

Lorenz and horseshoe knots

Ilya Kofman

College of Staten Island and The Graduate Center
City University of New York (CUNY)

Includes joint work with Joan Birman

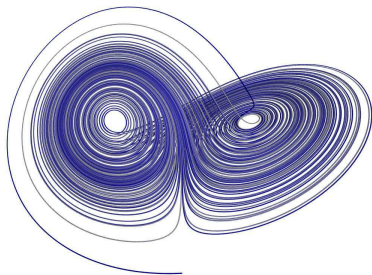
March 26, 2013

A tale of two dynamical systems

In 1963, the meteorologist Edward Lorenz discovered the first “strange attractor” in a system of ODEs:

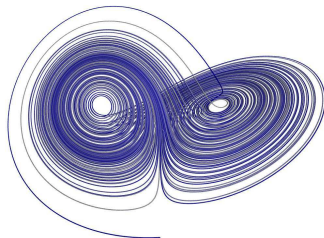
$$\frac{dx}{dt} = 10(y - x), \quad \frac{dy}{dt} = 28x - y - xz, \quad \frac{dz}{dt} = xy - \frac{8}{3}z$$

All orbits head for a region in $\mathbb{R}^3$ called the *Lorenz attractor*. Wildly different orbits may start at nearby points, but they will converge to the attractor.



Lorenz attractor

$$\frac{dx}{dt} = 10(y - x), \quad \frac{dy}{dt} = 28x - y - xz, \quad \frac{dz}{dt} = xy - \frac{8}{3}z$$



Chaotic behavior:

Very sensitive to initial conditions.
Popularized the *butterfly effect*.

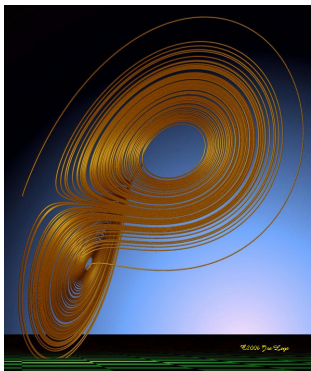
Robust dynamical system:

If we perturb the ODE coefficients,
the orbits still approach strange attractor.

Lorenz template

Guckenheimer and Williams introduced the *Lorenz template* – an embedded branched surface in $\mathbb{R}^3$ with a semi-flow.

Tucker rigorously justified this geometric model for the original Lorenz parameters.

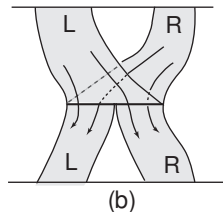
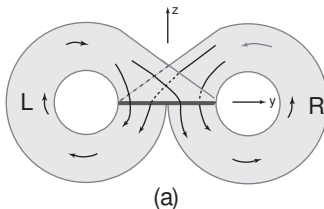
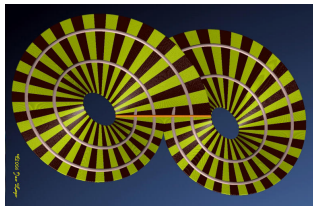


Lorenz attractor



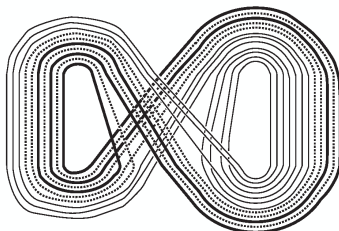
Lorenz template

A *Lorenz knot* is a periodic orbit in the Lorenz flow on $\mathbb{R}^3$.



Three orbits described by their words in R and L :

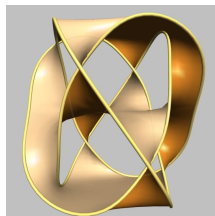
Thick line	$W_1 = LRLRL$	$(2, 3)$ -torus knot
Dotted line	$W_2 = LRLRLRL$	$(3, 5)$ -torus knot
Thin line	$W_3 = LRLRRRLRRR$	$(-2, 3, 7)$ -pretzel knot



What types of knots can occur?

Birman and Williams (1983) showed Lorenz knots and links are peculiar:

- Lorenz knots are prime.
- Lorenz links are fibered.
- Non-trivial Lorenz links have positive signature.



