Random walks on the mapping class group

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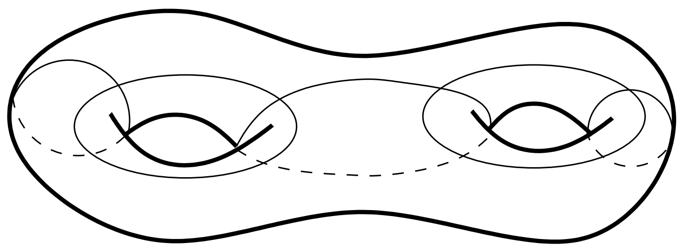
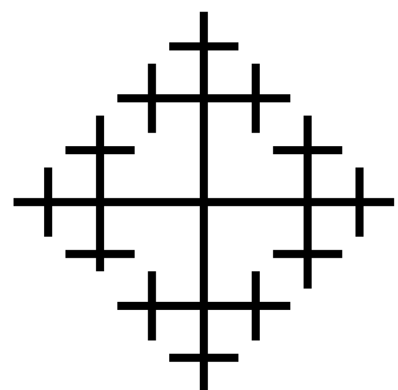
Σ closed orientable surface



Def: $G = \mathrm{MCG}(\Sigma)$
$= \mathrm{Diff}^+(\Sigma)/\mathrm{Diff}_0(\Sigma)$



Let Γ be a Cayley graph for G, consider nearest neighbour random walk on Γ

[Thurston] Classification of elements of G:

- Periodic
- Reducible
- Pseudo-Anosov

Thm: Let w_n be a random walk of length n, then $\mathbb{P}(w_n$ is pseudo-Anosov$) \to 1$ as $n \to \infty$

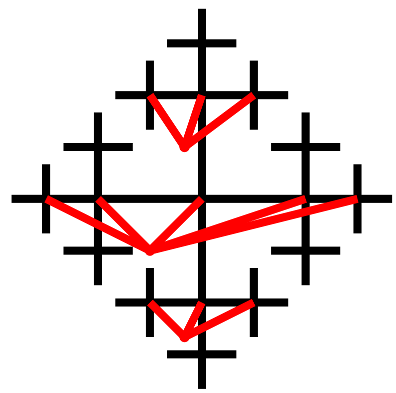
cf [Rivin, math.NT/0604489, Rivin-I. Kapovich]

More generally, pick probability distibution μ on G, Markov chain with $p(x, y) = \mu(x^{-1}y)$

Require:

- μ symmetric, i.e. $\mu(x) = \mu(x^{-1})$
- supp(μ) non-elementary
- supp(μ) not contained in a centralizer

Example: supp(μ) = Torelli group



[Masur-Minsky] G is weakly relatively hyperbolic

i.e. cone off cosets of subgroups H_i to produce relative space $\widehat{\Gamma}$

$\widehat{\Gamma}$ quasi-isometric to complex of curves, δ-hyperbolic

$|x|$ distance in Γ

$|\widehat{x}|$ distance in $\widehat{\Gamma}$

[Klarreich] $\partial\widehat{\Gamma} =$ minimal foliations $\subset \mathcal{PML}$

[Kaimanovich-Masur] $w_n \to \lambda \in \mathcal{PML}$ a.s.

This gives a measure ν on $\mathcal{PML}$,
$\nu(\overline{X}) = 0$ implies X transient

• Centralizers have measure zero

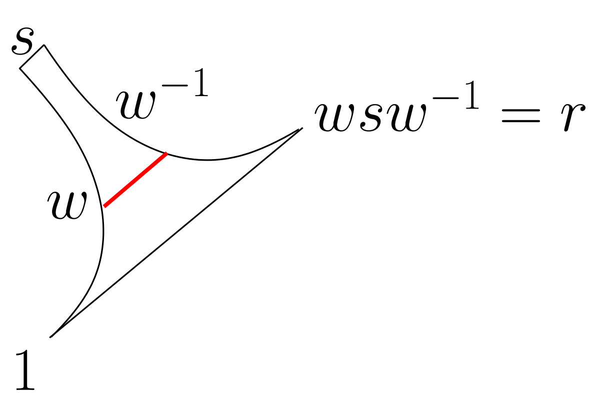
• Relative conjugacy bounds in G

i.e. if a,b conjugate, then $\exists w$ such that $a = wbw^{-1}$, and

$$|\widehat{w}| \leqslant K(|\widehat{a}| + |\widehat{b}|)$$

• Periodic, reducible elements are conjugate to relatively short elements $r = wsw^{-1}$

