Problem 1. *(10 pts.) State whether the following statements are* **Always** *true,* **Sometimes** *true, or* **Never** *true. Please circle one of* **A, S, N** *below.*

(a) *An invertible matrix can be written as a product of symmetric matrices.*

*(**S**) True for $I_n = I_n * I_n$. False, e.g., for upper triangular invertible matrices.*

(b) *A homogeneous 3×5 linear system has a nontrivial solution.*

*(**A**) by Theorem 1.8 on page 77.*

(c) *If* $\det(A) = 0$, *then* $\det(A + B) = \det(B)$.

*(**S**) Let O_n be the $n \times n$ zero matrix. True for $A = O_n$, $B = I_n$. False, e.g., for*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ then } \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(d) *If* $\det(A) = 0$, *then* $\det(BA) = 0$.

*(**A**) because* $\det(BA) = (\det(B))(\det(A)) = 0$.

(e) *A square matrix which has two identical columns is invertible.*

*(**N**) If two columns are identical, the matrix is singular.*

Problem 2. *(15 pts.) Justify three out of the following four statements with a short general argument:*

(a) *If A is a non-singular $n \times n$ matrix then:*

$$\det\left(A^{-1}\right) = \frac{1}{\det(A)}.$$

$$\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(I_n) = 1$$

(b) *If A and B are non-singular $n \times n$ matrices, then AB is also non-singular.*

$\det(AB) = \det(A)\det(B) \neq 0$ *which implies that AB is non-singular.*

(c) *A non-singular matrix has a unique inverse.*

If B and C are two inverses of A (i.e., $BA = AC = I_n$) then

$$B = BI_n = B(AC) = (BA)C = I_nC = C$$

(d) *If A and B are symmetric matrices, then AB is also symmetric.*

This is only sometimes true – my mistake! For example, false for

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

Problem 3. *(20 pts.) Write "impossible" or give an example of:*

(a) *A 3×3 matrix with no zeros but which is not invertible.*

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

(b) *A system with two equations and three unknowns that is inconsistent.*

$$x + y + z = 0$$
$$x + y + z = 1$$

(c) *A system with two equations and three unknowns that has a unique solution.*

Impossible

(d) *A system with two equations and three unknowns that has infinitely many solutions.*

$$x + y + z = 0$$
$$2x + y + z = 1$$

In this case, the solutions are $x = -1, y = r, z = 2 - r$, for any real number r.

Problem 4. *(10 pts.) Consider the following linear system:*

$$\begin{cases} 2x_1 - x_2 + x_4 = 0 \\ -x_1 + 2x_2 - x_3 = 1 \\ -x_2 + 2x_3 = 0 \end{cases} .$$

Write its associated augmented matrix. Reduce the matrix to its row-echelon form. Use the procedure to solve the system.

$$\left(\begin{array}{cccc|c} 2 & -1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & 1 \\ 0 & -1 & 2 & 0 & 0 \end{array}\right)$$

Now, you have some choices how to reduce this augmented matrix to its row-echelon form. For example, you can multiply row 2 by -1*, and replace row 1 with row 2. Then add* $-2\times$ *row 1 to row 2. Then multiply row 2 by* $\frac{1}{3}$*. Then add row 2 to row 3. Then multiply row 3 by* $\frac{3}{4}$*. This is (one of many possible) row-echelon form:*

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & 0 & -1 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} \end{array}\right)$$

To solve the system using this row-echelon form:
Start with the last row. Let $x_4 = r$ *for any real number* r*.*

$$x_3 = -\frac{1}{4}r + \frac{1}{2}$$

$$x_2 = \frac{2}{3}x_3 - \frac{1}{3}x_4 + \frac{2}{3} = -\frac{1}{2}r + 1$$

$$x_1 = 2x_2 - x_3 - 1 = -\frac{3}{4}r + \frac{1}{2}$$

Problem 5. *(25 pts.)*

(a) *Let:*

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 5 \\ -4 & 2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 3 \\ 5 & 1 & 0 \end{pmatrix}.$$

Compute $A+B$, AB, B^t, $\det(A)$, $\det(A^t)$ *and* $\det(3A)$.

$$A+B = \begin{pmatrix} 4 & 1 & 3 \\ -1 & 0 & 8 \\ 1 & 3 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 18 & 4 & 3 \\ 26 & 4 & -3 \\ -5 & 3 & 6 \end{pmatrix}$$

$$B^t = \begin{pmatrix} 2 & -1 & 5 \\ 0 & 1 & 1 \\ 0 & 3 & 0 \end{pmatrix}$$

$\det(A) = \det(A^t) = -54$

$\det(3A) = (3^3)(-54)$

(b) *Use elementary operations to find the inverse of:*

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

Therefore,

$$C^{-1} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Problem 6. *(10 pts.) Use Cramer's rule to solve the following linear system:*

$$\begin{cases} 2x + y = 1 \\ x + 2y + z = 0 \\ y + 2z = 0 \end{cases} .$$

$$\begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 4$$

$$x = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}}{4} = \frac{3}{4}$$

$$y = \frac{\begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{vmatrix}}{4} = \frac{-2}{4} = -\frac{1}{2}$$

$$z = \frac{\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{vmatrix}}{4} = \frac{1}{4}$$

Problem 7. *(20 pts.)*

(a) *Let* $L : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ *be the linear transformation defined by* $L(\boldsymbol{x}) = A\boldsymbol{x}$*, where*

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 5 \\ 3 & 2 & 4 \end{pmatrix}.$$

Is the vector $(1, 2, 3)$ *in the range of* L*?*

The question is the same as solving the system $A\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

This system has the solution $\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

(b) *Let* $L : \boldsymbol{R}^2 \to \boldsymbol{R}^3$ *be defined by* $L(x, y) = (2x + 3y, -2x + 3y, x + y)$*. Find the standard matrix representing* L*.*

$$A = \begin{pmatrix} 2 & 3 \\ -2 & 3 \\ 1 & 1 \end{pmatrix}.$$

Then $L(\boldsymbol{x}) = A\boldsymbol{x}$*, where* $\boldsymbol{x} = (x, y)$*.*