MTH 232 Exam 2 11/26/2014 Solutions

(1) Geometric Series, $r = -\frac{3}{2}$, diverges ($|r|>1$) OR n-th term test

(2) Integral Test: $\int_1^\infty x^2 e^{-x^3}\,dx$ $\quad u = x^3,\ du = 3x^2\,dx$ $\quad = \frac{1}{3}\int_1^\infty e^{-u}\,du = \lim_{R\to\infty} -\frac{1}{3}e^{-u}\Big|_1^R$

$$= \lim_{R\to\infty} -\frac{1}{3}\left(\overset{\to 0}{e^{-R}} - e^{-1}\right) = \frac{1}{3e} < \infty \quad \text{Converges.}$$

(3) Ratio Test: $\lim_{n\to\infty}\left|\frac{n+6}{9^{n+1}}\cdot\frac{9^n}{n+5}\right| = \lim_{n\to\infty}\left|\overset{\to 1}{\frac{n+6}{n+5}}\right|\cdot\frac{1}{9} = \frac{1}{9} < 1$ Converges

(4) Ratio Test: $\lim_{n\to\infty}\left|\frac{4^{n+1}(n+1)^2}{(n+1)!}\cdot\frac{n!}{4^n n^2}\right| = \lim_{n\to\infty}\left|\overset{\to 0}{\frac{4}{n+1}}\cdot\overset{\to 1}{\left(\frac{n+1}{n}\right)^2}\right| = 0$ Converges.

(5) Limit Comparison with $\sum_{n=1}^\infty \frac{1}{n}$: $\lim_{n\to\infty}\frac{\frac{n^2+2}{n^3+3}}{1/n} = \lim_{n\to\infty}\frac{n^3+2n}{n^3+3} = 1$ diverges since harmonic series diverges

(6) (Limit) Comparison with $\sum_{n=1}^\infty \frac{1}{n^{3/2}}$: $\lim_{n\to\infty}\frac{\frac{\sqrt{n}}{n^2+7}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty}\frac{n^2}{n^2+7} = 1$ Converges since p-series $p>1$ converges

(7) Alternating Series Test: $\lim_{n\to\infty}\frac{\ln n}{\sqrt{n}} \overset{[\frac{\infty}{\infty}]}{=} \lim_{n\to\infty}\frac{1/n}{1/2\sqrt{n}} = \lim_{n\to\infty}\frac{2}{\sqrt{n}} = 0$

Converges.

(8) Geometric series, $r = \frac{\pi^7}{e^8} > 1$, diverges.

(9) $\sum_{n=1}^\infty \frac{2^{n+2}}{5^{n+1}} = \frac{a_1}{1-r} = \frac{2^3/5^2}{1-2/5} = \frac{8}{25}\cdot\frac{5}{3} = \frac{8}{15}$

(10) $\lim_{n\to\infty}\left|\frac{(2x-4)^{n+1}}{3(n+1)+5}\cdot\frac{3n+5}{(2x-4)^n}\right| = \lim_{n\to\infty}|2x-4|\overset{\to 1}{\frac{3n+5}{3n+8}} = |2x-4| \le 1$

$-1 \le 2x-4 \le 1$

$3 \le 2x \le 5$

$3/2 \le x \le 5/2$

If $x = \frac{3}{2}$, $\sum_{n=1}^\infty \frac{(-1)^n}{3n+5}$ converges by alt series test

If $x = \frac{5}{2}$, $\sum_{n=1}^\infty \frac{1}{3n+5}$ diverges by limit comparison with $\sum_{n=1}^\infty \frac{1}{n}$

So interval is $\left[\frac{3}{2}, \frac{5}{2}\right)$.

(11) $\lim\limits_{n\to\infty} \left| \dfrac{(x+1)^{n+1}}{\sqrt{n+1}\, 3^{n+1}} \cdot \dfrac{\sqrt{n}\, 3^n}{(x+1)^n} \right| = \lim\limits_{n\to\infty} |x+1| \cdot \dfrac{1}{3} \cdot \cancel{\sqrt{\dfrac{n}{n+1}}}^{\,\nearrow 1} = \dfrac{|x+1|}{3} \le 1$

$-3 \le x+1 \le 3$

$-4 \le x \le 2$

If $x = 2$, $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ diverges (p-series $p=\frac{1}{2}$)

If $x=-4$, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n}}$ converges (alt. series test)

So interval is $[-4, 2)$

(12) $\ln(1+x) = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1} x^n}{n} = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$

$\Rightarrow \ln(1+x^3) = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1} x^{3n}}{n} = x^3 - \dfrac{x^6}{2} + \dfrac{x^9}{3} - \dfrac{x^{12}}{4} + \cdots$

(13) $\int \cos(x^2) = \sum\limits_{n=0}^{\infty} \int \dfrac{(-1)^n x^{4n}}{(2n)!} = \int 1 - \dfrac{x^4}{2!} + \dfrac{x^8}{4!} - \dfrac{x^{12}}{6!} + \cdots \, dx$

$= \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{(2n)!} \cdot \dfrac{1}{4n+1} x^{4n+1} = x - \dfrac{1}{2!}\cdot\dfrac{1}{5}x^5 + \dfrac{1}{4!}\cdot\dfrac{1}{9}x^9 - \dfrac{1}{6!}\cdot\dfrac{1}{13}x^{13} + \cdots$

(14)

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	x^{-2}	$\frac{1}{4}$
1	$-2x^{-3}$	$-\frac{1}{4}$
2	$6x^{-4}$	$\frac{6}{16} = \frac{3}{8}$
3	$-24x^{-5}$	$\frac{-24}{32} = -\frac{3}{4}$

$T(x) = \dfrac{1}{4} - \dfrac{1}{4}(x-2) + \dfrac{3}{8}\cdot\dfrac{1}{2}(x-2)^2 - \dfrac{3}{4}\cdot\dfrac{1}{3!}(x-2)^3 + \cdots$

$= \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n (n+1)!}{2^{n+2}} \cdot \dfrac{1}{n!}(x-2)^n$

$= \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n\, n+1}{2^{n+2}} (x-2)^n$