4.1 Vectors

Vectors are mathematical objects that can be added, and multiplied by numbers, subject to certain rules. The real numbers are the simplest example of vectors, and the rules for sums and multiples of any vectors are just the following properties of sums and multiples of numbers:

$$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u} \qquad 1\mathbf{u}=\mathbf{u}$$
$$\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w} \qquad a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$$
$$\mathbf{u}+\mathbf{0}=\mathbf{u} \qquad (a+b)\mathbf{u}=a\mathbf{u}+b\mathbf{u}$$
$$\mathbf{u}+(-\mathbf{u})=\mathbf{0} \qquad a(b\mathbf{u})=(ab)\mathbf{u}.$$

These rules obviously hold when $a,b,1,\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{0}$ are all numbers, and $\mathbf{0}$ is the ordinary zero.

They also hold when $\mathbf{u},\mathbf{v},\mathbf{w}$ are *points in the plane* $\mathbb{R}^2$, if we interpret $\mathbf{0}$ as $(0,0)$, $+$ as the *vector sum* defined for $\mathbf{u}=(u_1,u_2)$ and $\mathbf{v}=(v_1,v_2)$ by

$$(u_1,u_2)+(v_1+v_2)=(u_1+v_1,u_2+v_2),$$

and $a\mathbf{u}$ as the *scalar multiple* defined by

$$a(u_1,u_2)=(au_1,au_2).$$

The vector sum is geometrically interesting, because $\mathbf{u}+\mathbf{v}$ is the fourth vertex of a parallelogram formed by the points $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$ (Figure 4.1).

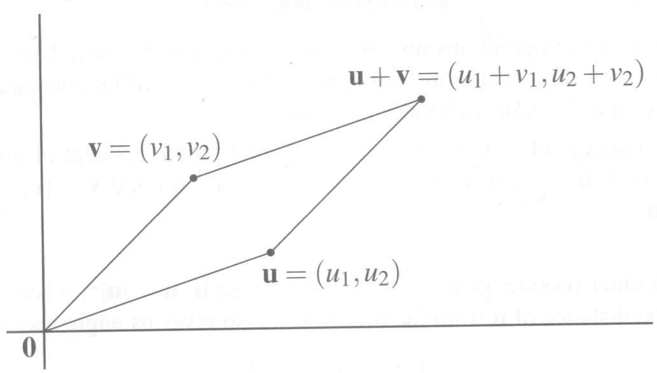

Figure 4.1: The parallelogram rule for vector sum

In fact, the rule for forming the sum of two vectors is often called the "parallelogram rule."

Scalar multiplication by a is also geometrically interesting, because it represents magnification by the factor a. It magnifies, or *dilates*, the whole plane by the factor a, transforming each figure into a similar copy of itself. Figure 4.2 shows an example of this with $a=2.5$.

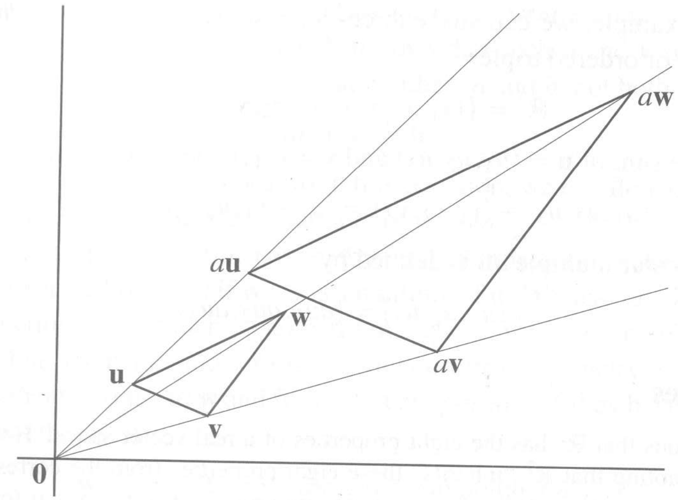

Figure 4.2: Scalar multiplication as a dilation of the plane

Real vector spaces

It seems that the operations of vector addition and scalar multiplication capture some geometrically interesting features of a space. With this in mind, we define a *real vector space* to be a set V of objects, called *vectors*, with operations of vector addition and scalar multiplication satisfying the following conditions:

- If $\mathbf{u}$ and $\mathbf{v}$ are in V, then so are $\mathbf{u}+\mathbf{v}$ and $a\mathbf{u}$ for any real number a.
- There is a *zero vector* $\mathbf{0}$ such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$ for each vector $\mathbf{u}$. Each $\mathbf{u}$ in V has a *additive inverse* $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
- Vector addition and scalar multiplication on V have the eight properties listed at the beginning of this section.

