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Quasi-tree expansion for the Bollobás–Riordan–Tutte polynomial

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Abstract

Bollobás and Riordan introduced a three-variable polynomial extending the Tutte polynomial to oriented ribbon graphs, which are multi-graphs embedded in oriented surfaces, such that complementary regions (faces) are disks. A quasi-tree of a ribbon graph is a spanning subgraph with one face, which is described by an ordered chord diagram. By generalizing Tutte's concept of activity to quasi-trees, we prove a quasi-tree expansion of the Bollobás–Riordan–Tutte polynomial.

1. *Introduction*

An *oriented ribbon graph* is a multi-graph (loops and multiple edges allowed) that is embedded in an oriented surface, such that its complement in the surface is a union of 2-cells. The embedding determines a cyclic order on the edges at every vertex. Terms for the same or closely related objects include: combinatorial maps, fat graphs, cyclic graphs, graphs with rotation systems and dessins d'enfant (see [**2**, **10**] and references therein).

The Tutte polynomial is a fundamental and ubiquitous invariant of graphs. Bollobás and Riordan [**2**] extended the Tutte polynomial to an invariant of oriented ribbon graphs in a way that takes into account the topology of the ribbon graph. In [**3**], they generalized it to a four-variable invariant of non-orientable ribbon graphs. We only consider the Bollobás–Riordan–Tutte polynomial for the orientable case, and henceforth all ribbon graphs will be oriented.

The Tutte polynomial can be defined by a state sum over all subgraphs, by contraction–deletion operations, and by a spanning tree expansion (see [**1**] for a detailed introduction).† Tutte's original definition in [**11**] was the spanning tree expansion, discussed below, which relies on the concept of *activity* of edges with respect to a spanning tree. In [**2**, **3**], the Bollobás–Riordan–Tutte polynomial was shown to satisfy many essential properties of the Tutte polynomial, including a spanning tree expansion using Tutte's activities.

For planar graphs, a spanning tree is a spanning subgraph whose regular neighborhood has one boundary component. For ribbon graphs, the analog of a spanning tree is a *quasi-tree*, which is a spanning subgraph with one face, introduced in [**6**]. Just as the spanning trees of a graph determine many of its important properties, topological properties of a ribbon graph are determined by the set of its quasi-trees. A natural question is whether the Bollobás–Riordan–Tutte polynomial has a quasi-tree expansion analogous to the spanning tree expansion for the Tutte polynomial.

In Section 2, we extend Tutte's concept of activity (with respect to a spanning tree) to *activity with respect to a quasi-tree* by expressing the quasi-tree as an ordered chord diagram. For a genus 0 ribbon graph, spanning trees and quasi-trees coincide, and the two notions of

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†The rank polynomial, formulated independently by Whitney [**12**], equals the Tutte polynomial after rescaling.

activity are the same. However, for ribbon graphs of higher genus, spanning trees are a proper subset of quasi-trees, and the two definitions of activity are quite distinct (see Remark 1 and Section 6).

In Section 3, we give an expansion of the Bollobás–Riordan–Tutte polynomial over quasi-trees. Each term in the expansion is determined by a particular quasi-tree as a product of factors with a topological meaning. In the genus 0 case, we recover Tutte's original spanning tree expansion. In general, our expansion is different from the spanning tree expansion given in [3]. For example, in the case of one-vertex ribbon graphs, the spanning tree expansion is the same as the expansion over all subgraphs, but the quasi-tree expansion has fewer terms (see Remark 2). In addition, we show that a specialization of the Bollobás–Riordan–Tutte polynomial gives the number of quasi-trees of every genus.

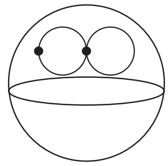
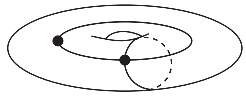
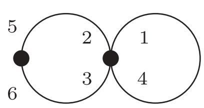
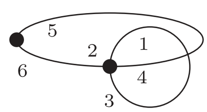
Together, Sections 4 and 5 prove the main theorem, Theorem 2. In Section 6, we compute the quasi-tree expansion for an example.

2. *Activities with respect to a quasi-tree*

A ribbon graph $\mathbb{G}$ can be considered both as a geometric and as a combinatorial object. Starting from the combinatorial definition, let $(\sigma_0, \sigma_1, \sigma_2)$ be permutations of $\{1, \dots, 2n\}$, such that σ_1 is a fixed-point free involution and $\sigma_0\sigma_1\sigma_2 = 1$. We define the orbits of σ_0 to be the vertex set $V(\mathbb{G})$, the orbits of σ_1 to be the edge set $E(\mathbb{G})$ and the orbits of σ_2 to be the face set $F(\mathbb{G})$. Let $v(\mathbb{G})$, $e(\mathbb{G})$ and $f(\mathbb{G})$ be the numbers of vertices, edges and faces of $\mathbb{G}$. The preceding data determine an embedding of $\mathbb{G}$ on a closed orientable surface, denoted by $S(\mathbb{G})$, as a cell complex. The set $\{1, \dots, 2n\}$ can be identified with the directed edges (or half-edges) of $\mathbb{G}$. Thus, $\mathbb{G}$ is connected if and only if the group generated by $\sigma_0, \sigma_1, \sigma_2$ acts transitively on $\{1, \dots, 2n\}$. The genus of $S(\mathbb{G})$ is called the genus of $\mathbb{G}$, $g(\mathbb{G})$. If $\mathbb{G}$ has $k(\mathbb{G})$ components, $2g(\mathbb{G}) = 2k(\mathbb{G}) - v(\mathbb{G}) + e(\mathbb{G}) - f(\mathbb{G}) = k(\mathbb{G}) + n(\mathbb{G}) - f(\mathbb{G})$, where $n(\mathbb{G}) = e(\mathbb{G}) - v(\mathbb{G}) + k(\mathbb{G})$ denotes the nullity of $\mathbb{G}$. Henceforth, we assume that $\mathbb{G}$ is a connected ribbon graph. See Table 1 for an example of distinct ribbon graphs with the same underlying graph.

Any subgraph H of the underlying graph G of $\mathbb{G}$ determines a ribbon subgraph $\mathbb{H}$ of $\mathbb{G}$ with underlying graph H. We can construct its embedding surface $S(\mathbb{H})$ as follows. A regular neighborhood of $\mathbb{H}$ can be constructed on the surface $S(\mathbb{G})$ by gluing disks at each vertex and rectangular bands whose midlines are the edges of $\mathbb{H}$. Let $\gamma_{\mathbb{H}}$ be the union of simple closed curves

TABLE 1. *Ribbon graphs described as graphs on surfaces and as permutations.*

	
	
$\sigma_0 = (1234)(56)$ $\sigma_1 = (14)(25)(36)$ $\sigma_2 = (246)(35)$	$\sigma_0 = (1234)(56)$ $\sigma_1 = (13)(26)(45)$ $\sigma_2 = (152364)$

