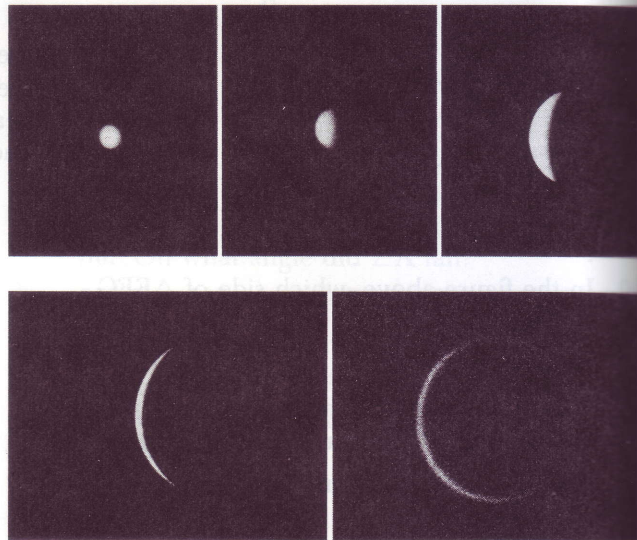
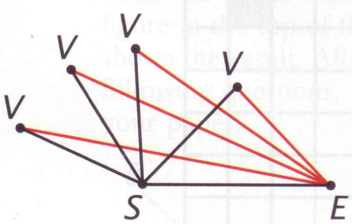


LESSON 4

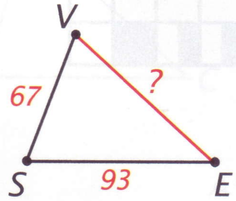
The Triangle Inequality Theorem

Venus is an interesting planet. Sometimes seen as the "morning star" and sometimes as the "evening star," it is often brighter than any object in the sky except the sun or moon.

The series of photographs above at the same scale show something first noticed by Galileo through his telescope. He saw that Venus has phases, like the moon. But, unlike the moon, Venus at its full phase appears only about a sixth as large as it does at its new phase. Galileo reasoned that, when Venus is in its full phase, it must be much farther from Earth than when it is in its new phase. Let's see how this could be so.



In the figure at the left, points S and E represent the sun and Earth, and the points labeled V represent some positions of Venus. In each sun-Earth-Venus triangle, SE and SV stay about the same, but the length VE changes a lot.



The average distance from the sun to Earth, SE, is 93 million miles; the average distance from the sun to Venus, SV, is 67 million miles. In the sun-Earth-Venus triangle at the left, what can we conclude about the length VE? The Triangle Inequality Theorem tells us that the sum of the lengths of any two sides of a triangle is greater than the length of the third side. From this theorem, we can write the inequalities

$$67 + 93 > VE,\ 67 + VE > 93, \text{ and } 93 + VE > 67.$$

The last of these inequalities doesn't tell us anything about VE, because $93 > 67$. From the other two, we can conclude that

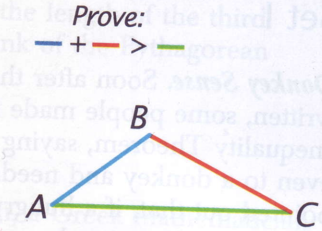
$$160 > VE \text{ and } VE > 26.$$

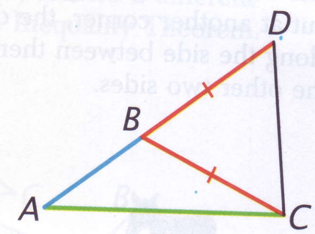
At its farthest, Venus is 160 million miles from Earth, more than six times the distance at its nearest, 26 million miles. This fact explains why the apparent diameter of Venus in its full phase is about a sixth of that in its new phase.

Our proof of the Triangle Inequality Theorem follows Euclid's. Its ingenuity helps explain why the *Elements* is such a significant book and why it is available today, not only in bookstores but also on the Internet!

Theorem 15. The Triangle Inequality Theorem
The sum of any two sides of a triangle is greater than the third side.

Given: ABC is a triangle.
Prove: AB + BC > AC.*

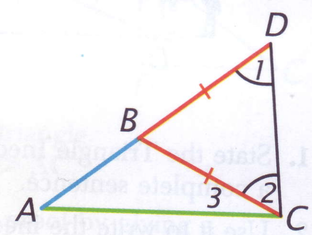


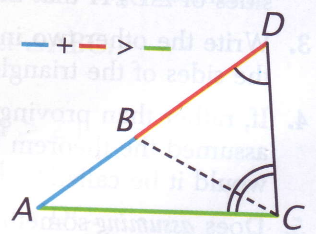


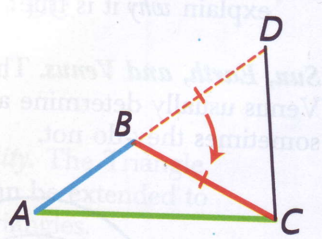
Before reading the proof, see if you can figure out the plan from the figures at the right.

Proof

Statements	*Reasons*
1. ABC is a triangle.	Given.
2. Draw line AB.	Two points determine a line.
3. Choose D beyond B on line AB so that BD = BC.	The Ruler Postulate.
4. Draw CD.	Two points determine a line.
5. $\angle 1 = \angle 2$.	If two sides of a triangle are equal, the angles opposite them are equal.
6. $\angle ACD = \angle 2 + \angle 3$.	Betweenness of Rays Theorem (CA-CB-CD).
7. $\angle ACD > \angle 2$.	"Whole Greater than Part" Theorem.
8. $\angle ACD > \angle 1$.	Substitution (steps 5 and 7).
9. In $\triangle ACD$, AD > AC.	If two angles of a triangle are unequal, the sides opposite them are unequal in the same order.
10. AB + BD = AD.	Betweenness of Points Theorem (A-B-D).
11. AB + BD > AC.	Substitution (steps 9 and 10).
12. AB + BC > AC.	Substitution (steps 3 and 11).







*Also, AC + CB > AB and BA + AC > BC. These inequalities can be proved in the same way.