Ghys showed among 1,701,936 prime knots with ≤ 16 crossings, only **20** appear as Lorenz knots, with only **7** of those non-torus knots:

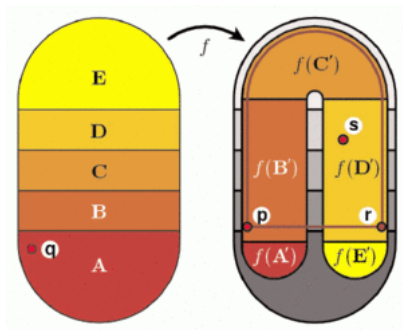
$3_1, 5_1, 7_1, 9_1, 8_{19}, 10_{124}, 11a_{367}, 12n_{242}, 12n_{725}, 13a_{4878}, 14n_{6022}, 14n_{21324}$
 $14n_{21881}, 15n_{41185}, 15a_{85263}, 16n_{184868}, 16n_{771177}, 16n_{783154}, 16n_{996934}$

Smale Horseshoe

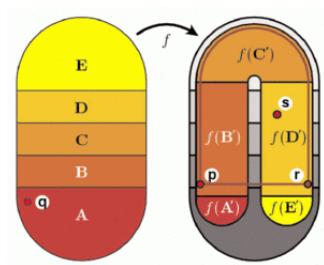
The Smale horseshoe is the “hallmark of chaos.”

Consider $f : D^2 \rightarrow D^2$ as shown below. Rectangles B , D are stretched vertically and shrunk horizontally by f , and D is also rotated.

The horseshoe is the invariant set $\Lambda = \{z : f^n(z) \in B \cup D \ \forall n \in \mathbb{Z}\}$. It is a product of two Cantor sets in $[0, 1]$.



Smale Horseshoe



There are fixed points p, q, s , stable and unstable manifolds:

$$W^s = \{z : f^n(z) \rightarrow p \text{ as } n \rightarrow \infty\} \quad W^u = \{z : f^n(z) \rightarrow p \text{ as } n \rightarrow -\infty\}$$

The point $r \in W^s \cap W^u$ is called a “homoclinic point.”

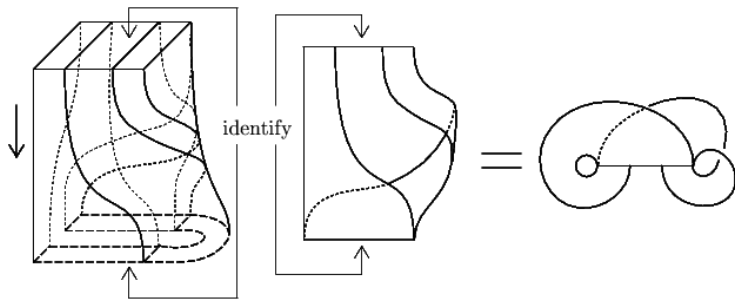
Thm. (Smale) If $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is any diffeomorphism with a transverse homoclinic point r , then for some $N > 0$, f^N has a horseshoe containing r

transverse homoclinic point $\Rightarrow$ horseshoe $\Rightarrow$ chaotic dynamics

These are now called hyperbolic dynamical systems.

Horseshoe template

Suspending the horseshoe map f yields a flow on the mapping torus $D^2 \times S^1$, embedded as a flow in $\mathbb{R}^3$. Holmes and Williams obtained the *horseshoe template* by collapsing each leaf of the stable (contracting) foliation to a point:



The horseshoe template is isotopic to the Lorenz-like template $\mathcal{L}(0, 1)$.

Subtemplates

A subtemplate is a branched submanifold (with boundary) for which the original semiflow restricts to an expanding semiflow.