It turns out that real vector spaces are a natural setting for Euclidean geometry. We must introduce extra structure, which is called the *inner product*, before we can talk about length and angle. But once the inner product is there, we can prove all theorems of Euclidean geometry, often more efficiently than before. Also, we can uniformly extend geometry to *any number of dimensions* by considering the space $\mathbb{R}^n$ of *ordered n-tuples* of real numbers $(x_1, x_2, \ldots, x_n)$.

For example, we can study three-dimensional Euclidean geometry in the space of ordered triples

$$\mathbb{R}^3 = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\},$$

where the sum of $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is defined by

$$(u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

and the scalar multiple $a\mathbf{u}$ is defined by

$$a(u_1, u_2, u_3) = (au_1, au_2, au_3).$$

Exercises

It is obvious that $\mathbb{R}^2$ has the eight properties of a real vector space. However, it is worth noting that $\mathbb{R}^2$ "inherits" these eight properties from the corresponding properties of real numbers. For example, the property $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (called the *commutative law*) for vector addition is inherited from the corresponding commutative law for number addition, $u + v = v + u$, as follows:

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (u_1, u_2) + (v_1 + v_2) \\ &= (u_1 + v_1, u_2 + v_2) \quad \text{by definition of vector addition} \\ &= (v_1 + u_1, v_2 + u_2) \quad \text{by commutative law for numbers} \\ &= (v_1, v_2) + (u_1, u_2) \quad \text{by definition of vector addition} \\ &= \mathbf{v} + \mathbf{u}. \end{aligned}$$

4.1.1 Check that the other seven properties of a vector space for $\mathbb{R}^2$ are inherited from corresponding properties of $\mathbb{R}$.

4.1.2 Similarly check that $\mathbb{R}^n$ has the eight properties of a vector space.

The term "dilation" for multiplication of all vectors in $\mathbb{R}^2$ (or $\mathbb{R}^n$ for that matter) by a real number a goes a little beyond the everyday meaning of the word in the case when a is smaller than 1 or negative.

4.1.3 What is the geometric meaning of the transformation of $\mathbb{R}^2$ when every vector is multiplied by -1? Is it a rotation?

4.1.4 Is it a rotation of $\mathbb{R}^3$ when every vector is multiplied by -1?

4.2 Direction and linear independence

Vectors give a concept of *direction* in $\mathbb{R}^2$ by representing lines through $\mathbf{0}$. If $\mathbf{u}$ is a nonzero vector, then the real multiples $a\mathbf{u}$ of $\mathbf{u}$ make up the line through $\mathbf{0}$ and $\mathbf{u}$, so we call them the points "in direction $\mathbf{u}$ from $\mathbf{0}$." (You may prefer to say that $-\mathbf{u}$ is in the direction *opposite* to $\mathbf{u}$, but it is simpler to associate direction with a whole line, rather than a half line.)

Nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, therefore, have *different directions from* $\mathbf{0}$ if neither is a multiple of the other. It follows that such $\mathbf{u}$ and $\mathbf{v}$ are *linearly independent*; that is, there are no real numbers a and b, not both zero, with

$$a\mathbf{u} + b\mathbf{v} = \mathbf{0}.$$

Because, if one of a, b is not zero in this equation, we can divide by it and hence express one of $\mathbf{u}$, $\mathbf{v}$ as a multiple of the other.

The concept of direction has an obvious generalization: $\mathbf{w}$ *has direction* $\mathbf{u}$ *from* $\mathbf{v}$ (or *relative to* $\mathbf{v}$) if $\mathbf{w} - \mathbf{v}$ is a multiple of $\mathbf{u}$. We also say that "$\mathbf{w} - \mathbf{v}$ has direction $\mathbf{u}$," and there is no harm in viewing $\mathbf{w} - \mathbf{v}$ as an abbreviation for the line segment from $\mathbf{v}$ to $\mathbf{w}$. As in coordinate geometry, we say that line segments from $\mathbf{v}$ to $\mathbf{w}$ and from $\mathbf{s}$ to $\mathbf{t}$ are *parallel* if they have the same direction; that is, if

$$\mathbf{w} - \mathbf{v} = a(\mathbf{t} - \mathbf{s}) \quad \text{for some real number } a \neq 0.$$

Figure 4.3 shows an example of parallel line segments, from $\mathbf{v}$ to $\mathbf{w}$ and from $\mathbf{s}$ to $\mathbf{t}$, both of which have direction $\mathbf{u}$.