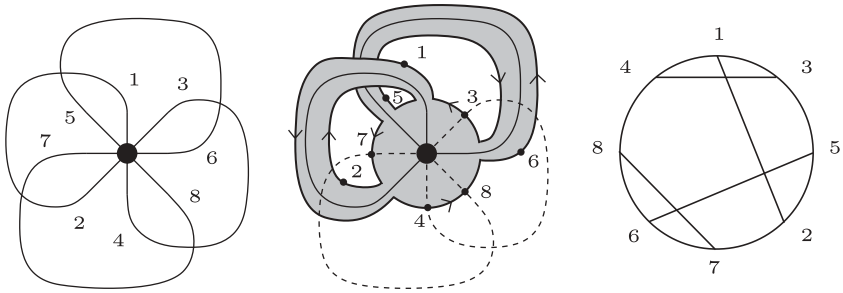


FIGURE 1. *Ribbon Graph* $\mathbb{G}$*, quasi-tree* $\mathbb{Q} = (12)(56)$ *with curve* $\gamma_{\mathbb{Q}}$*, chord diagram* $C_{\mathbb{Q}}$*.*

that bound such a regular neighborhood of $\mathbb{H}$ on $S(\mathbb{G})$. By attaching a disk to every boundary component of this regular neighborhood, we construct $S(\mathbb{H})$, whose genus $g(\mathbb{H})$ may be smaller than $g(\mathbb{G})$. By definition, the faces $F(\mathbb{H})$ are the complementary regions of $\mathbb{H}$ on $S(\mathbb{H})$. Thus, the components of $\gamma_{\mathbb{H}}$ correspond exactly to the faces $F(\mathbb{H})$. So if $|\gamma_{\mathbb{H}}|$ denotes the number of its components, $f(\mathbb{H}) = |\gamma_{\mathbb{H}}|$. In particular, $f(\mathbb{H}) \geqslant k(\mathbb{H})$. Note that ribbon subgraphs $\mathbb{H} \subseteq \mathbb{G}$ may be disconnected. Also note that an isolated vertex $\mathbb{H}$ cannot be represented by $(\sigma_0, \sigma_1, \sigma_2)$; in this case, $g(\mathbb{H}) = 0$ and $f(\mathbb{H}) = 1$.

A ribbon subgraph $\mathbb{H} \subseteq \mathbb{G}$ is called a spanning subgraph if $V(\mathbb{H}) = V(\mathbb{G})$. In this case, $\mathbb{H}$ is a ribbon graph formed from $\mathbb{G}$ by deleting some set of the edges, and keeping all vertices. The following concept was introduced and related to the determinant of a link in [**6**], and also related to Khovanov homology in [**4**]. Following [**6**, Definition 3.1], we have the following definition.

DEFINITION 1. A *quasi-tree* $\mathbb{Q}$ is a connected spanning subgraph of $\mathbb{G}$ with $f(\mathbb{Q}) = 1$.

Equivalently, a spanning subgraph $\mathbb{Q}$ of $\mathbb{G}$ is a quasi-tree if its regular neighborhood on $S(\mathbb{G})$ has exactly one boundary component $\gamma_{\mathbb{Q}}$. Also, a spanning connected ribbon graph $\mathbb{Q}$ is a quasi-tree if and only if $v(\mathbb{Q}) - e(\mathbb{Q}) + 2g(\mathbb{Q}) = 1$. If the genus is zero, then the underlying graph of $\mathbb{Q}$ is a spanning tree. In Table 1, only the ribbon graph on the right is itself a quasi-tree.

Geometrically, $\gamma_{\mathbb{Q}}$ is a simple closed curve on $S(\mathbb{G})$ that divides $S(\mathbb{G})$ as the connect sum of two surfaces with complementary genera. Traversing along $\gamma_{\mathbb{Q}}$, we can mark every half-edge of $\mathbb{G}$ on its first encounter. Therefore, $\gamma_{\mathbb{Q}}$ determines an *ordered chord diagram* $C_{\mathbb{Q}}$, which is a circle marked with $\{1, \ldots, 2n\}$ in some order, and chords joining all pairs $\{i, \sigma_1(i)\}$. We say that $\gamma_{\mathbb{Q}}$ is parameterized by $C_{\mathbb{Q}}$; for example, see Figure 1.

PROPOSITION 1. *Let* $\mathbb{G}$ *be a connected ribbon graph. For every quasi-tree* $\mathbb{Q}$ *of* $\mathbb{G}$*,* $\gamma_{\mathbb{Q}}$ *is parameterized by the ordered chord diagram* $C_{\mathbb{Q}}$*, whose consecutive markings in the positive direction are given by the permutation*

$$\sigma(i) = \begin{cases} \sigma_0(i) & i \notin \mathbb{Q}, \\ \sigma_2^{-1}(i) & i \in \mathbb{Q}. \end{cases}$$

Proof. Since $\mathbb{Q}$ is a quasi-tree, $\gamma_{\mathbb{Q}}$ is one simple closed curve. If we choose an orientation on $S(\mathbb{G})$, then we can traverse $\gamma_{\mathbb{Q}}$ along successive boundaries of bands and vertex disks, such that we always travel around the boundary of each disk in a positive direction (that is, the disk is on the left). If a half-edge is not in $\mathbb{Q}$, then $\gamma_{\mathbb{Q}}$ will pass across it traveling along the boundary of a vertex disk to the next band. If a half-edge is in $\mathbb{Q}$, then $\gamma_{\mathbb{Q}}$ traverses along one of the edges of its band. On $\gamma_{\mathbb{Q}}$, we mark a half-edge not in $\mathbb{Q}$ when $\gamma_{\mathbb{Q}}$ passes across it along the boundary of the vertex disk, and we mark a half-edge in $\mathbb{Q}$ when we traverse an edge of a band in the direction of the half-edge. If the half-edge i is not in $\mathbb{Q}$, traveling along the boundary of a vertex disk, the next half-edge is given by σ_0. If the half-edge i is in $\mathbb{Q}$, traversing the edge of its band to the vertex disk and then along the boundary of that disk, the next half-edge is given by $\sigma_0\sigma_1 = \sigma_2^{-1}$.

As $\mathbb{Q}$ is a quasi-tree, each of its half-edges must be in the orbit of its single face, whereas the complementary set of half-edges are met along the boundaries of the vertex disks. Since we mark all half-edges traversing $\gamma_{\mathbb{Q}}$, the chord diagram $C_{\mathbb{Q}}$ parameterizes $\gamma_{\mathbb{Q}}$. □

We now define *activity with respect to a quasi-tree*.

Definition 2. Fix a total order on the edges of a connected ribbon graph $\mathbb{G}$. For every quasi-tree $\mathbb{Q}$ of $\mathbb{G}$, this induces an order on the chords of $C_{\mathbb{Q}}$. A chord is *live* if it does not intersect lower-ordered chords, and otherwise it is *dead*. For any $\mathbb{Q}$, an edge e is *live* or *dead* when the corresponding chord of $C_{\mathbb{Q}}$ is live or dead; and e is *internal* or *external*, according to $e \in \mathbb{Q}$ or $e \in \mathbb{G} - \mathbb{Q}$, respectively.

If $\mathbb{G}$ is given by $(\sigma_0, \sigma_1, \sigma_2)$ as above, then we order the edges by $\min(i, \sigma_1(i))$, although any ordering convention will work as well. For every quasi-tree $\mathbb{Q}$ of $\mathbb{G}$, the induced order on chords of $C_{\mathbb{Q}}$ is also given by $\min(i, \sigma_1(i))$. In Figure 1, we show $C_{\mathbb{Q}}$ such that the only edge live with respect to $\mathbb{Q}$ is (12), which is internally live.