M. Sullivan (1993) showed:

$$\begin{array}{ccccccccccc} \cdots & \subset & \mathcal{L}(0,4) & \subset & \mathcal{L}(0,2) & \subset & \mathcal{L}(0,0) & \subset & \mathcal{L}(0,-2) & \subset & \mathcal{L}(0,-4) & \subset \cdots \\ & & & & & & & & \cap & & & \\ \cdots & \subset & \mathcal{L}(0,5) & \subset & \mathcal{L}(0,3) & \subset & \mathcal{L}(0,1) & \subset & \mathcal{L}(0,-1) & \subset & \mathcal{L}(0,-3) & \subset \cdots \end{array}$$

Ghrist (1995) constructed a *universal template* whose periodic orbits contain all knots. Generically, “most” chaotic flows contain all knots, including all $\mathcal{L}(0, n)$ for $n < 0$, such as $\mathcal{L}(0, -1)$:

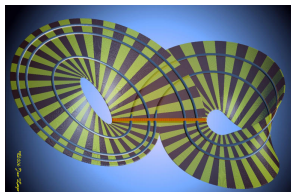
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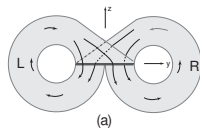


Symbolic dynamics

Smale (1967) used symbolic dynamics to study dynamics on the horseshoe, which are completely described by dynamical properties of the shift map on the space of bi-infinite sequences of two symbols.

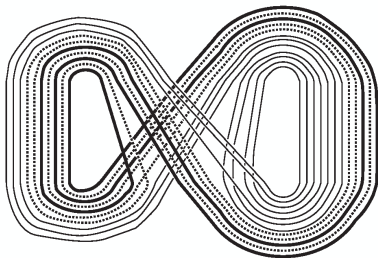
Symbolic dynamics

Lorenz knots were first studied in 1980's with symbolic dynamics using the Lorenz template.



Three orbits described by their words in R and L :

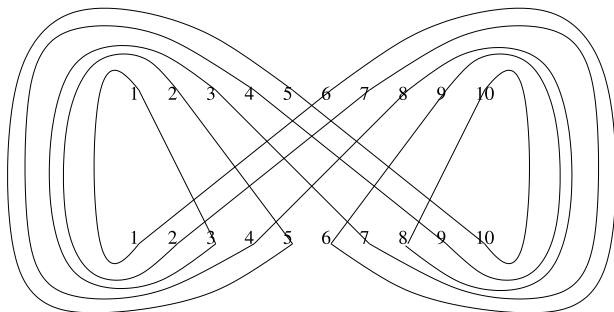
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How to construct the orbit
from the word:

Write all cyclic permutations
of the words, and order them by
lexicographical order $L < R$

$W_1 =$	$LRRLRLRLR$	4
$W_2 =$	$RRLRLRLR$	9
$W_3 =$	$RLLRLRLR$	6
$W_4 =$	$LLRLRLRLR$	1
$W_5 =$	$LRRLRLRLR$	3
$W_6 =$	$RLLRLRLR$	7
$W_7 =$	$LLRLRLRLR$	2
$W_8 =$	$LRRLRLRLR$	5
$W_9 =$	$RRLRLRLR$	10
$W_{10} =$	$RLRLRLRLR$	8

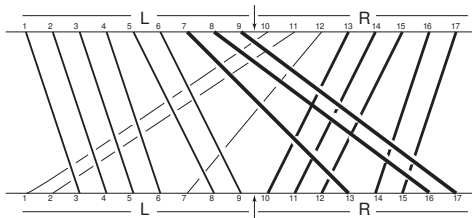
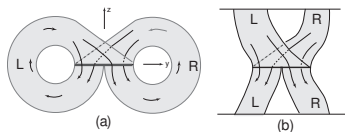


Lorenz braids

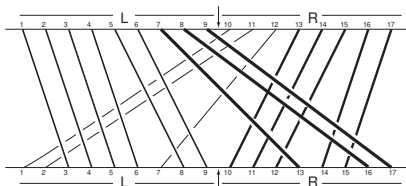
A *Lorenz braid* L is a positive braid lying on the Lorenz braid template.

Every Lorenz link is a closed Lorenz braid.

