Sample problems for Linear Algebra, Spring 2016, Exam 3

Problem 1.

$$W = \text{span}\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix}\} \qquad \mathbf{b} = \begin{bmatrix} 12 \\ 18 \\ 12 \\ 18 \end{bmatrix}$$

(a) Find a basis for $W^{\perp}$ in $\mathbf{R}^4$.

(b) Find the projection $\mathbf{p}$ onto W of $\mathbf{b}$; i.e., $\mathbf{p} = \text{proj}_W(\mathbf{b})$.
(Note this is an orthogonal basis of W.)

(c) Find $\mathbf{e} = \mathbf{b} - \mathbf{p}$. Verify that $\mathbf{e}$ is in $W^{\perp}$.

(d) Write $\mathbf{e}$ as a linear combination of your basis vectors from part (a).

Problem 2.

$$W = \text{span}\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}\} \qquad \mathbf{b} = \begin{bmatrix} 12 \\ 18 \\ 12 \\ 18 \end{bmatrix}$$

Find the projection $\mathbf{p}$ onto W of $\mathbf{b}$; i.e., $\mathbf{p} = \text{proj}_W(\mathbf{b})$.
(Note this is NOT an orthogonal basis of W.)

(a) Solve for $\hat{x}$, with $A^T A\hat{x} = A^T\mathbf{b}$.

(b) Solve for $\mathbf{p} = A\hat{x}$.

(c) Find $\mathbf{e} = \mathbf{b} - \mathbf{p}$. Verify that $\mathbf{e}$ is in $W^{\perp}$.

Problem 3. Find the best fit line using the least squares method for these points:

$$(-1, 1), \ (1, 2), \ (3, 4)$$

Problem 4. **(a)** Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are non-zero orthogonal vectors in $\mathbf{R}^n$.
Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

(b) Let Q be any orthogonal matrix. Show that Q preserves length; i.e., $||Q\mathbf{x}|| = ||\mathbf{x}||$.

Problem 5. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be orthonormal vectors in $\mathbf{R}^3$. Let $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$.
Find all possible values of the following determinants. Justify.

(a) Find A^{-1}.

(b) $\det[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$

(c) $\det[\mathbf{v}_1 + \mathbf{v}_2 \ \ \mathbf{v}_2 + \mathbf{v}_3 \ \ \mathbf{v}_3 + \mathbf{v}_1]$

Problem 6.

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$$

(a) Compute a determinant to show that S is a basis for $\mathbf{R}^3$. Justify.

(b) Use the Gram-Schmidt method to find an orthonormal basis.

Problem 7. Let A and B be $n \times n$ matrices.

(a) If $A^2 = A$, what are the possible values of $\det(A)$?

(b) If AB is invertible, is BA invertible?

(c) If all cofactors of A are zero, is A invertible?

(d) If $\det(A) = 0$, what is the possible rank of A?

Problem 8.

$$A = \begin{bmatrix} x & x & x & x \\ x & a & b & c \\ x & d & e & f \\ x & g & h & i \end{bmatrix}$$

(a) $\det A$ is a polynomial in x. Without computing $\det A$, what is the degree of the polynomial? Justify using permutations, and then using cofactors.

(b) If the bottom 3×3 matrix is the identity matrix, which values of x give $\det A = 0$?

Problem 9. Let P be the plane in $\mathbf{R}^3$ given by $3x + 2y - z = 0$.

(a) Find a basis for the subspace P in $\mathbf{R}^3$.

(b) Find a basis for the orthogonal complement $P^{\perp}$. Justify using a nullspace computation.

(c) Find a basis for the orthogonal complement $P^{\perp}$. Justify using the cross product.

Problem 10.

$$S = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 3 \\ 2 \end{bmatrix} \right\}$$

(a) Show that the vectors in S are coplanar.

(b) If A is a matrix with these columns, give the dimensions of the four fundamental subspaces of A.

Problem 11. Solve the following linear system using Cramer's Rule:

$$\begin{cases} 2x + y + z &= 3 \\ x - y - z &= 0 \\ x + 2y + z &= 0 \end{cases}$$