Tutte [**11**] originally defined activities as follows. For every spanning tree T of G, each edge $e \in G$ has an activity with respect to T. If $e \in T$, then $\mathrm{cut}(T, e)$ is the set of edges that connect $T \setminus e$. If $f \notin T$, then $\mathrm{cyc}(T, f)$ is the set of edges in the unique cycle of $T \cup f$. Note $f \in \mathrm{cut}(T, e)$ if and only if $e \in \mathrm{cyc}(T, f)$. An edge $e \in T$ or $e \notin T$ is *internally active* or *externally active* if it is the lowest edge in its cut or cycle, respectively, and otherwise it is *inactive*.

Because the two types of activities are distinct, we use the notation *active/inactive* when referring to activities in the sense of Tutte with respect to a spanning tree, and *live/dead* for activities with respect to a quasi-tree, as in Definition 2.

Remark 1. (i) If $g(\mathbb{G}) = 0$, then the underlying graph G is planar, and $\mathbb{G}$ is given by a fixed planar embedding of G. In this case, every quasi-tree $\mathbb{Q}$ of $\mathbb{G}$ is a spanning tree T of G. It is easy to check that live or dead edges of $\mathbb{G}$ with respect to $\mathbb{Q}$ are, respectively, active or inactive in G with respect to T.

(ii) A spanning tree of any ribbon graph is also a quasi-tree (of genus 0). In this case, the activities using Tutte's original definition are different from the activities using our definition. For the example in Figure 1, the only spanning tree is the one with no edges. Using Tutte's definition, all four edges are externally active, but using our definition, the activities are $\ell\ell dd$, where ℓ and d denote externally live and dead, respectively. See Section 6 for examples of non-trivial spanning trees whose activities are different from those of the corresponding quasi-trees.

(iii) As for planar graphs, the activities with respect to a quasi-tree depend on the edge order. In the case of a spanning tree T of a planar graph, when the edge order is changed, Tutte proved that there is a corresponding spanning tree T' whose activity in the new edge order matches the activity of T in the old order. However, for general quasi-trees, such a correspondence may not exist: In the example in Section 6, switching the edge order by the permutation $(1\,7)\,(2\,8)$ changes the activity of the unique genus 2 quasi-tree from $LDDDDD$ to $LLLDDD$.

3. *Main results*

The Bollobás–Riordan–Tutte polynomial $C(\mathbb{G}) \in \mathbb{Z}[X, Y, Z]$ is recursively defined by the disjoint union, $C(\mathbb{G}_1 \amalg \mathbb{G}_2) = C(\mathbb{G}_1) \cdot C(\mathbb{G}_2)$, and the following recursion for edges e of $\mathbb{G}$ and subgraphs $\mathbb{H}$ of $\mathbb{G}$, where $\mathbb{G} - e$ and $\mathbb{G}/e$ denote deletion and contraction, respectively:

$$C(\mathbb{G}) = \begin{cases} C(\mathbb{G} - e) + C(\mathbb{G}/e) & \text{if } e \text{ is neither a bridge nor a loop,} \\ X \cdot C(\mathbb{G}/e) & \text{if } e \text{ is a bridge,} \\ \sum_{\mathbb{H}} Y^{n(\mathbb{H})} Z^{g(\mathbb{H})} & \text{if } \mathbb{G} \text{ has one vertex,} \end{cases}$$

where an edge is a *bridge* if deleting it increases the number of components. Note that X is assigned to a bridge, and $1 + Y$ to a loop. For the Tutte polynomial $T_G(x, y)$, these are usually x and y, respectively. If G is the underlying graph of a ribbon graph $\mathbb{G}$, then $C(\mathbb{G}; X, Y, 1) = T_G(X, 1 + Y)$.

The Bollobás–Riordan–Tutte polynomial has a spanning subgraph expansion given by the following sum over all spanning subgraphs $\mathbb{H}$ of $\mathbb{G}$ (see [3, p. 85]):†

$$C(\mathbb{G}) = \sum_{\mathbb{H}} (X - 1)^{k(\mathbb{H}) - k(\mathbb{G})} Y^{n(\mathbb{H})} Z^{g(\mathbb{H})}. \tag{1}$$

The Tutte polynomial has a spanning tree expansion given by the following sum over all spanning trees T of a connected graph G with an order on its edges [11]:

$$T_G(x, y) = \sum_T x^{i(T)} y^{j(T)},$$

where $i(T)$ is the number of internally active edges and $j(T)$ is the number of externally active edges of G for a given spanning tree T of G. Similarly, the Bollobás–Riordan–Tutte polynomial has the following spanning tree expansion [3, p. 93]:

$$C(\mathbb{G}) = \sum_T X^{i(T)} \sum_{S \subset \varepsilon(T)} Y^{n(T \cup S)} Z^{g(T \cup S)}, \tag{2}$$

where $\varepsilon(T)$ is the set of externally active edges of $\mathbb{G}$ with respect to a spanning tree T of $\mathbb{G}$.

We use (1) to prove a quasi-tree expansion for the Bollobás–Riordan–Tutte polynomial, which is different from the expansion (2). Fix a total order on the edges of a connected ribbon graph $\mathbb{G}$. In Definition 2, we defined activities (*live* or *dead*) for edges of $\mathbb{G}$ with respect to $\mathbb{Q}$. Let $\mathcal{D}(\mathbb{Q})$ be the spanning subgraph whose edges are the dead edges in $\mathbb{Q}$ (*internally dead edges*). Let $\mathcal{I}(\mathbb{Q})$ be the set of live edges in $\mathbb{Q}$ (*internally live edges*). Let $\mathcal{E}(\mathbb{Q})$ be the set of live edges in $\mathbb{G} - \mathbb{Q}$ (*externally live edges*).

For a given quasi-tree, let $G_{\mathbb{Q}}$ denote the graph whose vertices are the components of $\mathcal{D}(\mathbb{Q})$ and whose edges are the internally live edges of $\mathbb{Q}$. Let $T_{G_{\mathbb{Q}}}(x, y)$ denote the Tutte polynomial of $G_{\mathbb{Q}}$. Our main result is the following theorem.

THEOREM 1. *Let $\mathbb{G}$ be a connected ribbon graph. The Bollobás–Riordan–Tutte polynomial is given by the following sum over all quasi-trees $\mathbb{Q}$ of $\mathbb{G}$:*

$$C(\mathbb{G}) = \sum_{\mathbb{Q}} Y^{n(\mathcal{D}(\mathbb{Q}))} Z^{g(\mathcal{D}(\mathbb{Q}))} (1 + Y)^{|\mathcal{E}(\mathbb{Q})|} T_{G_{\mathbb{Q}}}(X, 1 + YZ).$$

Let $\mathcal{B}(\mathbb{Q})$ and $\mathcal{N}(\mathbb{Q})$ be the set of internally live edges of $\mathbb{Q}$ that are, respectively, bridges and edges that join the same component of $\mathcal{D}(\mathbb{Q})$. Thus, $G_{\mathbb{Q}}$ has $|\mathcal{B}|$ bridges and $|\mathcal{N}|$ loops, which contribute factors $X^{|\mathcal{B}|}$ and $(1 + YZ)^{|\mathcal{N}|}$ to $T_{G_{\mathbb{Q}}}(X, 1 + YZ)$ in Theorem 1.