Any two strands of L cross at most once, and overcrossing strands never cross each other. So the *Lorenz vector* $\vec{d}_L = \langle d_1, \dots, d_p \rangle$ determines L .



$$\vec{d}_L = \langle 2, 2, 2, 2, 3, 3, 6, 8, 8 \rangle = \langle 2^4, 3^2, 6^1, 8^2 \rangle$$



$$\vec{d}_L = \langle 2, 2, 2, 2, 3, 3, 6, 8, 8 \rangle = \langle 2^4, 3^2, 6^1, 8^2 \rangle$$

If $W = \prod_{i=1}^t R^{n_i} L^{m_i}$ then t is the *trip number* $t(L)$. Using $\vec{d}_L$,

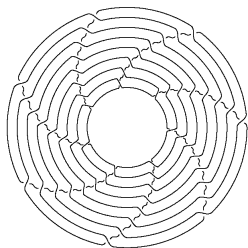
$$t(L) = \#\{ i \mid i + d_i > p \text{ where } 1 \leq i \leq p \}$$

Braid index of ℓ is the trip number of any Lorenz braid of ℓ (Williams).

In the example above, $p = 9$, $d_p = 8$, so $n = p + d_p = 17$ is the braid index of the Lorenz braid L . The trip number $t = 3$ is the braid index of ℓ .

Similarly, the genus of ℓ is determined by $2g = c - n + 2 - \mu$.

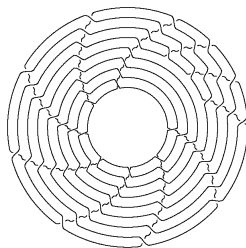
Torus links, twisted torus links, and T-links



$T(9, 7)$

$(9, 7)$ torus knot

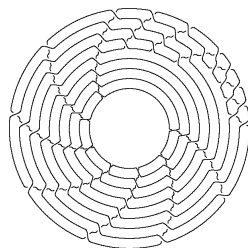
$$(\sigma_1 \sigma_2 \dots \sigma_8)^7$$



$T((5, 3), (9, 7))$

$(9, 7)$ torus knot
with type $(5, 3)$ twist

$$(\sigma_1 \sigma_2 \sigma_3 \sigma_4)^3 (\sigma_1 \sigma_2 \dots \sigma_8)^7$$



$T((3, 4), (5, 3), (9, 7))$

$(9, 7)$ torus knot with
 $(5, 3)$ and $(3, 4)$ twists

$$(\sigma_1 \sigma_2)^4 (\sigma_1 \sigma_2 \sigma_3 \sigma_4)^3 \cdot (\sigma_1 \sigma_2 \dots \sigma_8)^7$$

T-links with different roots

$\beta \in B_n$ is a *positive root* if $\beta^n = \Delta_n^2$, the full twist in B_n .

There are 2^{n-2} isotopy classes of positive n^{th} roots in B_n , e.g.

$$\begin{aligned}\delta_n &= \sigma_1 \cdots \sigma_{n-1} \quad (\text{the standard root}) \\ \bar{\delta}_n &= \sigma_{n-1} \cdots \sigma_1\end{aligned}$$

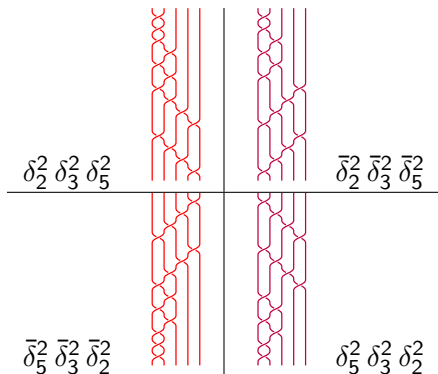
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Let $2 \leq r_1 < \cdots < r_k$ and $s_i \neq 0$ be integers for all $i = 1 \dots k$.

Define the T-link $T((r_1, s_1, \beta_1), \dots, (r_k, s_k, \beta_k))$ to be the the closure of $\beta_1^{s_1} \cdots \beta_k^{s_k} \in B_{r_k}$, each β_i is an r_i -th positive root on the first r_i strands.

If all $s_i > 0$, we call it a positive T-link.

If $|r_{i+1} - r_i| = 1$ for all $i = 1 \dots k - 1$, we call it a full T-link.

Lorenz links and T-links

Theorem 1 (Birman-K.)

Lorenz links are exactly the set of positive T-links with standard roots.

