

Chapter 25

Turaev Surfaces

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25.1 Introduction

The two most famous knot invariants, the Alexander polynomial (1923) and the Jones polynomial (1984), mark paradigm shifts in knot theory. After each polynomial was discovered, associating new structures to knot diagrams played a key role in understanding its basic properties. Using the Seifert surface, Seifert showed, for example, how to obtain knots with a given Alexander polynomial. For the Jones polynomial, even the simplest version of that problem remains open: Does there exist a non-trivial knot with trivial Jones polynomial?

Kauffman gave a state sum for the Jones polynomial, with terms for each of the 2^c states of a link diagram with c crossings. Turaev constructed a closed orientable surface from any pair of dual states with opposite markers.

For the Jones polynomial, the Turaev surface is a rough analog to the Seifert surface for the Alexander polynomial. For a given knot diagram, the Seifert genus and the Turaev genus are computed by separate algorithms to obtain each surface from the diagram. The invariants for a given knot K are defined as the minimum genera among all the respective surfaces for K . The Seifert genus is a topological measure of how far a given knot is from being unknotted. The Turaev genus is a topological measure of how far a given knot is from being alternating. (See [26], which discusses alternating distances.) For any alternating diagram, Seifert's algorithm produces the minimal genus Seifert surface. For any adequate diagram, Turaev's algorithm produces the minimal genus Turaev surface. Extending the analogy, we can determine the Alexander polynomial and the Jones polynomial of K from associated algebraic structures on the respective surfaces of K : the Seifert matrix for the Alexander polynomial, and the A -ribbon graph on the Turaev surface for the Jones polynomial.

The analogy is historical, as well. Like the Seifert surface for the Alexander polynomial, the Turaev surface was constructed to prove a fundamental conjecture related to the Jones polynomial. Tait conjectured that an alternating link always has an alternating diagram that has minimal crossing number among all diagrams for that link. A proof had to wait about a century until the Jones polynomial led to several new ideas used to prove Tait's Conjecture [21, 29, 35]. Turaev's later proof in [36] introduced Turaev surfaces and prompted interest in studying their properties.

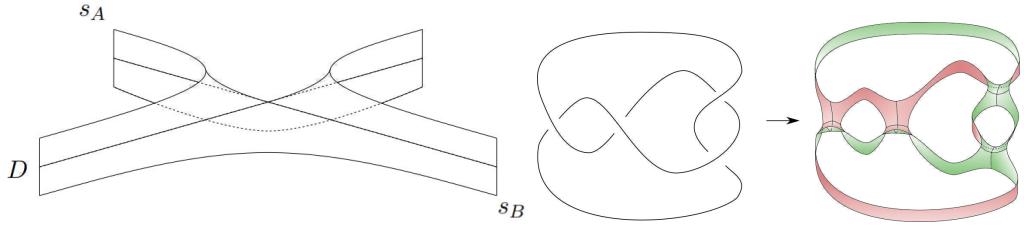


FIGURE 25.1: We put a saddle at each crossing of D to obtain a cobordism between the states s_A and s_B .

25.2 What Is the Turaev Surface?

Let D be the diagram of a link L drawn on S^2 . For any crossing \times , we obtain the A –smoothing as \diagup and the B –smoothing as \diagdown . The state s of D is a choice of smoothing at every crossing, resulting in a disjoint union of circles on S^2 . Let $|s|$ denote the number of circles in s . Let s_A be the all– A state, for which every crossing of D has an A –smoothing. Similarly, s_B is the all– B state. We will construct the Turaev surface from the dual states s_A and s_B .

At every crossing of D , we put a saddle surface which bounds the A –smoothing on the top and the B –smoothing on the bottom as shown in Figure 25.1. In this way, we get a cobordism between s_A and s_B , with the link projection Γ at the level of the saddles. The *Turaev surface* $F(D)$ is obtained by attaching $|s_A| + |s_B|$ discs to all boundary circles. See Figure 25.1 and [15] for an animation of the Turaev surface for the Borromean link.

