Heegaard splittings and virtual fibers

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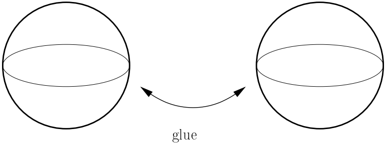
May 2008

Theorem: Let M be a closed hyperbolic 3-manifold, with a sequence of finite covers of bounded Heegaard genus. Then M is virtually fibered.

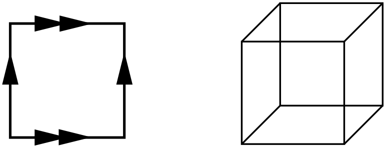
• 3-manifold: topological space locally homeomorphic to ordinary Euclidean space $\mathbb{R}^3$. Examples:

- 3-ball B^3 (has boundary)

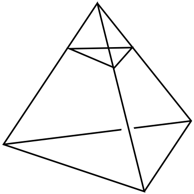


- 3-sphere $B^3 \cup B^3$ (closed = compact, no boundary)

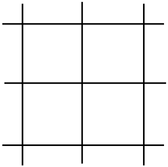
- $S^1 \times S^1 \times S^1$, 3-torus

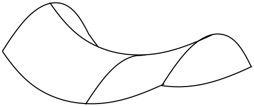


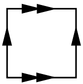
- glue some tetrahedra together such that links of vertices are 2-spheres (every three manifold can be obtained this way)

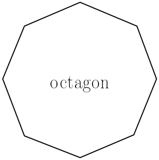


- hyperbolic: $M = \mathbb{H}^3/\Gamma$, $\Gamma < \mathrm{Isom}(\mathbb{H}^3)$ discrete cocompact.



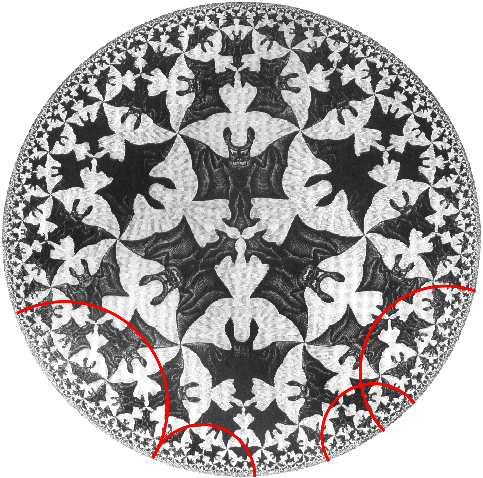






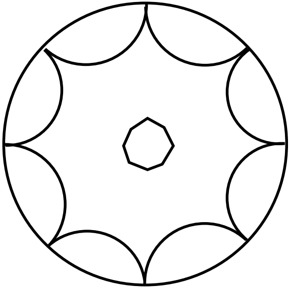
Identifying opposite sides of an octagon gives a genus two surface. This has a metric of negative curvature.

The Poincaré disc model:



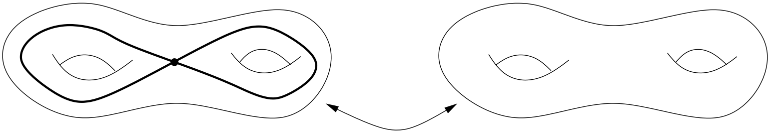
All angels/devils are the same size in the hyperbolic metric.
Straight lines are circles perpendicular to the boundary.

Regular octagons may have arbitrarily small angles:



so the hyperbolic plane may be tiled by regular octagons.
In three dimensions, the Poincaré model is a ball formed by rotating the Poincaré disc about a diameter. There are various tilings by polyhedra which give hyperbolic 3-manifolds.

• Heegaard splitting: $M = H_1 \cup H_2$, H_i = handlebody = regular neighbourhood of a graph in $\mathbb{R}^3$:

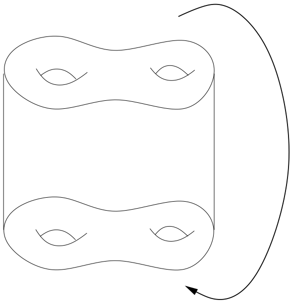


Graph is called a *spine* for the handlebody.

Every 3-manifold has a Heegaard splitting, in fact many Heegaard splittings.

We will normally be interested in a choice of minimal genus Heegaard splitting for the manifold.

