Casson invariants of random Heegaard splittings

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Mar 2014

Existence result

M closed orientable 3-manifold, integral homology sphere

Casson invariant: $\lambda(M) \in \mathbb{Z}$

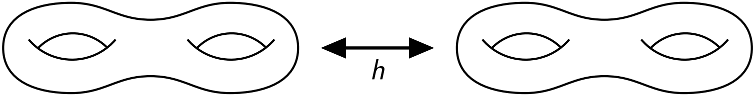
- $\lambda(S^3) = 0$
- $\lambda(\pm 1$ surgery on trefoil$) = \pm 1$
- additive under connect sum $\lambda(M_1 \# M_2) = \lambda(M_1) + \lambda(M_2)$

Thm[LMW]: For any $k \in \mathbb{Z}$, for any $g \in \mathbb{N}_{\geq 2}$, there are infinitely many distinct hyperbolic integral homology spheres M_i with $\lambda(M_i) = k$ and of Heegaard genus g.

Heegaard splittings

Heegaard splitting: decomposition of M^3 into union of two handlebodies

handlebody: regular neighbourood of a graph in $\mathbb{R}^3$, boundary S_g closed orientable surface of genus g



M determined by gluing map up to isotopy: $M(h)$

Heegaard genus(M): minimal genus of any Heegaard splitting

Random Heegaard splittings

Random walks on the mapping class group

$G = MCG(S_g) = Homeo^+(S_g)/\text{isotopy}$.

μ finitely supported probability distribution on G

s_i independent μ-distributed random variables

$$w_n = s_1 s_2 \dots s_n$$

$H = \langle \text{support}(\mu) \rangle$, require: H is a non-elementary subgroup of G
Let h_0 be a gluing map which produces the 3-sphere S^3.

A *random Heegaard splitting* is $M(w_n h_0)$

If $w_n \in \mathcal{T}$ Torelli subgroup $\ker(G \to Sp(2g, \mathbb{Z}))$, then $M(w_n h_0)$ is an integral homology 3-sphere

Let $\mathcal{K} < \mathcal{T}$ be the group generated by Dehn twists in separating curves, the (first) Johnson kernel

[Morita] The map $\lambda \colon \mathcal{K} \to \mathbb{Z}$ given by $f \mapsto \lambda(M(fh_0))$ is a (surjective) homorphism.

If w_n is a symmetric random walk on $H < \mathcal{K}$, then $\lambda(w_n)$ is a symmetric random walk on $\mathbb{Z}$

$$\mathbb{P}(\lambda(M(w_n h_0)) = k) \sim \frac{1}{\sqrt{n}}$$

Splitting distance d_{sp}

[LMW] There is a finitely generated subgroup $H < \mathcal{K}$ such that for any symmetric probability distribution μ generating H, the splitting distance of $M(w_n h_0)$ grows linearly with exponential decay, i.e. there are numbers c, K, L such that

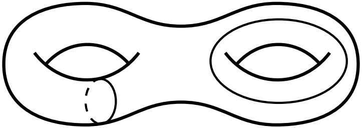
$$\mathbb{P}(d_{sp}(M(w_n h_0)) \leq Ln) \leq Kc^n$$

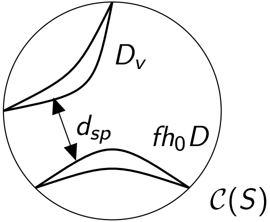
$\mathcal{C}(S)$ complex of curves: simplicial complex
- vertices: isotopy classes of simple closed curves
- simplices: spanned by disjoint simple closed curves

[Masur-Minsky] $\mathcal{C}(S)$ is Gromov hyperbolic

The disc set

Identify S_g with ∂V boundary of handlebody V. The disc set D_V is the set of all simple closed curves in $\mathcal{C}(S)$ which bound discs in V





[Masur-Minsky] D_V is a quasiconvex subset of $\mathcal{C}(S)$

A Heegaard splitting $M(fh_0)$ determines two disc sets D and $fh_0 D$.

