Statistics for Teichmüller geodesics

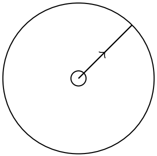
Joseph Maher, with Vaibhav Gadre and Giulio Tiozzo

joseph.maher@csi.cuny.edu

College of Staten Island, CUNY

Mar 2013

Q: What does a "typical" geodesic look like?

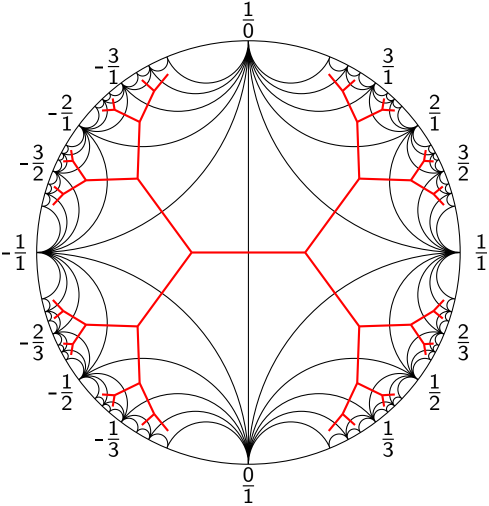


- visual measure/Lebesgue measure (Leb)
- random walks, hitting measure/harmonic measure ν

Often these are *mutually singular*, i.e. there are sets U_1, U_2 such that

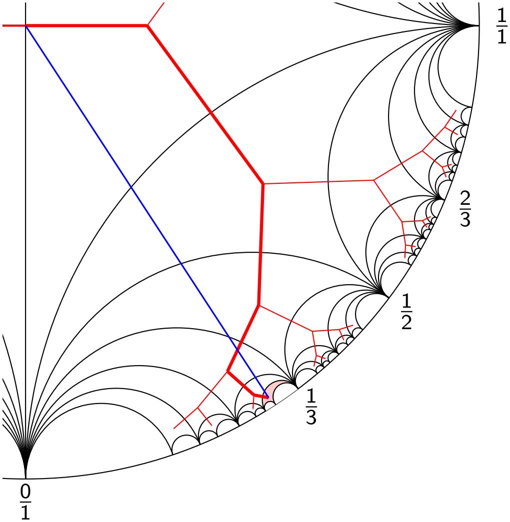
$$\begin{aligned} \mathrm{Leb}(U_1) = 1 \quad \nu(U_1) = 0 \\ \mathrm{Leb}(U_2) = 0 \quad \nu(U_2) = 1 \end{aligned}$$

$PSL(2, \mathbb{Z}) \curvearrowright \mathbb{H}^2$

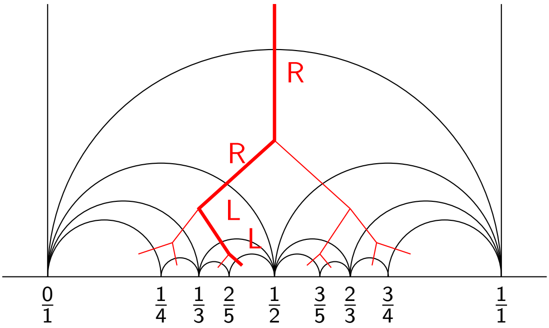


Leb: standard Lebesgue measure on S^1

ν: hitting measure from nearest neighbour random walk on dual tree

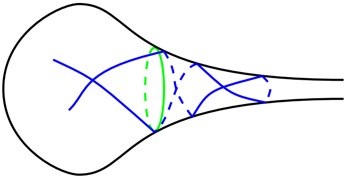


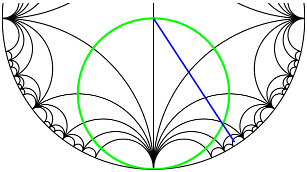
$x \in S^1 \leftrightarrow$ geodesic γ in $\mathbb{H}^2 \leftrightarrow LR$-sequence $\leftrightarrow$ continued fraction



$$\overbrace{RR\ldots R}^{a_1}\overbrace{L\ldots L}^{a_2}\overbrace{R\ldots R}^{a_3}\ldots \leftrightarrow \frac{1}{a_1+\frac{1}{a_2+\ldots}}$$

[Gauss] Leb: $\mathbb{P}(a_i = n) \sim \frac{1}{n^2}$, [Minkowski] $\nu : \mathbb{P}(a_i = n) \sim \frac{1}{2^n}$





$a_i \leftrightarrow$ length of cusp excursions in $\mathrm{SL}(2,\mathbb{R}) \leftrightarrow$ word length increase

word length d_G grows as $\sum_{i=1}^n a_i$, relative length d_{rel} grows as n

$$\text{ratio } \rho = \lim_{t\to\infty} \frac{d_G(1, g_t)}{d_{rel}(1, g_t)} = \frac{1}{n}\sum_{i=1}^{n} a_i = \begin{cases} \infty & \text{Leb} \\ c & \nu \end{cases}$$

Fuchsian groups, $\mathbb{H}^2/\Gamma$ finite volume, non-compact

$$[\text{Gadre-M-Tiozzo}]\ \rho = \lim_{t\to\infty} \frac{d_G(1,g_t)}{d_{rel}(1,g_t)} = \begin{cases} \infty & \text{Leb} \\ c & \nu \end{cases}$$

([Guivarc'h-Le Jan] singularity)

Mapping class groups, $\mathcal{T}(S)/G$

$$[\text{Gadre-M-Tiozzo}]\ \rho = \lim_{t\notin\mathcal{T}_\epsilon\to\infty} \frac{d_G(1,g_t)}{d_{rel}(1,g_t)} = \begin{cases} \infty & \text{Leb} \\ c & \nu \end{cases}$$

([Gadre] sinuglarity)

$\mathcal{T}(S)$ unit (co)-tangent space $\mathcal{Q}(S)$ is unit area quadratic differentials

Leb: Masur-Veech holonomy measure invariant under geodesic flow/$SL(2,\mathbb{R})$

$q \in \mathcal{Q}(S) \leftrightarrow$ flat surface, Teichmüller disc $D_q = SL(2,\mathbb{R}) \cdot q \sim \mathbb{H}^2$

q with metric cylinders of areas bounded below $\leftrightarrow$ horoball in D_q

[Masur] number of such flat cylinders of length $\leq T \sim cT^2$

[Rafi] excursion $\sim$ distance along horoball $\sim$ twist parameter $\sim$ subsurface projection distance

[Masur-Minsky] word length $d_G(1, g_t) \sim \sum_{Y \subseteq S} \lfloor d_Y(1, g_t) \rfloor_A$

ν: [Kesten][Day] $d_G(1, w_n) \sim n$, [M] $d_{rel}(1, w_n) \sim n$

random walk with finite support tracks a geodesic sublinearly, $\frac{1}{n} d(1, g_t) \to 0$ as $t \to \infty$

[Tiozzo] d_n sequence of numbers with $|d_n - d_{n+1}| \leq D$, d_n have asymptotic distribution, then $\frac{1}{n} d_n \to 0$

Examples: $d_n = d_G(1, g_t)$, $d_n = d_{\mathcal{T}}(x_0, g_t x_0)$

Proof: for any $\epsilon \in (0, 1)$ there is M such that $\dfrac{|d_n \geq M|}{n} \to \epsilon$ as $n \to \infty$