†In [3], this expansion is given for $R(\mathbb{G})$. To relate $R(\mathbb{G})$ to $C(\mathbb{G})$, we replace Z by $Z^{1/2}$ (see [3, p. 89]).

In the case when $\mathbb{G}$ has a single vertex, there are only loops, so we have the following simplification.

Corollary 1. *Let $\mathbb{G}$ be a connected ribbon graph with one vertex. Taking the sum over all quasi-trees $\mathbb{Q}$ of $\mathbb{G}$,*

$$C(\mathbb{G}) = \sum_{\mathbb{Q}} Y^{n(\mathcal{D}(\mathbb{Q}))} Z^{g(\mathcal{D}(\mathbb{Q}))} (1+Y)^{|\mathcal{E}(\mathbb{Q})|} (1+YZ)^{|\mathcal{I}(\mathbb{Q})|}.$$

Remark 2. (i) If $g(\mathbb{G}) = 0$, by Remark 1(i), quasi-trees of $\mathbb{G}$ are spanning trees of the underlying graph G, and live or dead reduces to active or inactive, respectively. In this case, $G_{\mathbb{Q}}$ is a tree with $|\mathcal{I}(\mathbb{Q})|$ edges. After substituting $Y = y - 1$ and $Z = 1$ in $C(\mathbb{G})$, we recover Tutte's original spanning tree expansion for $T_G(x, y)$ from Theorem 1.

(ii) For one-vertex ribbon graphs, the only spanning tree is the subgraph with no edges. All edges are loops, so all edges are externally active in the sense of Tutte. The spanning tree expansion (2) becomes the expansion (1) over all subgraphs. In contrast, the quasi-tree expansion in Corollary 1 has fewer terms because some subgraphs are not quasi-trees.

(iii) Dewey [**7**] has generalized both activity with respect to a quasi-tree and Theorem 1 to the non-orientable case.[†]

3.1. *Counting quasi-trees*

The Tutte polynomial counts the number of spanning trees of a connected graph G by the specialization $T_G(1, 1)$. Below, we show that specializing the Bollobás–Riordan–Tutte polynomial counts the number of quasi-trees of every genus.

Proposition 2. *Let $q(\mathbb{G}; t, Y) = C(\mathbb{G}; 1, Y, tY^{-2})$. Then $q(\mathbb{G}; t, Y)$ is a polynomial in t and Y such that*

$$q(\mathbb{G}; t, 0) = \sum_j a_j t^j,$$

where a_j is the number of quasi-trees of genus j. Consequently, $q(\mathbb{G}; 1, 0)$ equals the number of quasi-trees of $\mathbb{G}$.

Proof. The surviving terms in the expansion (1) of $C(\mathbb{G}; 1, Y, Z)$ satisfy $k(\mathbb{H}) = k(\mathbb{G}) = 1$, so they correspond to connected spanning subgraphs. Hence,

$$q(\mathbb{G}; t, Y) = C(\mathbb{G}; 1, Y, tY^{-2}) = \sum_{\mathbb{H}} t^{g(\mathbb{H})} Y^{n(\mathbb{H}) - 2g(\mathbb{H})},$$

where the sum is taken over connected spanning subgraphs. Since $2g(\mathbb{H}) = k(\mathbb{H}) + n(\mathbb{H}) - f(\mathbb{H})$, it follows that $n(\mathbb{H}) - 2g(\mathbb{H}) = f(\mathbb{H}) - k(\mathbb{H}) = f(\mathbb{H}) - 1 \geqslant 0$. This proves that $q(\mathbb{G}; t, Y)$ is a polynomial. The terms of $q(\mathbb{G}, t, 0)$ are those whose Y exponent vanishes, which come from spanning subgraphs $\mathbb{H}$ with $f(\mathbb{H}) = 1$. These are precisely quasi-trees, whose genus $g(\mathbb{H})$ is given by the exponent on t. □

[†]This work was part of the NSF-supported Research Experience for Undergraduates at LSU.

3.2. *Duality*

The Tutte polynomial satisfies an important duality property; for a dual graph G^*, $T_G(x, y) = T_{G^*}(y, x)$. Since a ribbon graph is embedded in a surface, there is a natural dual ribbon graph. Bollobás and Riordan [**3**] found a one-variable specialization of the Bollobás–Riordan–Tutte polynomial that is invariant under this duality.

Building on the work of Ellis-Monaghan and Moffat, Chmutov found that the Bollobás–Riordan–Tutte polynomial satisfies a much more general duality with respect to any subset of edges of a ribbon graph (see [**5**] and references therein). When all the edges are dualized, this construction yields the usual dual ribbon graph. Let g denote the genus of $\mathbb{G}$. In our notation, we have

$$(X-1)^g C_{\mathbb{G}}(X, Y, Z)|_{(X-1)YZ=1} = Y^g C_{\mathbb{G}^*}(Y, X, Z)|_{(X-1)YZ=1}.$$

More recently, Krushkal [**9**] introduced a four-variable polynomial invariant of orientable ribbon graphs that satisfies a duality relation like the Tutte polynomial, and that specializes to the Bollobás–Riordan–Tutte polynomial.

The quasi-trees of a ribbon graph and its dual are in one-to-one correspondence. Since $\gamma_{\mathbb{Q}}$ is a simple closed curve on $S(\mathbb{G})$ that divides $S(\mathbb{G})$ as the connect sum of two surfaces with complementary genera, the genus of the dual quasi-tree $g(\mathbb{Q}^*) = g(\mathbb{G}) - g(\mathbb{Q})$ (see [**6**, Theorem 4.1]). It is an interesting question to understand the above duality in terms of the quasi-tree expansion, and whether this expansion gives rise to new duality properties.

4. *Binary tree of spanning subgraphs*

The spanning subgraphs of a given ribbon graph $\mathbb{G}$ form a poset (of states) $\mathcal{P}$ isomorphic to the boolean lattice $\{0, 1\}^{E(\mathbb{G})}$ of subsets of the set of edges. The partial order is given by $\mathcal{E} = (e_i) \preceq \mathcal{E}' = (e'_i)$, provided $e_i \leqslant e'_i$ for all i. In this section, we define a binary tree $\mathcal{T}$ that is similar to the skein resolution tree for diagrams widely used in knot theory (see, for example, [**8**]). By the construction below, the leaves of $\mathcal{T}$ correspond exactly to quasi-trees of $\mathbb{G}$.

A *resolution* of $\mathbb{G}$ is a function $s : E(\mathbb{G}) \to \{0, 1\}$, that determines a spanning subgraph $\mathbb{H}_s = \{e \in \mathbb{G} \mid s(e) = 1\}$. Let $\rho : E(\mathbb{G}) \to \{0, 1, *\}$ be a *partial resolution* of $\mathbb{G}$, with edges called *unresolved* if they are assigned $*$. Let $\mathbb{H}_\rho = \{e \in \mathbb{G} \mid \rho(e) = 1\}$. A partial resolution determines an interval in the poset $[\rho] = \{s \mid s(e_i) = \rho(e_i) \text{ if } \rho(e_i) \in \{0, 1\}\} = [\rho \wedge 0, \rho \wedge 1]$, which is the interval between $\rho \wedge 0$ with all unresolved edges of ρ set to zero and $\rho \wedge 1$ with all unresolved edges of ρ set to one. Given a partial resolution ρ, we call both ρ and $\mathbb{H}_\rho$ *split* if $f(\mathbb{H}_\rho \cup U) > 1$ for all subsets U of unresolved edges.