The *Turaev genus* of D is the genus of $F(D)$, given by

$$g_T(D) = g(F(D)) = (c(D) + 2 - |s_A| - |s_B|)/2.$$

The Turaev genus $g_T(L)$ of any non-split link L is the minimum of $g_T(D)$ among all diagrams D of L . By [36, 12], L is alternating if and only if $g_T(L) = 0$, and if D is an alternating diagram then $F(D) = S^2$. In general, for any link diagram D , it follows that (see [12]):

1. $F(D)$ is a Heegaard surface of S^3 ; i.e., an unknotted closed orientable surface in S^3 .
2. D is alternating on $F(D)$, and the faces of D can be checkerboard colored on $F(D)$, with discs for s_A and s_B colored white and black, respectively.
3. $F(D)$ has a Morse decomposition, with D and crossing saddles at height zero, and the $|s_A|$ and $|s_B|$ discs as maxima and minima, respectively.

Conversely, in [5], conditions were given for a Heegaard surface with cellularly embedded alternating diagram on it to be a Turaev surface.

25.3 The Turaev Surface and the Jones Polynomial

A diagram D is A -adequate if at each crossing, the two arcs of s_A from that crossing are in different state circles. In other words, $|s_A| > |s|$ for any state s with exactly one B -smoothing. Similarly, we define a B -adequate diagram by reversing the roles of A and B above. If D is both A -adequate and B -adequate it is called *adequate*. If D is neither A -adequate nor B -adequate it is called *inadequate*. A link L is *adequate* if it has an adequate diagram, and is *inadequate* if all its diagrams are inadequate. Any reduced alternating diagram is adequate, hence every alternating link is adequate.

Adequacy implies that s_A and s_B contribute the extreme terms $\pm t^\alpha$ and $\pm t^\beta$ of the Jones polynomial $V_L(t)$, which determine the span $V_L(t) = |\alpha - \beta|$, which is a link invariant. Let $c(L)$ be the minimal crossing number among all diagrams for L . In [36], Turaev proved

$$\text{span } V_L(t) \leq c(L) - g_T(L)$$

with equality if L is adequate. If D is a prime non-alternating diagram, then $g_T(D) > 0$. Thus, $\text{span } V_L(t) = c(L)$ if and only if L is alternating, from which Tait's Conjecture follows.

Therefore, for any adequate link L with an adequate diagram D (see [1]),

$$g_T(L) = g_T(D) = \frac{1}{2} (c(D) - |s_A(D)| - |s_B(D)|) + 1 = c(L) - \text{span } V_L(t).$$

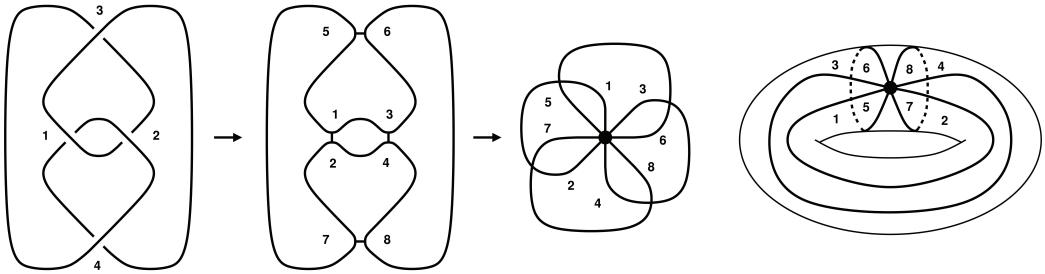
So for the connect sum $L \# L'$ of adequate links, $g_T(L \# L') = g_T(L) + g_T(L')$.

Turaev genus and knot homology. Khovanov homology and knot Floer homology categorify the Jones polynomial and the Alexander polynomial, respectively. The *width* of each bigraded knot homology, $w_{KH}(K)$ and $w_{HF}(K)$, is the number of diagonals with non-zero homology. The Turaev genus bounds the width of both knot homologies [10, 27]:

$$w_{KH}(K) - 2 \leq g_T(K) \quad \text{and} \quad w_{HF}(K) - 1 \leq g_T(K). \quad (25.1)$$

For adequate knots, $w_{KH}(K) - 2 = g_T(K)$ [1]. These inequalities have been used to obtain families of knots with unbounded Turaev genus (see [8]).

Ribbon graph invariants. Like the Seifert surface, the Turaev surface provides much more information than its genus. An oriented *ribbon graph* is a graph with an embedding in an oriented surface, such that its faces are discs. Turaev's construction determines an oriented ribbon graph G_A on $F(D)$: We add an edge for every crossing in s_A , and collapse each state circle of s_A to a vertex of G_A , preserving the cyclic order of edges given by the checkerboard coloring (see [8]).

FIGURE 25.2: Ribbon graph G_A for an inadequate diagram of the trefoil.