Precisely:

If ℓ is a closed Lorenz braid $L \in B_{p+r_k}$ with $2 \leq r_1 < \dots < r_k$,

$$d_L = \langle d_1, \dots, d_p \rangle = \langle r_1^{s_1}, \dots, r_k^{s_k} \rangle$$

then ℓ is the T-link $T((r_1, s_1, \delta_{r_1}), \dots, (r_k, s_k, \delta_{r_k}))$ given by

$$\delta_{r_1}^{s_1} \delta_{r_2}^{s_2} \dots \delta_{r_k}^{s_k} =$$

$$(\sigma_1 \sigma_2 \dots \sigma_{r_1-1})^{s_1} (\sigma_1 \sigma_2 \dots \sigma_{r_2-1})^{s_2} \dots (\sigma_1 \sigma_2 \dots \sigma_{r_k-1})^{s_k} \in B_{r_k}$$

Lorenz links and T-links

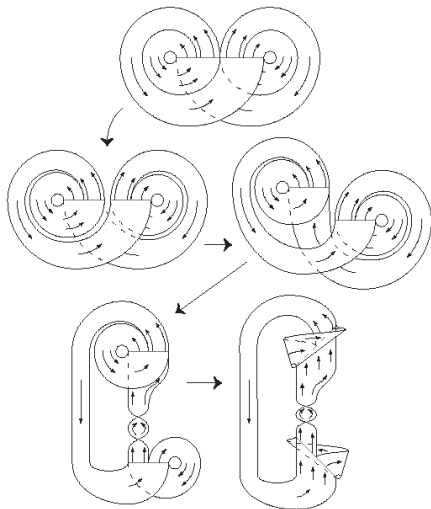
Theorem 1 (Birman-K.)

Lorenz links are exactly the set of positive T-links with standard roots.

Idea of proof:

Cut open Lorenz template
to create a braid template.

Use braid isotopy to get
precise 1-1 correspondence
between positive Lorenz braids
and positive T-braids.



Horseshoe links and T-links

Theorem 2

Horseshoe links are exactly the set of full positive T-links with $\bar{\delta}_n$ roots.

Proof: By the proof of Theorem 1, every link on $\mathcal{L}(0, 1)$ is the closure of $\beta \Delta_{r_k}$ where $\beta = \delta_{r_1}^{s_1} \cdots \delta_{r_k}^{s_k} \in B_{r_k}$ is a Lorenz T-braid.

So using $\Delta_n \bar{\delta}_n = \delta_n \Delta_n$, we get

$$\begin{aligned} \beta \Delta_{r_k} &= \delta_{r_1}^{s_1} \cdots \delta_{r_k}^{s_k} \Delta_{r_k} \\ &= \bar{\delta}_2 \cdots \bar{\delta}_{r_1-1} \bar{\delta}_{r_1}^{s_1+1} \bar{\delta}_{r_1+1} \cdots \bar{\delta}_{r_k}^{s_k+1} \end{aligned}$$

Thus for any $r \geq 2$ and any $s_i \geq 1$ for $i = 2 \dots r$,

$$\mathcal{H} = T((2, s_2, \bar{\delta}_2), (3, s_3, \bar{\delta}_3), \dots, (r, s_r, \bar{\delta}_r))$$

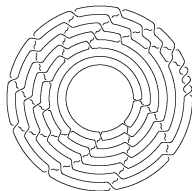
For any horseshoe link $\mathcal{H} = T((2, s_2, \bar{\delta}_2), (3, s_3, \bar{\delta}_3), \dots, (r, s_r, \bar{\delta}_r))$, let $\vec{d}_{\mathcal{H}} = \langle 2^{s_2}, 3^{s_3}, \dots, r^{s_r} \rangle$.

