Growth rates for stable commutator length

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G group

commutator $[g,h] = ghg^{-1}h^{-1}$

commutator subgroup $G' = [G,G]$

Def: commutator length:

$$\mathrm{cl}(g) = \min\{n \mid g \text{ is a product of } n \text{ commutators }\}$$

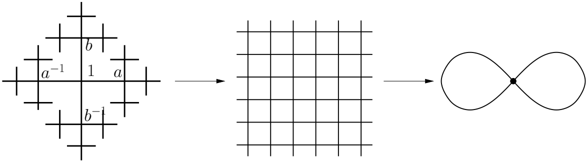
- cl word metric on G' wrt to generating set of all commutators

Def: stable commutator length:

$$\mathrm{scl}(g) = \lim_{n\to\infty} \frac{1}{n} cl(g^n)$$

- scl is translation length of g on G'
- monotonic: $\phi : G \to H$ homomorphism, $scl(\phi(g)) \leqslant scl(g)$

Example: $F_2 = \langle a, b \mid \rangle$



$g = [a, b] = aba^{-1}b^{-1}$

$\mathrm{cl}(g) = 1, \quad \mathrm{cl}(g^2) = 2, \quad cl(g^3) = 2$:

$[a, b]^3 = [aba^{-1}, b^{-1}aba^{-2}][b^{-1}ab, b^2]$

so $\mathrm{scl}([a, b]) \leqslant 2/3$

Topological version:

- Group G $\leftrightarrow$ space X, $\pi_1 X = G$
- $g \in G \leftrightarrow$ homotopy class of loop γ in X
- conjugacy class of $g \leftrightarrow$ free homotopy class of γ
- $g \in G' \leftrightarrow [\gamma] = 0 \in H_1(X) \leftrightarrow \gamma$ bounds a surface S
- $\text{cl}(g) \leftrightarrow \inf\{\text{genus}(S) \mid \partial S \to \gamma \text{ degree } 1\}$
- $\text{scl}(g) \leftrightarrow \inf\{\frac{1}{n}\text{genus}(S) \mid \partial S \to \gamma \text{ degree } n\}$

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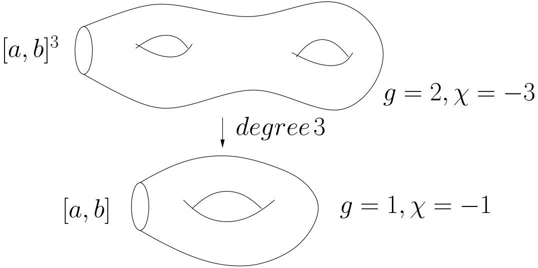

genus not multiplicative under covers, use Euler characteristic:

Def: $\mathrm{scl}(g) = \inf\{\frac{-1}{2n}\chi(S) \mid \partial S \to \gamma \text{ degree } n\}$

An *extremal* surface realizes scl, in fact $\mathrm{scl}([a,b]) = \frac{1}{2}$

Generic elements in G:

- random word of length n

  S_n all elements of length n, uniform measure
  B_n all elements of length $\leqslant n$, uniform measure

- random walk of length n

  uniform measure on symmetric generating set A
  $w_n = s_1 s_2 \dots s_n$, s_i chosen from A independently

Thm[Calegari-M]: G Gromov hyperbolic, there are constants c_1, c_2 such that

$$\mathbb{P}\left(c_1 n / \log n \leqslant \mathrm{scl}(g_n) \leqslant c_2 n / \log n\right) \to 1 \text{ as } n \to \infty$$

[Calegari-Walker] F_k, $c = \log(2k-1)/6$

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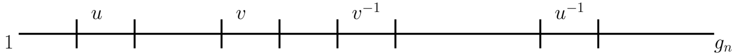
$\phi : G \to H$ homorphism, then $\phi(w_n)$ random walk on H, so $\mathrm{scl}(w_n) \leqslant c_2 n / \log n$ for *any* group.

F_2 word length version

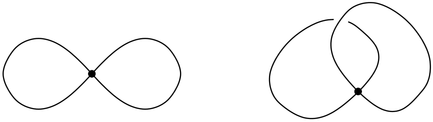
- estimates: $\mathbb{P}(g_n \in F_2') \sim \frac{1}{n}$, [Sharp][Rivin]
- corrections: $g \in F_2$, there is h such that $gh \in F_2'$, with $|h| \leqslant |\alpha(g)| + 4$, $\alpha(g)$ abelianization of g
- reorderings: $uvw = vu[u^{-1}, v^{-1}]w = vuw[w^{-1}u^{-1}w, w^{-1}v^{-1}w]$



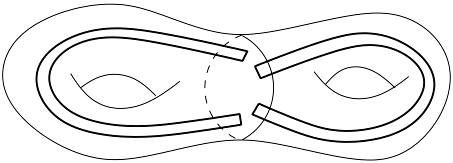
Matching at size $\log n$ means $\mathrm{scl}(g_n) \leqslant n/\log n$

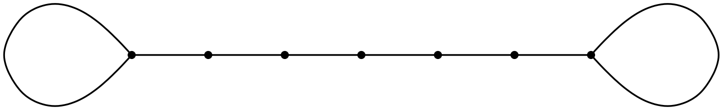
Thm[Culler]: If $g^n \in F_2'$ (cyclically reduced) then there is a fatgraph S with $\chi(S) = 1 - \mathrm{cl}(g^n)$ and ∂S labelled by g^n

Def: A *fatgraph* is a graph with a cyclic ordering at each vertex.



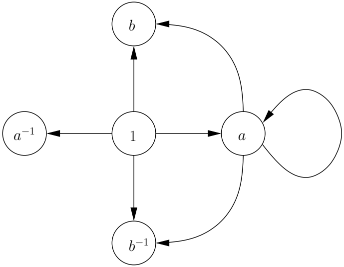
fatgraphs $\leftrightarrow$ surfaces





$scl(g_n) \leqslant n/k(n)$ implies matching at scale $k(n)$

lower bound: random words in F_2 generated by a Markov process.



upper bound: words of length $m = c_3 \log n$, derived Markov chain with 3^m states

For each word v of length m, number of v's = number of v^{-1}'s up to error of size $\sqrt{n}$, so $\text{scl}(g) \leqslant n/m + \sqrt{n}$

mixing [Kahane]

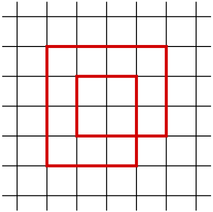
Hyperbolic groups, word length

- fat graphs in handlebodies $\leftrightarrow$ pleated surfaces in Mineyev's flow space
- unique geodesics in F_2 $\leftrightarrow$ combing for regular language
- F_2 Markov chain $\leftrightarrow$ Markov chain on automata
- derived Markov chain $\leftrightarrow$ derived Markov chain on automata

scl dual to quasimorphisms

Def: $\phi\colon G \to R$ such that $|\phi(gh) - \phi(g) - \phi(h)| \leqslant D(\phi)$

Thm[Bavard Duality]



$$\mathrm{scl}(g) = \frac{1}{2} \sup_{\phi} \frac{|\phi(g)|}{D(\phi)}$$

Example: $\phi : F_2 \to \mathbb{Z}$, winding number

$\phi(g) =$ (number of left turns - number of right turns)/4

Claim: $D(\phi) = 1$, so $\mathrm{scl}([a, b]) \geqslant \frac{1}{2}$