If L is alternating, then $V_L(t) = T_G(-t, -1/t)$, where $T_G(x, y)$ is the Tutte polynomial [35]. For any L , $V_L(t)$ is a specialization of the Bollobás–Riordan–Tutte polynomial of G_A [12]. These ideas extend to virtual links and non-orientable ribbon graphs [11]. In [13], a unified description is given for all these knot and ribbon graph polynomial invariants.

25.4 Turaev Genus One Links

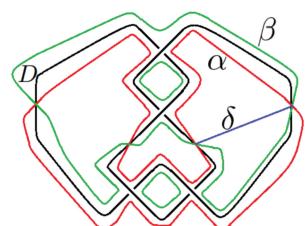
The Seifert genus is directly computable for alternating and positive links, and has been related to many classical invariants. Moreover, knot Floer homology detects the Seifert genus of knots. In contrast, for most non-adequate links, computing the Turaev genus is an open problem.

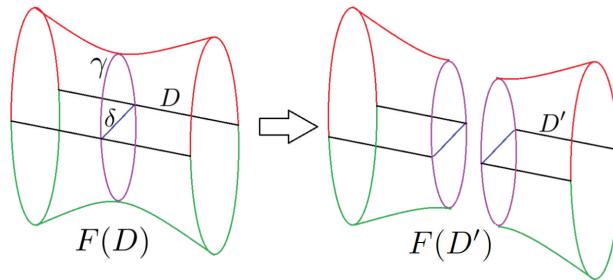
The Turaev genus of a link can be computed when the upper bounds in the inequalities (25.1) or those in [26] match the Turaev genus of a particular diagram, which gives a lower bound. So it is useful to know which diagrams realize a given Turaev genus. Link diagrams with Turaev genus one and two were classified in [6, 22].

This classification uses the decomposition of any prime, connected link diagram $D \subset S^2$ into alternating tangles. As in [22], an edge in D is *non-alternating* when it joins two overpasses or two underpasses. If D is non-alternating, we can isotope the state circles in s_A and s_B to intersect exactly at the midpoints of all non-alternating edges of D . In the figure shown to the right, $\alpha \in s_A$, $\beta \in s_B$. The arc δ joining the points in $\alpha \cap \beta$ is called a *cutting arc* of D ([22]).

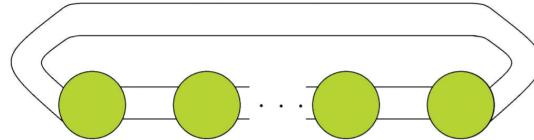
A cutting arc is the intersection of S^2 with a compressing disc of the Turaev surface $F(D)$, which intersects D at the endpoints of δ .

The boundary γ of this compressing disc is called a *cutting loop*. Every cutting arc of D has a corresponding cutting loop on $F(D)$, and surgery of D along a cutting arc corresponds to surgery of $F(D)$ along a compressing disc, as shown in the following figure.

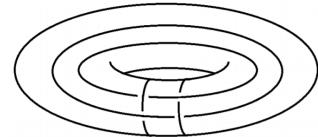




If D' is obtained by surgery from D , the surgered surface is its Turaev surface $F(D')$ with genus $g_T(D') = g_T(D) - 1$. So if $g_T(D) = 1$, then γ is a meridian of the torus $F(D)$, and surgery along all cutting arcs of D cuts the diagram into alternating 2-tangles [22]. Hence, if $g_T(D) = 1$, then D is a cycle of alternating 2-tangles:



This also implies that for any alternating diagram D on its Turaev surface $F(D)$, if $g_T(D) \geq 1$ there is an essential simple loop γ on F which intersects D twice and bounds a disc in a handlebody bounded by F . Thus, the link on the surface in Example 1.3.1 of [26] cannot come from Turaev's construction. However, this condition is not sufficient; for example, the diagram at right satisfies the condition, but cannot be a Turaev surface because any planar diagram D for this link has more than four crossings, which would remain as crossings on $F(D)$.



Hayashi [16] and Ozawa [30] considered more general ways to quantify the complexity of the pair (F, D) , which has prompted recent interest in *representativity* of knots (see, e.g., [4, 7, 18, 24, 31, 32, 33]).

25.5 Open Problems

Below, we consider open problems in two broad categories:

Question 1. How do you determine the Turaev genus of a knot or link?

Does the Turaev genus always equal the dealternating number of a link? This is true in many cases, and no lower bounds are known to distinguish these invariants (see [26]).