Then $\mathcal{H}$ has a *Lorenz ghost* $\mathcal{G}$, given by $\vec{d}_{\mathcal{H}}$:

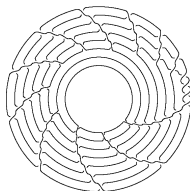
$$\mathcal{G} = T((2, s_2, \delta_2), (3, s_3, \delta_3), \dots, (r, s_r, \delta_r))$$

For any horseshoe link $\mathcal{H}$, its minimal braid index t , minimal crossing number c , and minimal braid $\mathbb{M}$ are determined by its Lorenz ghost $\mathcal{G}$. But the number of components of $\mathcal{H}$ and $\mathcal{G}$ are often different.

e.g., $\vec{d}_{\mathcal{H}} = \langle 2^3, 3, 4, 5, 6^4, 7, 8^3 \rangle$



$\mathcal{G}$ is a 3-component Lorenz link



$\mathcal{H}$ is a horseshoe knot

$$\mathcal{H} = T((2, s_2, \bar{\delta}_2), (3, s_3, \bar{\delta}_3), \dots, (r, s_r, \bar{\delta}_r))$$

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For any horseshoe link $\mathcal{H}$, its minimal braid index t , minimal crossing number c , and minimal braid $\mathbb{M}$ are determined by its Lorenz ghost $\mathcal{G}$. But the number of components of $\mathcal{H}$ and $\mathcal{G}$ are often different.

Theorem 3 $t(\mathcal{H}) = t(\mathcal{G})$ and $c(\mathcal{H}) = c(\mathcal{G})$.

Moreover, $t \leq r \leq 2t - 1$ and for knots $c \geq 3t(t - 1)/2$.

From the $(t, \vec{n}, \vec{m})$ data for $\mathcal{G}$, the minimal braids are as follows:

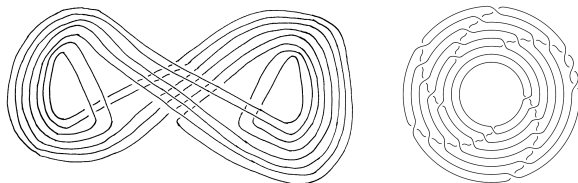
$$\mathbb{M}_{\mathcal{G}} = \Delta_t^2 \prod_{i=1}^{t-1} (\sigma_1 \cdots \sigma_i)^{n_i} \prod_{i=1}^{t-1} (\sigma_{t-1} \cdots \sigma_{t-i})^{m_i}$$

$$\mathbb{M}_{\mathcal{H}} = \Delta_t^2 \prod_{i=1}^{t-1} (\sigma_1 \cdots \sigma_{t-i})^{n_{t-i}} \prod_{i=1}^{t-1} (\sigma_i \cdots \sigma_1)^{m_i}$$

Example

The Lorenz link ℓ given by $\vec{d}_\ell = \langle 2, 2, 6, 6, 6, 8, 8 \rangle$ has $t = 5$.

ℓ is the T-link $T((2, 2, \delta_2), (6, 3, \delta_6), (8, 2, \delta_8))$, with T-braid $\mathbb{T} = \delta_2^2 \delta_6^3 \delta_8^2$.

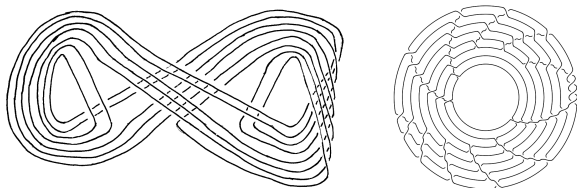


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Adding a half-twist to the right lobe of the Lorenz template, from ℓ we get horseshoe knot $\mathcal{H}$. By Theorem 1, $\mathcal{H}$ is given by the 8-braid $\mathbb{H} = \Delta_8 \mathbb{T}$.



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By Theorem 2, $\vec{d}_{\mathcal{H}} = \langle 2, 2, 2, 3, 4, 5, 6, 6, 6, 6, 7, 8, 8, 8 \rangle$. Thus,

$$\mathcal{H} = T((2, 3, \bar{\delta}_2), (3, 1, \bar{\delta}_3), (4, 1, \bar{\delta}_4), (5, 1, \bar{\delta}_5), (6, 4, \bar{\delta}_6), (7, 1, \bar{\delta}_1), (8, 3, \bar{\delta}_8))$$