Definition 3. If e is an unresolved edge in a partial resolution ρ, let ρ^e_0 and ρ^e_1 be partial resolutions obtained from ρ by resolving e to be 0 and 1, respectively. Then e is called *nugatory* if either one of $\mathbb{H}_{\rho_0}$ or $\mathbb{H}_{\rho_1}$ is split.

Note that an unresolved edge e of ρ is nugatory if and only if one of the intervals $[\rho^e_0]$ or $[\rho^e_1]$ contains no quasi-trees. Figure 2 shows two possibilities for a nugatory edge.

For example, when $g(\mathbb{G}) = 0$ and ρ is not split, an edge e is nugatory in ρ if and only if adding it completes a cycle in ρ^e_1, or ρ^e_0 is disconnected and no unresolved edges can connect it back.

Theorem 2. *For any connected ribbon graph $\mathbb{G}$ with ordered edges, there exists a rooted binary tree $\mathcal{T}$ whose nodes are partial resolutions ρ of $\mathbb{G}$, and whose leaves correspond to quasi-trees $\mathbb{Q}$ of $\mathbb{G}$. If the leaf ρ corresponds to $\mathbb{Q}$, then its unresolved edges are nugatory,*

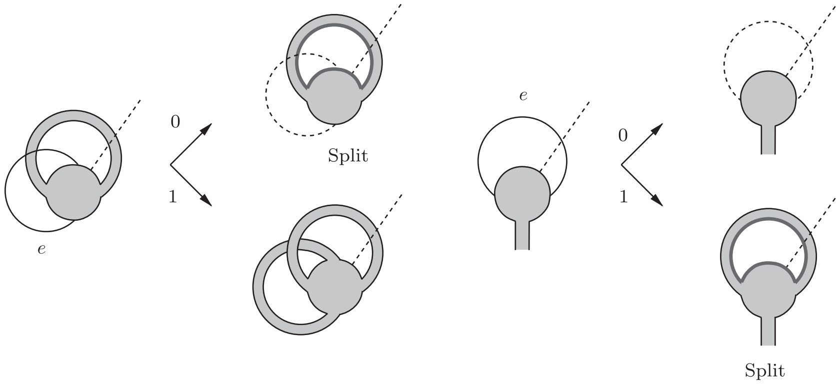


FIGURE 2. *Two possibilities for a nugatory edge e: When e is resolved as indicated, the thicker boundary component remains disjoint for all choices of unresolved edges, resulting in at least two boundary components in the 'split' cases.*

and they can be uniquely resolved to obtain $\mathbb{Q}$. In $\mathbb{G}$, these are exactly the live edges with respect to $\mathbb{Q}$.

Proof. We prove this theorem in a sequence of two lemmas below.

Let the root of $\mathcal{T}$ be the totally unresolved partial resolution, for which $\rho(e) = *$ for all e. We resolve edges by changing $*$ to 0 or 1 in the reverse order (starting with highest ordered edge). If an edge is nugatory, then the edge is left unresolved, and we proceed to the next edge. For a given node ρ in $\mathcal{T}$, if e is not nugatory, then the left child is ρ_0^e and the right child is ρ_1^e. We terminate this process at a leaf when all subsequent edges are nugatory, and return as far back up $\mathcal{T}$ as necessary to a node with a non-nugatory edge still left to be resolved. Therefore, the leaves of $\mathcal{T}$ are spanning subgraphs of $\mathbb{G}$ all of whose unresolved edges are nugatory.

Let $\gamma_\rho = \gamma_{\mathbb{H}_\rho}$, which was defined previously as the boundary of a certain regular neighborhood of $\mathbb{H}_\rho$, and let $|\gamma_\rho|$ denote the number of its components. By definition, $f(\mathbb{H}_\rho) = |\gamma_\rho|$, which is the number of faces on $S(\mathbb{H}_\rho)$, the associated surface for $\mathbb{H}_\rho$.

Let $\Gamma(\rho) = \gamma_\rho \cup \operatorname{Int}(\rho^{-1}(*))$, where $\operatorname{Int}(\rho^{-1}(*))$ denotes the set of interiors of all unresolved edges on $S(\mathbb{G})$. Note that $\Gamma(\rho)$ is connected if and only if we can join the components of γ_ρ by resolving some edges of ρ. Since $f(\mathbb{H}_\rho) = |\gamma_\rho|$, it follows that ρ is split if and only if $\Gamma(\rho)$ is disconnected.

LEMMA 1. *Let ρ be any partial resolution that is not split, with an unresolved edge $e \in \mathbb{G}$. For $i \in \{0, 1\}$, let $\rho_i = \rho_i^e$, and let $\Gamma_i(\rho, e) = \Gamma(\rho_i^e)$. The edge e is nugatory if and only if either $\Gamma_0(\rho, e)$ or $\Gamma_1(\rho, e)$ is disconnected on $S(\mathbb{G})$. If $\Gamma_0(\rho, e)$ is disconnected, then $|\gamma_{\rho_1}| = |\gamma_\rho| - 1$ and $|\gamma_{\rho_0}| = |\gamma_\rho|$. If $\Gamma_1(\rho, e)$ is disconnected, then $|\gamma_{\rho_0}| = |\gamma_\rho|$ and $|\gamma_{\rho_1}| = |\gamma_\rho| + 1$.*

Proof. For $i \in \{0, 1\}$, ρ_i is split if and only if $\Gamma_i(\rho, e)$ is disconnected. Since ρ is not split, $\mathbb{H}_{\rho_0}$ or $\mathbb{H}_{\rho_1}$ is split if and only if deleting e or cutting along e, respectively, disconnects $\Gamma(\rho)$.

If $\Gamma_0(\rho, e)$ is disconnected, then e is the only edge connecting two components of γ_ρ. Hence, these two components are connected in γ_{ρ_1}. This gives $|\gamma_{\rho_1}| = |\gamma_\rho| - 1$ and $|\gamma_{\rho_0}| = |\gamma_\rho|$. On the other hand, if $\Gamma_1(\rho, e)$ is disconnected, then e intersects a component of γ_ρ twice without linking any other unresolved edge, so this component becomes disconnected in γ_{ρ_1}. This gives $|\gamma_{\rho_0}| = |\gamma_\rho|$ and $|\gamma_{\rho_1}| = |\gamma_\rho| + 1$. □

We can now see that the partial resolution of a leaf can be resolved uniquely to give a quasi-tree. By construction, for a leaf ρ of $\mathcal{T}$, $\mathbb{H}_\rho$ is not split, so there exists a resolution $s \in [\rho]$ such that $f(\mathbb{H}_s) = |\gamma_{\mathbb{H}_s}| = 1$. In particular, since all unresolved edges are nugatory, by Lemma 1, there is a unique resolution $s \in [\rho]$ such that $|\gamma_{\mathbb{H}_s}|$ is minimized. Including nugatory edges e for which $\Gamma_1(\rho, e)$ is connected, and excluding nugatory edges e for which $\Gamma_1(\rho, e)$ is disconnected, $|\gamma_{\mathbb{H}_s}| = 1$. Hence, $\mathbb{H}_s$ is a quasi-tree.