The lower bounds (25.1) vanish for quasi-alternating links. For any $g > 1$, does there exist a quasi-alternating link with Turaev genus g ?

The Turaev genus is additive under connect sum for adequate knots, and invariant under mutation if the diagram is adequate [1]. In general, for any K and K' , is $g_T(K \# K') = g_T(K) + g_T(K')$? If K and K' are mutant knots, is $g_T(K) = g_T(K')$? The latter question is open even for adequate knots; if D is a non-adequate diagram of an adequate knot K , then for a mutant D' of D , it might be possible that $g_T(K) < g_T(D) = g_T(D') = g_T(K')$.

If K is a positive knot with Seifert genus $g(K)$, then $g_T(K) \leq g(K)$. Is this inequality strict; i.e., is $g_T(K) < g(K)$ for a positive knot? It is known to be strict for $g(K) = 1, 2$ [20, 34] and for adequate positive knots [25].

In general, how do you compute the Turaev genus, which is a link invariant, without using link diagrams? Is it determined by some other link invariants?

Question 2. How do you characterize the Turaev surface?

From the construction in Section 25.2, it is hard to tell whether a given pair (F, D) is a Turaev surface. The existence of a cutting loop implies that the alternating diagram D on F must have minimal complexity; i.e., there exists an essential simple loop on F which intersects D twice. But this condition is not sufficient. What are the sufficient conditions for a given pair (F, D) to be a Turaev surface?

Alternating, almost alternating, and toroidally alternating knots have been characterized topologically using a pair of spanning surfaces in the knot complement [14, 17, 19, 23]. Turaev genus one knots are toroidally alternating, and they contain almost-alternating knots, but they have not been characterized topologically as a separate class of knots. What is a topological characterization of Turaev genus one knots, or generally, of knots with any given Turaev genus?

Any non-split, prime, alternating link in S^3 is hyperbolic, unless it is a closed 2-braid [28]. This result was recently generalized to links in a thickened surface $F \times I$. If the link L in $F \times I$ admits a diagram on F which is alternating, cellularly embedded, and satisfies an appropriate generalization of “prime,” then $(F \times I) - L$ is hyperbolic [2, 9, 18]. Now, for a given Turaev surface $F(D)$, let L be a link in $F(D) \times I$ which projects to the alternating diagram on F . It follows that typically the complement $(F(D) \times I) - L$ is hyperbolic, assuming there are no essential annuli. If $g_T(D) = 1$ then $(F(D) \times I) - L$ has finite hyperbolic volume. If $g_T(D) > 1$, then there is a well-defined finite volume if the two boundaries are totally geodesic (see [3]). How do the geometric invariants of $(F(D) \times I) - L$ depend on the original diagram D in S^2 ?

25.6 Bibliography

- [1] Tetsuya Abe. The Turaev genus of an adequate knot. *Topology Appl.*, 156(17):2704–2712, 2009.
- [2] C. Adams, C. Albors-Riera, B. Haddock, Z. Li, D. Nishida, B. Reinoso, and L. Wang. Hyperbolicity of links in thickened surfaces. *Topology Appl.*, 256:262–278, 2019.