$\Rightarrow wsw^{-1}$ is quasi-geodesic for $|\widehat{s}| < B$



Let $R =$ reducibles $\cup$ periodics

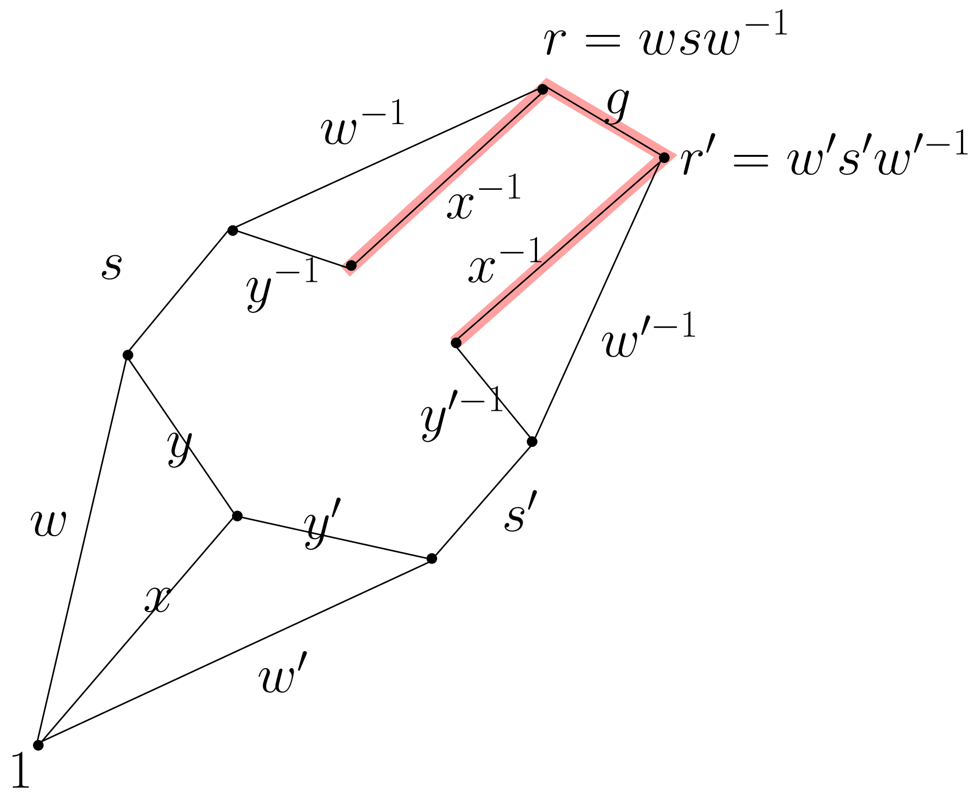
$R_k = k$-close elements of R (close in Γ!)

$R_k = \{r \in R \mid \exists\ r' \in R \setminus r$ with $d_\Gamma(r, r') \leqslant k\}$

Claim: $\nu(\overline{R_k}) = 0$

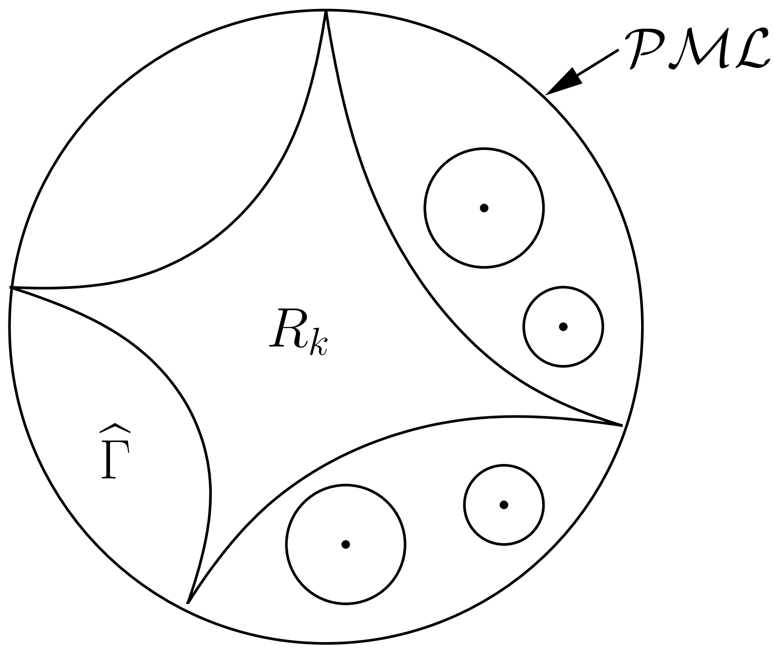
Proof: if $r_n \in R_k$ and $r_n \to \lambda \in \mathcal{PML}$, then there is $r'_n \in R_k \setminus r_n$, with $d_\Gamma(r_n, r'_n) \leqslant k$ and $r'_n \to \lambda$

$B_k \subset \Gamma$ finite, pass to subsequence with $r_n^{-1} r'_n$ constant, i.e. consider $R^g = \{r \in R \ \mid\ rg \in R\}$



quasi-geodesic paths fellow travel,
so $w = xy$, $w' = xy'$ for $|\widehat{y}|, |\widehat{y'}|$ short
$\Rightarrow |\widehat{x^{-1}gx}|$ short

Conjugacy bounds $\Rightarrow x^{-1}gx = zgz^{-1}, |\widehat{z}|$ short
$\Rightarrow g(xz) = (xz)g$, so $w \in N_K(C(g))$
$\Rightarrow \overline{R^g} \subset \overline{C(g)}$



$\mathbb{P}(w_n \in R_k) \to 0$ as $n \to \infty$

$\mathbb{P}(w_n \in R \setminus R_k) \leqslant 1/k$

Random 3-manifolds: use w_n as the gluing map for a Heegaard splitting

Thm: $\mathbb{P}(M(w_n)$ is hyperbolic$) \to 1$ as $n \to \infty$

Previous argument shows translation distance of $w_n \to \infty$

Claim: splitting distance $\to \infty$

[Kerckhoff] Disc set has measure zero

[Masur-Minsky] Disc set is quasiconvex

[Hempel, Kobayashi] Distance $> 2 \Rightarrow$ hyperbolic, assuming geometrization