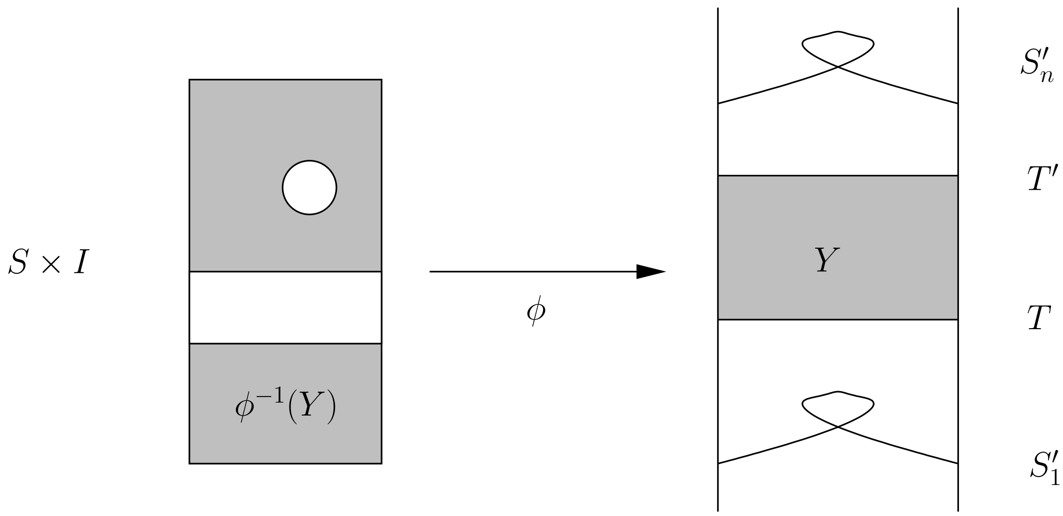


Replace surfaces S_i with S'_i so that the homotopy from S'_n to S'_i is disjoint from S'_j for $j < i$.



[Gabai] Singular norm ⇒ embedded surfaces

Homotopic ⇒ isotopic



Finiteness ⇒ virtual fiber