• fibered 3-manifold: S closed surface, $\phi : S \to S$, take $S \times I$ and glue top to bottom by ϕ. $M_\phi = S \times I / \sim$, $(x, 1) \sim (\phi(x), 0)$



Fact: if the gluing map is not periodic or reducible, then the resulting fibered manifold is hyperbolic.

reducible = fixes a disjoint collection of simple closed curves

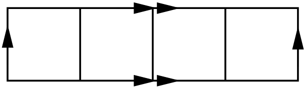
• virtually fibered: some finite cover is fibered

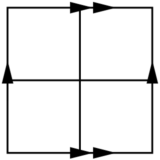
Conjecture [Thurston]: every hyperbolic 3-manifold is virtually fibered

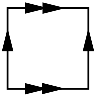
Finite covers: $\widetilde{M} \to M$ local homeomorphism, globally of degree d

Finite covers $\leftrightarrow$ finite index subgroups

Examples:







[Lubotzky] number of subgroups grows exponentially in d, proportion of congruence covers $\to 0$ as index $\to \infty$

Congruence covers:

$\text{Isom}(\mathbb{H}^3) = PSL(2, \mathbb{C})$, so $M = \mathbb{H}^3/\Gamma$, Γ discrete cocompact subgroup.

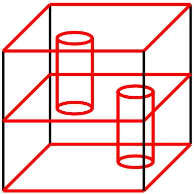
In fact $\Gamma \subset PSL(2, A)$, A some algebraic field, so can reduce mod p, $PSL(2, A) \to PSL(2, \mathbb{F}_p)$.

This gives infinitely many covers.

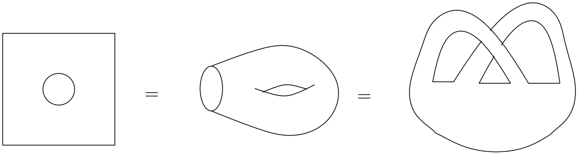
Let M_i be a finite cover of M of degree d_i. The pre-image of a Heegaard surface of M in M_i is a Heegaard surface for M_i, but there may be Heegaard splittings of lower genus. Let χ_i be the Euler characteristic of the minimal genus Heegaard surface of M_i.

Definition: the *Heegaard gradient* of a collection of covers M_i is $\liminf \chi_i / d_i$.

Example: cyclic covers of fibered manifolds have bounded Heegaard genus.



Red surface = two copies of a fiber tubed together. Red surface is a Heegaard splitting surface.
Complementary regions to red surface are (surface with boundary) $\times I$ = handlebody.



Theorem: [Lackenby] Let M be a 3-manifold with a sequence of regular covers M_i of degree d_i, with Heegaard surfaces of Euler characteristic χ_i. If $\chi_i/\sqrt[4]{d_i} \to 0$ then M is virtually fibered.

Lackenby, using [Lubotzky, Sarnack] showed Heegaard genus grows linearly in congruence covers $\Gamma \to PSL(2, \mathbb{F}_q)$ of arithmetic manifolds. [Long-Lubotzky-Reid] general case.

[Ichihara] A Seifert fibered 3-manifold has zero Heegaard gradient iff it is virtually fibered.

Theorem: M closed hyperbolic 3-manifold with a sequence of finite covers with

- bounded Scharlemann-Thompson width
- Heegaard gradient $\chi_i/d_i \to 0$

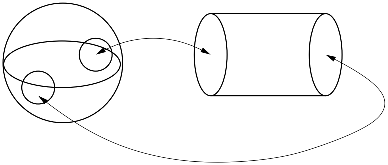
then all but finitely many M_i are fibered over S^1 or I^*

Independently announced by Agol

• Scharlemann-Thompson width:

Heegaard splitting = handle decomposition

0-handle B^3
1-handle $B^1 \times B^2$
2-handle $B^2 \times B^1$
3-handle B^3



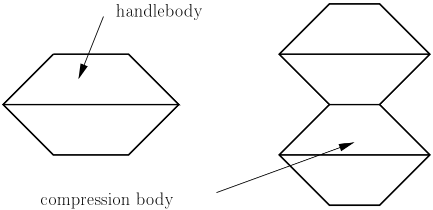
first handlebody = 0-handle $\cup$ 1-handles
second handlebody = 2-handles $\cup$ 3-handle

Think of the handles as being glued on in some order.

May be able to re-arrange the handles, i.e. add some two handles before adding all of the 1-handles. Width at handle t is (number of 1-handles) - (number of 2-handles). Width of splitting is max width over all handles.

Note this differs from original definition which involved writing down the width for each handle addition in descending order and ordering these lexicographically.

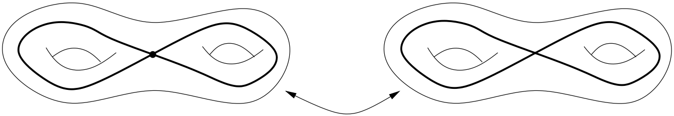
Splitting of genus g has width at most g.