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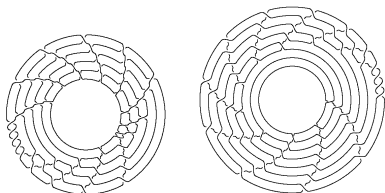
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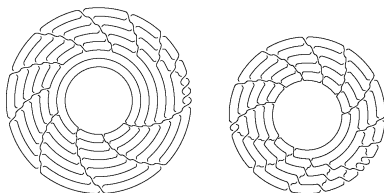
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By Theorem 3, minimal braid for $\mathcal{H}$ is determined by Lorenz data for

$$\mathcal{G} = T((2, 3, \delta_2), (3, 1, \delta_3), (4, 1, \delta_4), (5, 1, \delta_5), (6, 4, \delta_6), (7, 1, \delta_1), (8, 3, \delta_8))$$



$\mathcal{G}$ is a 3-component Lorenz link



$\mathcal{H}$ is a hyperbolic Lorenz knot

Previously, Holmes and Williams showed that torus knots $T(p, q)$, $p < q$, which are horseshoe knots must satisfy $3p \leq 2q$. Now, we can tabulate all Horseshoe links for given bounds on braid index and crossing number.

Corollary

For any given (t, c) , a complete enumeration of Horseshoe links is given by closed braids determined by $\vec{d}$ such that $t \leq r \leq 2t - 1$ and $(c - t) \leq S - 2r \leq 2(c - t)$, where S is the sum of the entries of $\vec{d}$.

Results for low crossing number:

For $c \leq 16$, there are only 14 horseshoe knots, three of which are hyperbolic:

$$P(-2, 3, 7), \quad T((2, 6), (3, 4)), \quad T((2, 6), (3, 5))$$

For $c \leq 24$, there are only 33 horseshoe knots, 13 of which are hyperbolic.

Lorenz horseshoe links

All our experiments suggest the following:

Conjecture: Every horseshoe link is Lorenz.

The horseshoe template $\mathcal{L}(0, 1)$ is non-orientable, so it cannot embed as a subtemplate of the Lorenz template $\mathcal{L}(0, 0)$, which is orientable. So the conjecture can only be proved “non-uniformly” for each link.

Stronger conjecture:

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Infinite families of Lorenz horseshoe examples:

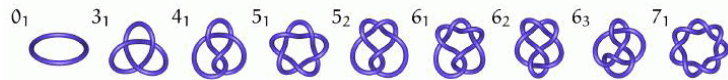
$T(p, q)$ with $3p \leq 2q$,

$T(3, q, 2, s)$ with $q, s \geq 1$

In these examples, $\mathcal{H} = \mathcal{G}$ but that's usually not true.

Simplest hyperbolic knots

Historically, complexity for knots measured by crossing number:



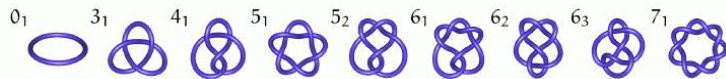
For a hyperbolic knot K , a natural measure of **geometric complexity** is the number of ideal tetrahedra in $S^3 - K$.

Hyperbolic knots with geometric complexity up to 6 tetrahedra were found by Callahan-Dean-Weeks (1999), extended to 7 tetrahedra by Champanerkar-Kofman-Paterson (2004). REU project by Tim Mullen to extend this census to 8 tetrahedra (2011).

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Knots	0	1	2	4	22	43	129	301	502