Heegaard splitting distance: $d_{sp}(M(fh_0)) = d_{\mathcal{C}(S)}(D, fh_0 D)$

[Kobayashi][Hempel][Perelman] if $d_{sp}(M(fh_0)) > 2$ then $M(fh_0)$ is hyperbolic

[Scharlemann-Tomova] if $d_{sp}(M(fh_0)) > 2g$ then the Heegaard genus of $M(fh_0)$ is equal to g

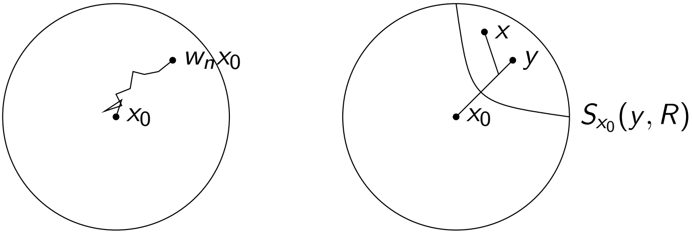
Proof of existence result:

$$\mathbb{P}(\lambda(M(w_n h_0)) = k) \sim \frac{1}{\sqrt{n}}$$

$$\mathbb{P}(d_{sp}(M(w_n h_0)) \leq 2g) \leq Kc^n$$

Linear progress in $\mathcal{C}(S)$:

$$\mathbb{P}(d_{\mathcal{C}(S)}(x_0, w_n x_0) \leq Ln) \leq Kc^n$$



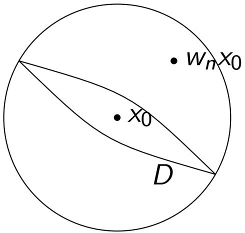
Exponential decay for shadows:

$$S_{x_0}(y, R) = \{x \in \mathcal{C}(S) \mid (x \cdot y)_{x_0} \geq d_{\mathcal{C}(S)}(x_0, y) - R\}$$

$$\mu_n(S_{x_0}(y, R)) \leq Kc^{d_{\mathcal{C}(S)}(x_0,y)-R}$$

Linear progress away from the disc set

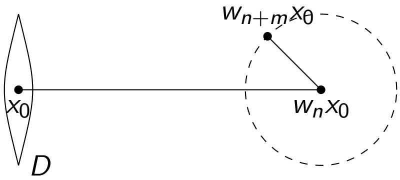
Lemma: $\mathbb{P}(d_{\mathcal{C}(S)}(D, w_n x_0) \leq Ln) \leq Kc^n$



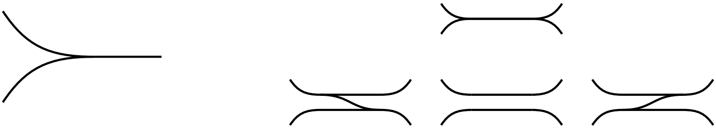
$$X_n = \lfloor d_{\mathcal{C}(S)}(D, w_n x_0)/R \rfloor$$

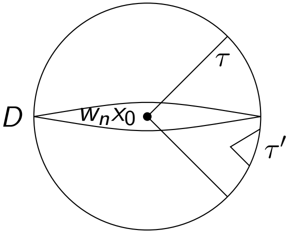
Local estimates:

- X_n large: $\mathbb{P}(X_{n+m} \leq X_n - k) \leq Kc^k$



- X_n small: [Kerckhoff] there is a maximal recurrent traintrack τ such that for any $w_n h_0 D$, τ may be split at most $N(S_g)$ times to τ' disjoint from $w_n h_0 D$





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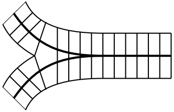
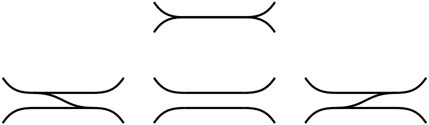
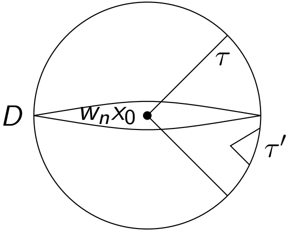
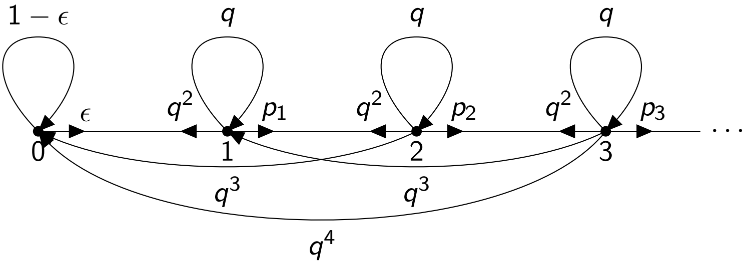






Comparison Markov chain



This Markov chain has linear progress with exponential decay. (Calculate spectral radius)

Stochastic domination implies X_n has exponential progress with linear decay.