- [3] Colin Adams, Or Eisenberg, Jonah Greenberg, Kabir Kapoor, Zhen Liang, Kate O’connor, Natalia Pacheco-Tallaj, and Yi Wang. TG-Hyperbolicity of virtual links, arXiv:1912.09435 [math.GT], 2019.
- [4] Román Aranda, Seungwon Kim, and Maggy Tomova. Representativity and waist of cable knots. *J. Knot Theory Ramifications*, 27(4):1850025, 8, 2018.
- [5] Cody Armond, Nathan Druivenga, and Thomas Kindred. Heegaard diagrams corresponding to Turaev surfaces. *J. Knot Theory Ramifications*, 24(4):1550026, 14, 2015.
- [6] Cody W Armond and Adam M Lowrance. Turaev genus and alternating decompositions. *Algebraic & Geometric Topology*, 17(2):793–830, 2017.
- [7] Ryan Blair and Makoto Ozawa. Height, trunk and representativity of knots. *J. Math. Soc. Japan*, 71(4):1105–1121, 2019.
- [8] Abhijit Champanerkar and Ilya Kofman. A survey on the Turaev genus of knots. *Acta Math. Viet.*, 39(4):497–514, 2014.
- [9] Abhijit Champanerkar, Ilya Kofman, and Jessica S. Purcell. Geometry of biperiodic alternating links. *J. Lond. Math. Soc. (2)*, 99(3):807–830, 2019.
- [10] Abhijit Champanerkar, Ilya Kofman, and Neal Stoltzfus. Graphs on surfaces and Khovanov homology. *Algebr. Geom. Topol.*, 7:1531–1540, 2007.
- [11] Sergei Chmutov. Generalized duality for graphs on surfaces and the signed Bollobás-Riordan polynomial. *J. Combin. Theory Ser. B*, 99(3):617–638, 2009.
- [12] Oliver T. Dasbach, David Futer, Efstratia Kalfagianni, Xiao-Song Lin, and Neal W. Stoltzfus. The Jones polynomial and graphs on surfaces. *J. Combin. Theory Ser. B*, 98(2):384–399, 2008.
- [13] Joanna A. Ellis-Monaghan and Iain Moffatt. *Graphs on surfaces: Dualities, polynomials, and knots*. Springer Briefs in Mathematics. Springer, New York, 2013.
- [14] Joshua Evan Greene. Alternating links and definite surfaces. *Duke Mathematical Journal*, 166(11):2133–2151, 2017.
- [15] M. Hajij. Turaev Surface Borromean rings - REMIX. [YouTube video]. <http://www.youtube.com/watch?v=j431ionQD9w>, 2010.
- [16] Chuichiro Hayashi. Links with alternating diagrams on closed surfaces of positive genus. *Math. Proc. Cambridge Philos. Soc.*, 117(1):113–128, 1995.
- [17] Joshua Howie. A characterisation of alternating knot exteriors. *Geometry & Topology*, 21(4):2353–2371, 2017.
- [18] Joshua A. Howie and Jessica S. Purcell. Geometry of alternating links on surfaces. *Trans. Amer. Math. Soc.*, 373(4):2349–2397, 2020.
- [19] Tetsuya Ito. A characterization of almost alternating knots. *Journal of Knot Theory and Its Ramifications*, 27(01):1850009, 2018.
- [20] In Dae Jong and Kengo Kishimoto. On positive knots of genus two. *Kobe J. Math.*, 30(1-2):1–18, 2013.

- [21] L. Kauffman. State models and the Jones polynomial. *Topology*, 26(3):395–407, 1987.
- [22] Seungwon Kim. Link diagrams with low Turaev genus. *Proceedings of the American Mathematical Society*, 146(2):875–890, 2018.
- [23] Seungwon Kim. A topological characterization of toroidally alternating knots, arXiv:1608.00521 [math.GT], 2016.
- [24] Thomas Kindred. Alternating links have representativity 2. *Algebr. Geom. Topol.*, 18(6):3339–3362, 2018.
- [25] Sang Youl Lee, Chan-Young Park, and Myoungsoo Seo. On adequate links and homogeneous links. *Bulletin of the Australian Mathematical Society*, 64(3):395–404, 2001.
- [26] Adam Lowrance. Alternating distances of knots, to appear in *A Concise Encyclopedia of Knot Theory*.
- [27] Adam M. Lowrance. On knot Floer width and Turaev genus. *Algebr. Geom. Topol.*, 8(2):1141–1162, 2008.
- [28] William Menasco. Closed incompressible surfaces in alternating knot and link complements. *Topology*, 23(1):37–44, 1984.
- [29] Kunio Murasugi. Jones polynomials and classical conjectures in knot theory. *Topology*, 26(2):187–194, 1987.
- [30] Makoto Ozawa. Non-triviality of generalized alternating knots. *Journal of Knot Theory and Its Ramifications*, 15(03):351–360, 2006.
- [31] Makoto Ozawa. Bridge position and the representativity of spatial graphs. *Topology and its Applications*, 159(4):936–947, 2012.
- [32] Makoto Ozawa. The representativity of pretzel knots. *Journal of Knot Theory and Its Ramifications*, 21(02):1250016, 2012.
- [33] John Pardon. On the distortion of knots on embedded surfaces. *Annals of mathematics*, pages 637–646, 2011.
- [34] A Stoimenow. Knots of genus one or on the number of alternating knots of given genus. *Proceedings of the American Mathematical Society*, 129(7):2141–2156, 2001.
- [35] M. Thistlethwaite. A spanning tree expansion of the Jones polynomial. *Topology*, 26(3):297–309, 1987.
- [36] V. G. Turaev. A simple proof of the Murasugi and Kauffman theorems on alternating links. *Enseign. Math. (2)*, 33(3-4):203–225, 1987.