Homework 1

Topology I, Math 70800, Spring 2018

Instructor: Ilya Kofman

Due: Thursday Feb 8 before class

Reading [1]

1. Read Chapter 0, Section *Cell Complexes* pages 5-8. Read examples of cell structures on orientable and non-orientable surfaces, the spheres S^n, the real and complex projective space $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$.
2. Read at least one proof of classification of surfaces, available at: `http://www.math.csi.cuny.edu/~ikofman/topology_GC_F15.html`
3. Read section on "Collapsing subspaces" and Examples 0.7, 0.8 and 0.9 on pages 11-12 in Chapter 0.
4. An important ingredient in collapsing and deforming spaces is the following property: If (X, A) is a CW pair consisting of a CW complex X and a contractible subcomplex A , then the quotient map $X \to X/A$ is a homotopy equivalence. Read the proof of this in Propositions 0.16 and 0.17 on page 15-16 in Chapter 0.

Problems

1. Let X be the space obtained from S^2 by attaching n 2-cells along any collection of n disjoint circles in S^2. Show that $X \simeq \bigvee_{n+1} S^2$.
2. Let G be a graph with v vertices and e edges. (a) Show that $G \simeq \bigvee_n S^1$ and find n in terms of v and e. (b) Show that if $G \simeq \bigvee_m S^1$, then $m = n$.

A compact surface S with boundary is a 2-manifold with boundary. Let $\overline{S}$ be the closed surface obtained by gluing disks to the boundary of S. Then the genus of S is the genus of the closed surface $\overline{S}$. You can model a surface with boundary as the standard surface model with the interiors of disjoint discs deleted.

3. (a) Find the genus of the cylinder and the Mobius strip.

[1] All section, chapter, page and example numbers refer to the book "Algebraic Topology" by Allen Hatcher freely available at `http://www.math.cornell.edu/~hatcher/AT/ATpage.html`

(b) Show that the torus with a disc removed (punctured torus) is homotopic to the figure eight.

(c) Let S be a compact surface with boundary with genus g and b boundary components. Show that $S \simeq \bigvee_n S^1$ and find n in terms of g and b.

4. Identify the following surfaces from the given surface symbols.
(a) $abca^{-1}b^{-1}c^{-1}$ (b) $abca^{-1}db^{-1}c^{-1}d^{-1}$ (c) $ae^{-1}a^{-1}bdb^{-1}cc$

5. Let X be a CW complex.

   (a) Show that attaching a 2-cell to X adds a relation to $\pi_1(X)$.

   (b) Show that attaching an n-cell to X for $n \geq 3$ does not change the fundamental group.

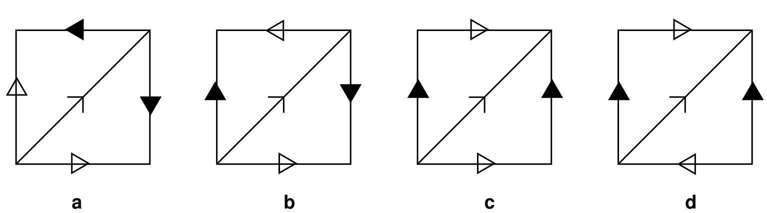
6. (a) Show that the space obtained by identifying the edges of a polygon in pairs is a closed surface.

   (b) Show that the space obtained by identifying pairs of edges of a finite collection of disjoing 2-simplicies is a closed surface.

   (c) Show that the edges in (b) can always be oriented so as to define a Δ-complex structure on the resulting surface. (harder!)

7. Find a Δ-complex structure and use it to compute the simplicial homology for the following spaces:
(a) $X = \vee_n S^1$ (b) $X = \vee_n S^2$
(c) X obtained by identifying the verticies of Δ^2 to a point.

8. (a) Identify the surfaces given below, and (b) Compute their simplicial homology.



Hand-in: Problems 3c, 5, 7b, 8(a).