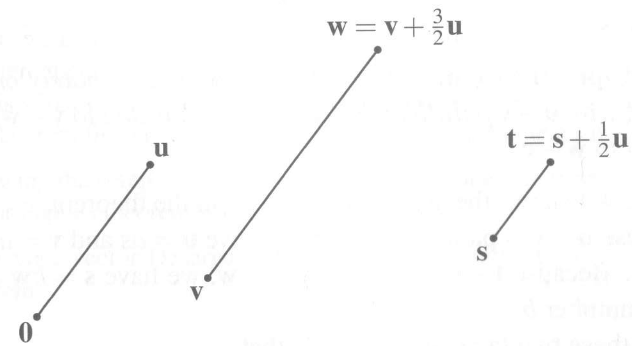

Figure 4.3: Parallel line segments with direction **u**

Here we have

$$\mathbf{w} - \mathbf{v} = \frac{3}{2}\mathbf{u} \quad \text{and} \quad \mathbf{t} - \mathbf{s} = \frac{1}{2}\mathbf{u}, \quad \text{so} \quad \mathbf{w} - \mathbf{v} = 3(\mathbf{t} - \mathbf{s}).$$

Now let us try out the vector concept of parallels on two important theorems from previous chapters. The first is a version of the Thales theorem that parallels cut a pair of lines in proportional segments.

Vector Thales theorem. *If* $\mathbf{s}$ *and* $\mathbf{v}$ *are on one line through* $\mathbf{0}$, $\mathbf{t}$ *and* $\mathbf{w}$ *are on another, and* $\mathbf{w}-\mathbf{v}$ *is parallel to* $\mathbf{t}-\mathbf{s}$, *then* $\mathbf{v}=a\mathbf{s}$ *and* $\mathbf{w}=a\mathbf{t}$ *for some number* a.

If $\mathbf{w}-\mathbf{v}$ is parallel to $\mathbf{t}-\mathbf{s}$, then

$$\mathbf{w}-\mathbf{v}=a(\mathbf{t}-\mathbf{s})=a\mathbf{t}-a\mathbf{s} \quad \text{for some real number } a.$$

Because $\mathbf{v}$ is on the same line through $\mathbf{0}$ as $\mathbf{s}$, we have $\mathbf{v}=b\mathbf{s}$ for some b, and similarly $\mathbf{w}=c\mathbf{t}$ for some c (this is a good moment to draw a picture). It follows that

$$\mathbf{w}-\mathbf{v}=c\mathbf{t}-b\mathbf{s}=a\mathbf{t}-a\mathbf{s},$$

and therefore,

$$(c-a)\mathbf{t}+(a-b)\mathbf{s}=\mathbf{0}.$$

But $\mathbf{s}$ and $\mathbf{t}$ are in different directions from $\mathbf{0}$, hence linearly independent, so

$$c-a=a-b=0.$$

Thus, $\mathbf{v}=a\mathbf{s}$ and $\mathbf{w}=a\mathbf{t}$, as required. □

As in axiomatic geometry (Exercise 1.4.3), the Pappus theorem follows from the Thales theorem. However, "proportionality" is easier to handle with vectors.

Vector Pappus theorem. *If* $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ *lie alternately on two lines through* $\mathbf{0}$, *with* $\mathbf{u}-\mathbf{v}$ *parallel to* $\mathbf{s}-\mathbf{r}$ *and* $\mathbf{t}-\mathbf{s}$ *parallel to* $\mathbf{v}-\mathbf{w}$, *then* $\mathbf{u}-\mathbf{t}$ *is parallel to* $\mathbf{w}-\mathbf{r}$.

Figure 4.4 shows the situation described in the theorem.

Because $\mathbf{u}-\mathbf{v}$ is parallel to $\mathbf{s}-\mathbf{r}$, we have $\mathbf{u}=a\mathbf{s}$ and $\mathbf{v}=a\mathbf{r}$ for some number a. Because $\mathbf{t}-\mathbf{s}$ is parallel to $\mathbf{v}-\mathbf{w}$, we have $\mathbf{s}=b\mathbf{w}$ and $\mathbf{t}=b\mathbf{v}$ for some number b.