Lemma 2. *Let ρ be a leaf of $\mathcal{T}$, and let $\mathbb{Q} \in [\rho]$ be the corresponding quasi-tree. If $\rho(e) = *$, then e is live with respect to $\mathbb{Q}$, and otherwise e is dead with respect to $\mathbb{Q}$.*

Proof. If e_i and e_j are any edges of ρ, then we say that e_i and e_j link each other if, when uniquely resolved to obtain $\mathbb{Q}$, their endpoints alternate on $\gamma_{\mathbb{Q}}$. Equivalently, their corresponding chords intersect in $C_{\mathbb{Q}}$. This notion does not depend on whether the edges are resolved in ρ. If $g(\mathbb{G}) = 0$, then $\mathbb{Q}$ is a spanning tree, and edges link each other if and only if they satisfy a cut-cycle condition with respect to $\mathbb{Q}$: $e_i \in \mathrm{cut}(\mathbb{Q}, e_j)$ or $e_i \in \mathrm{cyc}(\mathbb{Q}, e_j)$.

Let e_i and e_j be unresolved edges of ρ, which are therefore nugatory. Let $s \in [\rho]$ be the unique resolution such that $\mathbb{H}_s = \mathbb{Q}$. Let s' be the resolution obtained from s by changing the states of both e_i and e_j. If e_i and e_j link each other, then $|\gamma_{s'}| = |\gamma_s| = 1$. Hence, $\mathbb{H}_{s'}$ is a quasi-tree for a second resolution $s' \in [\rho]$, which is a contradiction. Thus, unresolved edges can only link resolved edges.

Suppose that e_i is unresolved and links a resolved edge e_j with $j < i$. There exists a unique closest parent $\tilde{\rho}$ of ρ in $\mathcal{T}$, such that e_j is a non-nugatory unresolved edge in $\tilde{\rho}$. Since edges are resolved in the reverse order, e_i is nugatory in $\tilde{\rho}$. As e_i links e_j, it follows that $\Gamma_0(\tilde{\rho}, e_i)$ and $\Gamma_1(\tilde{\rho}, e_i)$ are both connected, which contradicts Lemma 1. Thus, if e_i and e_j are linked, then $i < j$, so e_i is live.

Now, let e_i be a resolved edge of ρ. There exists a unique closest parent $\tilde{\rho}$ of ρ in $\mathcal{T}$, such that e_i is a non-nugatory unresolved edge in $\tilde{\rho}$. By Lemma 1, $\Gamma_0(\tilde{\rho}, e_i)$ and $\Gamma_1(\tilde{\rho}, e_i)$ are both connected. Hence, there exists e_j, which is unresolved in $\tilde{\rho}$, such that e_i and e_j are linked. If e_j is resolved after e_i in $\mathcal{T}$, then $j < i$. Since e_i and e_j are linked, e_i is dead. On the other hand, if e_j is left unresolved in $\mathcal{T}$, then e_j is live by the argument in the previous paragraph with i and j reversed. Since e_i and e_j are linked, and e_j is live, it follows that e_i is dead. □

This completes the proof of Theorem 2. □

5. *Proof of Theorem* 1

Let $\mathbb{H} \subseteq \mathbb{G}$ be a spanning subgraph. Let $n(\mathbb{H})$, $g(\mathbb{H})$ and $k(\mathbb{H})$ denote the nullity, genus and number of components of $\mathbb{H}$, respectively. Since $v(\mathbb{H}) = v(\mathbb{G})$,

$$n(\mathbb{H}) = k(\mathbb{H}) - v(\mathbb{G}) + e(\mathbb{H}), \quad g(\mathbb{H}) = \frac{2k(\mathbb{H}) - v(\mathbb{G}) + e(\mathbb{H}) - f(\mathbb{H})}{2}.$$

Let $\mathbb{Q}$ be a quasi-tree of $\mathbb{G}$. Let $\mathcal{I} = \mathcal{I}(\mathbb{Q})$ and $\mathcal{E} = \mathcal{E}(\mathbb{Q})$ be, respectively, the internally and externally live edges with respect to $\mathbb{Q}$. Let $\mathcal{D} = \mathcal{D}(\mathbb{Q})$ be the spanning subgraph whose edges are the dead edges in $\mathbb{Q}$.

By Theorem 2, there is a unique partial resolution ρ of $\mathbb{G}$ that is a leaf of $\mathcal{T}$, for which $\mathbb{Q} \in [\rho]$, and all resolutions $\mathbb{H}_s$ for $s \in [\rho]$ are of the form $\mathcal{D} \cup S$ where $S \subseteq \mathcal{I} \cup \mathcal{E}$. All resolutions $\mathbb{H}_s$ are elements of the state poset $\mathcal{P}$, so the sum in (1) is a state sum for $\mathcal{P}$. The sum in Theorem 1 is a state sum for $\mathcal{T}$. Below, we prove that these two state sums are equal.

Lemma 3. *For a quasi-tree* $\mathbb{Q}$ *of* $\mathbb{G}$, *let* $S = S_1 \cup S_2$, *where* $S_1 \subseteq \mathcal{I}(\mathbb{Q})$ *and* $S_2 \subseteq \mathcal{E}(\mathbb{Q})$.

(i) $k(\mathcal{D}(\mathbb{Q}) \cup S) = k(\mathcal{D}(\mathbb{Q}) \cup S_1)$;
(ii) $n(\mathcal{D}(\mathbb{Q}) \cup S) = n(\mathcal{D}(\mathbb{Q}) \cup S_1) + |S_2|$;
(iii) $g(\mathcal{D}(\mathbb{Q}) \cup S) = g(\mathcal{D}(\mathbb{Q}) \cup S_1)$.

Proof. Let $e \in \mathcal{E}(\mathbb{Q})$. By Theorem 2, $\mathbb{Q}$ corresponds to ρ such that e is nugatory. By Lemma 1, $\Gamma_1(\rho, e)$ is disconnected, so $\Gamma_0(\rho, e)$ is connected. Hence, e intersects only one component of $\gamma_{\mathcal{D}}$. Thus, $k(\mathcal{D} \cup e) = k(\mathcal{D})$, and part (i) follows.

$$\begin{aligned} n(\mathcal{D} \cup S) &= k(\mathcal{D} \cup S) - v(\mathbb{G}) + e(\mathcal{D} \cup S) \\ &= k(\mathcal{D} \cup S_1) - v(\mathbb{G}) + e(\mathcal{D} \cup S_1) + |S_2| \\ &= n(\mathcal{D} \cup S_1) + |S_2|. \end{aligned}$$