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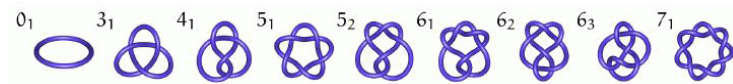
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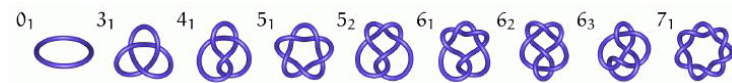
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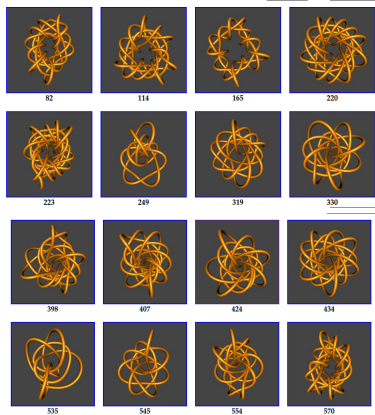
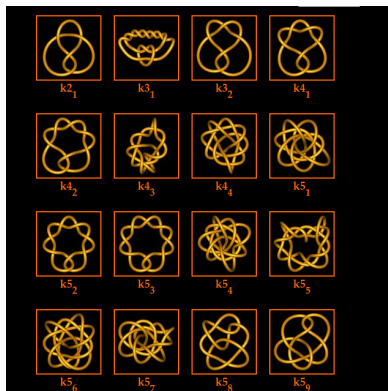
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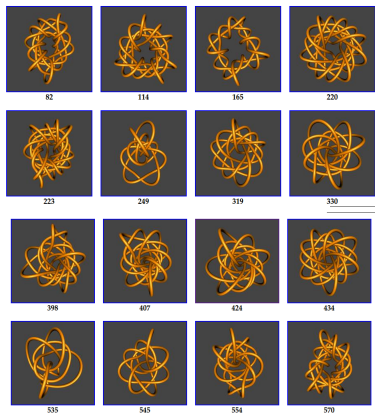
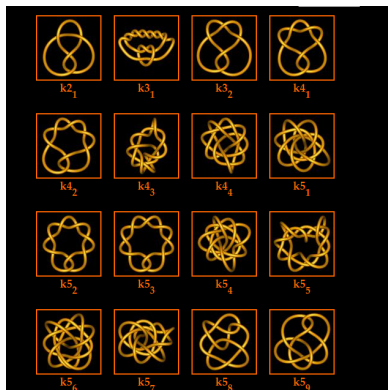
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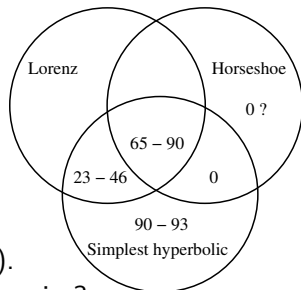
Of the 201 knots with ≤ 7 tetrahedra, 108 are Lorenz, 90 are not Lorenz, and we do not know about the other 3 knots.

Of the 201 knots with ≤ 7 tetrahedra, 65 are horseshoe knots.
Moreover, **all 65 of the horseshoe knots are Lorenz!**

At least 23 of the 108 Lorenz knots are not horseshoe, and we do not know about the other 20 Lorenz knots.

Question:

Knots in center of Venn diagram are
periodic orbits of dynamical systems:
Horseshoe, Lorenz and modular flow (Ghys).
Is “geometrically simple” related to the dynamics?



Upper bounds on hyperbolic volume

Theorem. (Champanerkar-Futer-K-Neumann-Purcell, 2011)

Let $T(p, q, r, s)$ be a (p, q) -torus knot twisted with δ_r^s or $\bar{\delta}_r^s$. Then

$$\text{Vol}(T(p, q, r, s)) < 10v_3 \quad \text{if } r = 2,$$

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We proved similar volume bounds for all T-links, thus for all Lorenz and horseshoe links. Even though need many parameters to specify such a link, just the initial number of strands of $T(p, q)$ being twisted ($= r$) suffices to bound its volume.

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Question: Which Lorenz and horseshoe knots are hyperbolic?

Non-examples: $T(3, 4, 2, 2) = T(3, 5)$, $T(6, 5, 3, 2) = T(5, 7)$, etc.

Image credits

- p.2 www.ecopolis.org/wp-content/uploads/2008/04/lorenz_attractor.jpg
- p.4,9 Jos Leys, *AMS Feature Column*
- p.6 Produced with *SeifertView* by Jarke J. van Wijk
- p.7 Bill Casselman, *AMS Notices*
- p.8 Robert Ghrist, Philip Holmes, Michael Sullivan, *Knots and links in three-dimensional flows*
- p.14,19 Produced with latex code for closed braids by Volker Gebhardt
- p.15 Produced with PGF/TikZ code for braids by Andrew Stacey
- p.16 Pierre Dehornoy, thesis
- p.22 Dror Bar-Natan, *Knot Atlas*
- p.23 Rob Sharein, *KnotPlot*