Homework 2

Topology I, Math 70800, Spring 2018

Instructor: Ilya Kofman

Due: Thursday Feb 22 before class

Reading [1]

1. Read about the idea and motivation for homology, pages 97-101.
2. Proof of Proposition 2.9 on page 111, Section 2.1.
3. Proof of the Five-Lemma on page 129.
4. Read about *Barycentric Subdivision of Simplices* from pages 119 - 120, and *Iterated Barycentric Subdivision* from page 123.
5. Read about the *Naturality of Exact sequences* on page 127.

Problems

1. (a) For an exact sequence $A \xrightarrow{f} B \to C \to D \xrightarrow{g} E$ show that $C = 0$ iff f is surjective and g is injective.

   (b) Using this prove that the inclusion $A \xrightarrow{i} X$ induces isomorphisms on all homology groups iff $H_n(X, A) = 0$ for all n.

2. Let $r : X \to A$ be a retraction and let $i : A \to X$ be the inclusion map. Show that $i_* : H_*(A) \to H_*(X)$ is a monomorphism onto a direct summand.

3. Show that chain homotopy of chain maps is an equivalence relation.

4. If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ is exact, then f is surjective if and only if h is surjective.

5. (a) If $0 \to A \to B \to C \to 0$ is a short exact sequence of of vector spaces and linear maps, then show that $\dim \mathrm{B} = \dim \mathrm{A} + \dim \mathrm{C}$.

   (b) If $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated abelian groups, then show that $\mathrm{rank\ B} = \mathrm{rank\ A} + \mathrm{rank\ C}$. (Hint: Extend a maximally independent subset of A to a maximally independent subet of B).

[1] *Algebraic Topology* by Allen Hatcher, at `http://www.math.cornell.edu/~hatcher/AT/ATpage.html`

(c) If $0 \to A_n \to A_{n-1} \to \ldots \to A_1 \to A_0 \to 0$ is an exact squence of finitely generated abelian groups, then $\sum_{i=0}^{n}(-1)^i \mathrm{rank}\ \mathrm{A_i} = 0$.

6. For each of the following exact sequences say as much as possible about the abelian group G and/or the unknown homomorphism α.

   (a) $0 \to \mathbb{Z} \to G \to \mathbb{Z} \to 0$

   (b) $0 \to \mathbb{Z} \to G \to \mathbb{Z}_2 \to 0$

   (c) $0 \to \mathbb{Z} \to G \to \mathbb{Z}_n \to 0$

   (d) $0 \to \mathbb{Z}_4 \to G \to \mathbb{Z}_2 \to 0$

   (e) $0 \to \mathbb{Z}_{p^m} \to G \to \mathbb{Z}_{p^n} \to 0$

   (f) $0 \to \mathbb{Z}_3 \to G \to \mathbb{Z}_2 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to 0$

   (g) $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_2 \to 0$

   (h) $0 \to G \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_2 \to 0$

7. Show that $H_0(X, A) = 0$ iff A meets eah path-component of X.

8. Compute the relative homology groups for the following pairs (X, A).

   (a) $X = S^2$ and $A = \{a_1, \ldots, a_n\}$

   (b) $X = T^2$ and $A =$meridian.

   (c) $X = \mathbb{R}$ and $A = \mathbb{Q}$.

   (d) $X = T^2$ and $A =$ meridian and longitude.

9. Compute the homology of space X gives below, by finding a homotopy equivalent space (usually a deformation retract) Y whose homology you have computed. Justify the homotopy equivalence (or deformation retract).

   (a) X is orientable surface of genus g with b boundary components.

   (b) X is non-orientable surface of genus g with b boundary components.

   (c) $X = \mathbb{R}^3 - \{(0,0,z) | z \in \mathbb{R}\}$

   (d) $X = \mathbb{R}^3 - \big(\bigcup_{i=0}^{n} \{(i,0,z) | z \in \mathbb{R}\}\big)$

   (e) $X = \mathbb{R}^3 - \{(x,y,0) | x^2 + y^2 = 1\}$

   (f) X is a torus with n meridional disks attached (a meridional disk a disk which bounds a meridian inside the torus).

   (g) $X = \mathbb{R}^3 - \big(\{(0,0,z) | z \in \mathbb{R}\} \cup \{(x,y,0) | x^2 + y^2 = 1\}\big)$

Hand-in: 2, 5c, 8ad, 9fg