Compression body: take $S \times I$ and glue on 2-handles to one side only (lower side), cap off 2-spheres with 3-balls. Call upper side disjoint from 2-handles ∂_+, other boundary components ∂_-.

Proof (of main theorem):

Sweepouts: $f : S \times I \to M$,
$f_* : H_3(S \times I, \partial) \to H_3(M, \text{spines})$, isomorphism



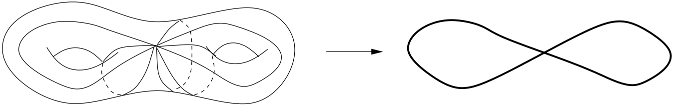
Simplicial sweepouts [Bachman, Cooper, White] [Canary, Thurston, et al]:

A sweepout such that $S \times t$ has a triangulation, which varies continuously with t, and the image of each triangle is a geodesic triangle (may be immersed, degenerate).

Assume spines have one vertex. Homotope spines to have vertex at same point, and homotope edges to be geodesic arcs.

They are not loops as homotoping rel fixed basepoint.

Triangulate $S \times 0$ with one-vertex triangulation such that each edge is mapped either to the vertex, or exactly once around a single edge of spine of H_1.

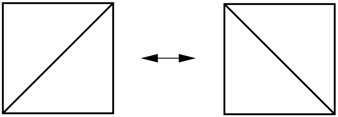


Triangulate $S \times 1$ such that each edge is mapped either to the vertex, or exactly once around a single edge of spine of H_2.

Triangulations of S may be very different.

Flip complex:

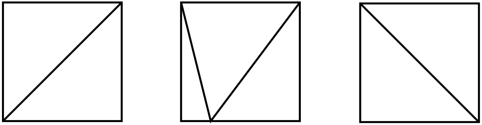
- vertices: one-vertex triangulations of S up to isotopy
- edges: connect two triangulations if they are connected by a flip



[Hatcher-Thurston] flip complex is connected

Choose a (shortest) path of flips connecting the triangulation of $S \times 0$ to the triangulation of $S \times 1$.

Construct continuous family of triangulations, with bounded number of triangles.



S_0 S_t S_1

Straighten triangles in hyperbolic metric by homotopies fixing the vertex of the spine.

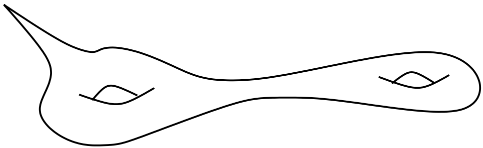
$\overline{f} : S \times I \to M$, each $S \times t$ triangulated, and image of each triangle is a geodesic triangle.

Image is negatively curved away from vertex of spine, as extra vertex has angle $\geqslant 2\pi$.

Total number of triangles bounded (at most $2g$).

Area of each hyperbolic triangle is at most π.

Get sweepout by immersed surfaces with area bound, but may have large diameter

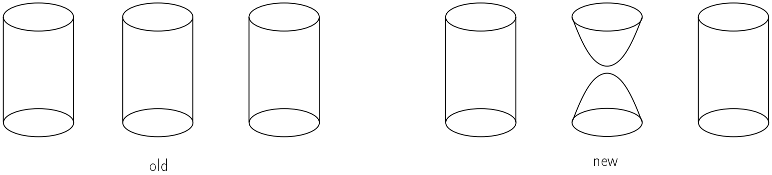


Want: sweepout by surfaces of bounded diameter.

Definition: Generalised sweepout.

Σ 3-manifold with boundary
$f : (\Sigma, \partial\Sigma) \to (M, \text{spines})$ degree 1
$h : \Sigma \to \mathbb{R}$ Morse function

Apply local modifications to sweepout surfaces with large diameter:



Cut out annulus $\times I$ ($=$ solid torus, with particular height function) on the right and replace it with the surfaces on the left ($=$ solid torus with particular height function).

Local move doesn't change degree.

Metric version:

Negatively curved surface has large diameter $\Rightarrow$ there is a short curve.

Injectivity radius of M = half length of shortest essential curve = ϵ say. Injectivity radius of cover $M_i \geqslant$ injectivity radius of M.

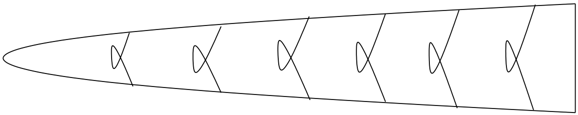
So if a simple closed curve in S_t has length $\leqslant \epsilon$, then it is inessential in M_i.