Since $f(\mathbb{H}) = |\gamma_{\mathbb{H}}|$, by Lemma 1, $f(\mathcal{D} \cup e) = f(\mathcal{D}) + 1$, hence

$$\begin{aligned} 2g(\mathcal{D} \cup S) &= 2k(\mathcal{D} \cup S) - v(\mathbb{G}) + e(\mathcal{D} \cup S) - f(\mathcal{D} \cup S) \\ &= 2k(\mathcal{D} \cup S_1) - v(\mathbb{G}) + (e(\mathcal{D} \cup S_1) + |S_2|) - (f(\mathcal{D} \cup S_1) + |S_2|) \\ &= 2k(\mathcal{D} \cup S_1) - v(\mathbb{G}) + e(\mathcal{D} \cup S_1) - f(\mathcal{D} \cup S_1) \\ &= 2g(\mathcal{D} \cup S_1). \end{aligned}$$ □

Lemma 4. *For a quasi-tree* $\mathbb{Q}$ *of* $\mathbb{G}$, *let* $S_1 \subseteq \mathcal{I}(\mathbb{Q})$. *Let* W *be the spanning subgraph of* $G_{\mathbb{Q}}$ *whose edges are the edges in* S_1. *Then we have:*

(i) $n(\mathcal{D}(\mathbb{Q}) \cup S_1) = n(\mathcal{D}(\mathbb{Q})) + n(W)$;
(ii) $g(\mathcal{D}(\mathbb{Q}) \cup S_1) = g(\mathcal{D}(\mathbb{Q})) + n(W)$.

Proof. For spanning subgraph W of $G_{\mathbb{Q}}$, $k(W) = k(\mathcal{D} \cup S_1)$. Hence,

$$\begin{aligned} n(W) &= k(W) - v(G_{\mathbb{Q}}) + e(W) = k(\mathcal{D} \cup S_1) - k(\mathcal{D}) + |S_1|. \\ n(\mathcal{D} \cup S_1) &= k(\mathcal{D} \cup S_1) - v(\mathbb{G}) + e(\mathcal{D} \cup S_1) \\ &= (k(\mathcal{D}) - v(\mathbb{G}) + e(\mathcal{D})) + (k(\mathcal{D} \cup S_1) - k(\mathcal{D}) + |S_1|) \\ &= n(\mathcal{D}) + n(W). \end{aligned}$$

Let $e \in \mathcal{I}(\mathbb{Q})$. By Theorem 2, $\mathbb{Q}$ corresponds to ρ such that e is nugatory, and by Lemma 1, $\Gamma_1(\rho, e)$ is connected. Since $f(\mathbb{H}) = |\gamma_{\mathbb{H}}|$, by Lemma 1, $f(\mathcal{D} \cup e) = f(\mathcal{D}) - 1$. Since live edges do not link each other, we can iterate this to obtain $f(\mathcal{D} \cup S_1) = f(\mathcal{D}) - |S_1|$. Therefore,

$$\begin{aligned} 2g(\mathcal{D} \cup S_1) &= 2k(\mathcal{D} \cup S_1) - v(\mathbb{G}) + e(\mathcal{D} \cup S_1) - f(\mathcal{D} \cup S_1) \\ &= 2k(\mathcal{D}) - v(\mathbb{G}) + (e(\mathcal{D}) + |S_1|) - (f(\mathcal{D}) - |S_1|) + 2k(\mathcal{D} \cup S_1) - 2k(\mathcal{D}) \\ &= 2g(\mathcal{D}) + 2(k(\mathcal{D} \cup S_1) - k(\mathcal{D}) + |S_1|) \\ &= 2g(\mathcal{D}) + 2n(W). \end{aligned}$$ □

Proof of Theorem 1. The sum in Theorem 1 is over quasi-trees, which correspond to leaves $[\rho]$ of $\mathcal{T}$. It suffices to show that, for any quasi-tree, its summand in Theorem 1 equals the sum over all $\mathbb{H}_s$ for $s \in [\rho]$ in equation (1).

Let $S = S_1 \cup S_2$, where $S_1 \subseteq \mathcal{I}$ and $S_2 \subseteq \mathcal{E}$. By Lemma 3, the contribution from $[\rho]$ to the sum in equation (1) is

$$\begin{aligned}\sum_{S\subseteq\mathcal{I}\cup\mathcal{E}} &(X-1)^{k(\mathcal{D}\cup S)-1} Y^{n(\mathcal{D}\cup S)}\ Z^{g(\mathcal{D}\cup S)}\\ &= \sum_{S_2\subseteq\mathcal{E}} Y^{|S_2|} \sum_{S_1\subseteq\mathcal{I}} (X-1)^{k(\mathcal{D}\cup S_1)-1}\ Y^{n(\mathcal{D}\cup S_1)}\ Z^{g(\mathcal{D}\cup S_1)}\\ &= (1+Y)^{|\mathcal{E}|} \sum_{S_1\subseteq\mathcal{I}} (X-1)^{k(\mathcal{D}\cup S_1)-1}\ Y^{n(\mathcal{D}\cup S_1)}\ Z^{g(\mathcal{D}\cup S_1)}.\end{aligned}$$

In the following, we use the spanning subgraph expansion of the Tutte polynomial (see, for example, [**1**, p. 339]),

$$T_G(x,y) = \sum_{W\subseteq G} (x-1)^{k(W)-k(G)}(y-1)^{n(W)}.$$

Let $G_{\mathbb{Q}}$ denote the graph whose vertices are the components of $\mathcal{D}$ and whose edges are the edges in $\mathcal{I}$. The quasi-tree $\mathbb{Q}$ is a connected subgraph of $\mathbb{G}$, so $G_{\mathbb{Q}}$ is a connected graph, hence $k(G_{\mathbb{Q}}) = 1$. The subgraphs $\{\mathcal{D}\cup S_1 \mid S_1 \subseteq \mathcal{I}\}$ are in one-to-one correspondence with spanning subgraphs $W \subseteq G_{\mathbb{Q}}$. Let $n_0 = n(\mathcal{D})$ and $g_0 = g(\mathcal{D})$. By Lemma 4,

$$\begin{aligned}\sum_{S_1\subseteq\mathcal{I}} (X-1)^{k(\mathcal{D}\cup S_1)-1} Y^{n(\mathcal{D}\cup S_1)} Z^{g(\mathcal{D}\cup S_1)} &= \sum_{W\subseteq G_{\mathbb{Q}}} (X-1)^{k(W)-1} Y^{n(\mathcal{D})+n(W)} Z^{g(\mathcal{D})+n(W)}\\ &= Y^{n_0} Z^{g_0} \sum_{W\subseteq G_{\mathbb{Q}}} (X-1)^{k(W)-k(G_{\mathbb{Q}})} (YZ)^{n(W)}\\ &= Y^{n_0} Z^{g_0} T_{G_{\mathbb{Q}}}(X, 1+YZ).\end{aligned}$$

The last step is obtained from the spanning subgraph expansion of the Tutte polynomial with $x = X$ and $y = 1 + YZ$.

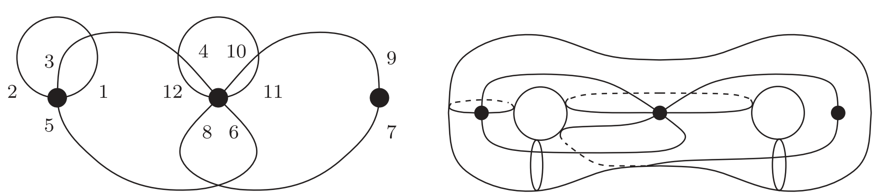
Therefore, for each $\mathbb{Q}$, the contribution to the sum in (1) is

$$Y^{n(\mathcal{D}(\mathbb{Q}))} Z^{g(\mathcal{D}(\mathbb{Q}))} (1+Y)^{|\mathcal{E}(\mathbb{Q})|} T_{G_{\mathbb{Q}}}(X, 1+YZ).$$

This completes the proof of Theorem 1. □

6. *Example*

We compute the quasi-tree and spanning tree expansions for a ribbon graph $\mathbb{G}$ with twelve quasi-trees having a variety of topological types. The ribbon graph $\mathbb{G}$ has three vertices and six edges, given by $\sigma_0 = (1,3,2,5)\,(7,9)\,(10,4,12,8,6,11)$, $\sigma_1 = (1,2)\,(3,4)\,(5,6)\,(7,8)\,(9,10)\,(11,12)$, so $\sigma_2 = (1,6,7,10,12,3,2,4,9,8,11,5)$. We order the edges of $\mathbb{G}$ by $\min(i, \sigma_1(i))$. The ribbon graph $\mathbb{G}$ and its surface are shown in the following:



In the table below, we denote quasi-trees using the edge order; for example, 001010 denotes $\mathbb{Q}$ consisting of only the third and fifth edges, $(5,6)$ and $(9,10)$. For each $\mathbb{Q}$, we compute the chord diagram, activities (L and ℓ, respectively, for internally and externally live; D and d, respectively, for internally and externally dead), numbers $\{g, n, \bar{g}, \varepsilon\} = \{g(\mathbb{Q}), n(\mathcal{D}(\mathbb{Q})), g(\mathcal{D}(\mathbb{Q})), |\mathcal{E}(\mathbb{Q})|\}$, graph $G_{\mathbb{Q}}$ and its weight in the sum of Theorem 1. For the chord diagrams, we give the cyclic permutation of the half-edges. The types of graphs $G_{\mathbb{Q}}$ that occur in this example are as follows: (1) vertex, (2) edge, (3) two edges with a vertex in common, (4) two edges with both vertices in common, (5) 2-cycle joined to a bridge, (6) loop, and (7) loop joined to a bridge.

$\mathbb{Q}$	$C_{\mathbb{Q}}$	Activity	$g, n, \bar{g}, \varepsilon$	$G_{\mathbb{Q}}$	Weight
001010	$(1, 3, 2, 5, 11, 10, 7, 9, 4, 12, 8, 5)$	$\ell dDdDd$	$0, 0, 0, 1$	1	$(1 + Y)$
001100	$(1, 3, 2, 5, 11, 10, 4, 12, 8, 9, 7, 6)$	$\ell dDLdd$	$0, 0, 0, 1$	2	$X(1 + Y)$
001111	$(1, 3, 2, 5, 11, 8, 9, 4, 12, 10, 7, 6)$	$\ell dDDDD$	$1, 2, 1, 1$	1	$Y^2Z(1 + Y)$
010010	$(1, 3, 12, 8, 6, 11, 10, 7, 9, 4, 2, 5)$	$\ell LddDd$	$0, 0, 0, 1$	2	$X(1 + Y)$
010100	$(1, 3, 12, 8, 9, 7, 6, 11, 10, 4, 2, 5)$	$\ell LdLdd$	$0, 0, 0, 1$	3	$X^2(1 + Y)$
010111	$(1, 3, 12, 10, 7, 6, 11, 8, 9, 4, 2, 5)$	$\ell LdDDD$	$1, 2, 1, 1$	2	$XY^2Z(1 + Y)$
011011	$(1, 3, 12, 10, 7, 9, 4, 2, 5, 11, 8, 6)$	$\ell LLdDD$	$1, 1, 0, 1$	4	$Y(1 + Y)(X + 1 + YZ)$
011101	$(1, 3, 12, 10, 4, 2, 5, 11, 8, 9, 7, 6)$	$\ell LLLdD$	$1, 1, 0, 1$	5	$XY(1 + Y)(X + 1 + YZ)$
011110	$(1, 3, 12, 8, 9, 4, 2, 5, 11, 10, 7, 6)$	$\ell LLDDd$	$1, 1, 0, 1$	4	$Y(1 + Y)(X + 1 + YZ)$
111010	$(1, 5, 11, 10, 7, 9, 4, 2, 3, 12, 8, 6)$	$LDDdDd$	$1, 1, 0, 0$	6	$Y(1 + YZ)$
111100	$(1, 5, 11, 10, 4, 2, 3, 12, 8, 9, 7, 6)$	$LDDLdd$	$1, 1, 0, 0$	7	$XY(1 + YZ)$
111111	$(1, 5, 11, 8, 9, 4, 2, 3, 12, 10, 7, 6)$	$LDDDDD$	$2, 3, 1, 0$	6	$Y^3Z(1 + YZ)$

Adding the weights in the last column, the Bollobás–Riordan–Tutte polynomial of $\mathbb{G}$ is

$$C(\mathbb{G}) = Z^2Y^4 + 2XZY^3 + 4ZY^3 + X^2Y^2 + 3XY^2 + 3XZY^2 + 4ZY^2 + 2Y^2 + 2X^2Y + 6XY + 4Y + X^2 + 2X + 1.$$

By Proposition 2, $q(\mathbb{G}; t, Y) = C(\mathbb{G}; 1, Y, tY^{-2}) = 4 + 7t + t^2$, which counts the quasi-trees of every genus.

As an example, let $\mathbb{Q}$ be the eighth quasi-tree, denoted by 011101. The associated partial resolution is $\rho = * * * * 01$. We see that $\mathcal{D}(\mathbb{Q})$ has three components, consisting of two isolated vertices and a loop; $G_{\mathbb{Q}}$ has three vertices and three edges, two connected in parallel and a second edge to the remaining vertex. The Tutte polynomial $T_{G_{\mathbb{Q}}}(x, y) = x(x + y)$. Thus, the contribution from $\mathbb{Q}$ is $XY(1 + Y)(X + 1 + YZ)$, which is also the contribution from the sixteen terms in the state sum (1) for all $s \in [\rho]$.

We now compute the spanning tree expansion (2) of the Bollobás–Riordan–Tutte polynomial for this example. Using the notation above, the spanning trees are the genus 0 quasi-trees. In the table below, we give the spanning trees T, their activities in the sense of Tutte, their weights given by the inner sum in (2) and the factor $X^{i(T)}$ in (2). Note that the activities for the spanning trees below are different in every case from the activities given above for the corresponding quasi-trees.

T	Activity	Weight	$X^{i(T)}$
001010	$\ell\ell D\ell D\ell$	$1 + 4Y + 2Y^2 + 4Y^2Z + 4Y^3Z + Y^4Z^2$	1
001100	$\ell\ell DLd\ell$	$1 + 3Y + Y^2 + 2Y^2Z + Y^3Z$	X
010010	$\ell Ld\ell D\ell$	$(1 + Y)(1 + 2Y + Y^2Z)$	X
010100	$\ell LdLd\ell$	$(1 + Y)^2$	X^2

Taking the sum according to (2), we obtain $C(\mathbb{G})$ as above.

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