Choose annular neighbourhoods of short curves which are disjoint.
Key points:
- universal cover of $S\backslash$ basepoint is CAT(-1)
- annuli vary continuously
- the components of the complement have bounded diameter

Get generalised sweepout by immersed surfaces with diameter bound and genus bound.

Volume of cover $= d \times$(volume of M), and large volume implies large diameter. In fact as Heegaard gradient $\chi_i/d_i \to 0$, there is a sequence of compression bodies in the M_i whose volume becomes arbitrarily large.

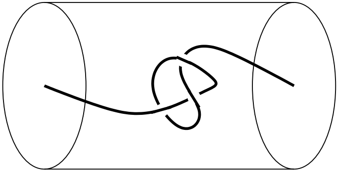
For M_i of large degree there is a compression body in M_i with many disjoint nested sweepout surfaces.



Nested: even though the sweepout surfaces are immersed, it still makes sense to say one surface separates another from the boundary of the handlebody, by taking algebraic intersection number.

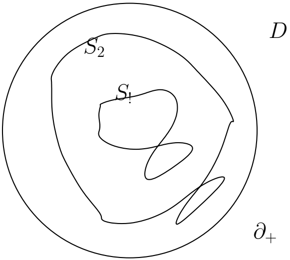
Assume surfaces have the same genus.

We will now show nested $\Rightarrow$ homotopic

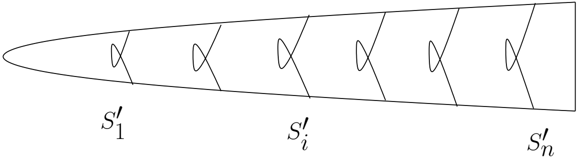


Lemma: Let S_1 and S_2 be nested surfaces in a compression body obtained by surgering ∂_+ along collections of discs Δ_1 and Δ_2, and assume S_2 separates S_1 from ∂_+. Then we may choose the compressing discs such that $\Delta_2 \subset \Delta_1$.

Proof: Suppose D is not a compressing disc for ∂_+ that does not lie in any family of compressing discs Δ_1 which may be used to produce S_1. Then $S_1 \cap D \neq \varnothing$, so there is at least one simple closed curve of intersection. But S_2 separates S_1 from ∂_+, so there must also be intersections between D and S_2, so D is not a compressing disc for S_2. $\square$



Replace surfaces S_i with S_i' so that the homotopy from S_n' to S_i' is disjoint from S_j' for $j < i$.



[Gabai] Singular norm $\Rightarrow$ embedded surfaces

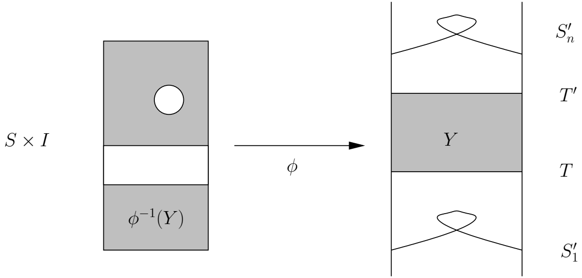
We can replace the immersed surfaces with embedded surfaces in the same homology class, and the new surfaces will be contained in a regular neighbourhood of the original immersed surface, and the genus will be no larger.

Homotopic ⇒ isotopic

Take $f : S \times I$ to be the homotopy - we can make it have bounded diameter by construction above.

Change collection of surfaces if necessary such that homotopy from S'_n to S'_k does not hit S'_j for $j < k$.

Let T and T' be embedded surfaces, which together bound Y in the compression body.



Consider the pre-image of Y in the homotopy from S_1' to S_n', which we may assume is degree one onto Y.

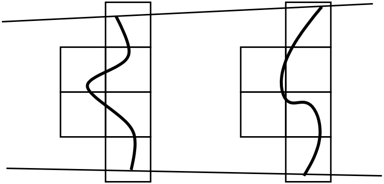
If the boundary components of the pre-image in $S \times I$ are incompressible, we may change the homotopy to compress them, and if there are any S^2 components, again we may change the homotopy to remove them.

We may then make the pre-images horizontal in $S \times I$.

As Y has incompressible boundaries, which are homotopic, and Y is homotopy equivalent to a surface, Waldhausen implies Y is a product.

Finiteness $\Rightarrow$ virtual fiber

A choice of fundamental domain for M gives a tiling of any cover M_i.



Each parallel surface T_i is contained in finitely many fundamental domains. As there are only finitely many ways of gluing finitely many fundamental domains together, if there are enough T_i, there are two that hit the same pattern of fundamental domains, so we may cut the manifold M_i along two of these collections of fundamental domains and reglue to obtain a fibered manifold.