From these two facts, we conclude that

$$\mathbf{u}=a\mathbf{s}=ab\mathbf{w} \quad \text{and} \quad \mathbf{t}=b\mathbf{v}=ba\mathbf{r},$$

hence,

$$\mathbf{u}-\mathbf{t}=ab\mathbf{w}-ba\mathbf{r}=ab(\mathbf{w}-\mathbf{r}),$$

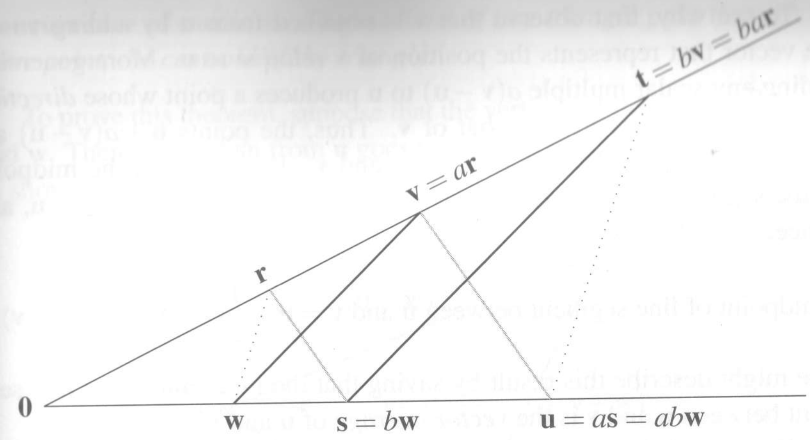

Figure 4.4: The parallel Pappus configuration, labeled by vectors

and therefore, $\mathbf{u}-\mathbf{t}$ is parallel to $\mathbf{w}-\mathbf{r}$. □

The last step in this proof, where we exchange ba for ab, is of course a trifle, because $ab=ba$ for any real numbers a and b. But it is a big step in Chapter 6, where we try to develop geometry without numbers. There we have to build an arithmetic of line segments, and the Pappus theorem is crucial in getting multiplication to behave properly.

Exercises

In Chapter 1, we mentioned that a second theorem about parallels, the Desargues theorem, often appears alongside the Pappus theorem in the foundations of geometry. This situation certainly holds in vector geometry, where the appropriate Desargues theorem likewise follows from the vector Thales theorem.

4.2.1 Following the setup explained in Exercise 1.4.4, and the formulation of the vector Pappus theorem above, formulate a "vector Desargues theorem."

4.2.2 Prove your vector Desargues theorem with the help of the vector Thales theorem.

4.3 Midpoints and centroids

The definition of a real vector space does not include a definition of distance, but we can speak of the midpoint of the line segment from $\mathbf{u}$ to $\mathbf{v}$ and, more generally, of the point that divides this segment in a given ratio.

To see why, first observe that $\mathbf{v}$ is obtained from $\mathbf{u}$ by adding $\mathbf{v}-\mathbf{u}$, the vector that represents the position of $\mathbf{v}$ *relative* to $\mathbf{u}$. More generally, adding any scalar multiple $a(\mathbf{v}-\mathbf{u})$ to $\mathbf{u}$ produces a point whose *direction* relative to $\mathbf{u}$ is the same as that of $\mathbf{v}$. Thus, the points $\mathbf{u}+a(\mathbf{v}-\mathbf{u})$ are precisely those on the line through $\mathbf{u}$ and $\mathbf{v}$. In particular, the midpoint of the segment between $\mathbf{u}$ and $\mathbf{v}$ is obtained by adding $\frac{1}{2}(\mathbf{v}-\mathbf{u})$ to $\mathbf{u}$, and hence,

$$\text{midpoint of line segment between } \mathbf{u} \text{ and } \mathbf{v} = \mathbf{u}+\frac{1}{2}(\mathbf{v}-\mathbf{u}) = \frac{1}{2}(\mathbf{u}+\mathbf{v}).$$

One might describe this result by saying that the midpoint of the line segment between $\mathbf{u}$ and $\mathbf{v}$ is the *vector average* of $\mathbf{u}$ and $\mathbf{v}$.

This description of the midpoint gives a very short proof of the theorem from Exercise 2.2.1, that the diagonals of a parallelogram bisect each other. By choosing one of the vertices of the parallelogram at $\mathbf{0}$, we can assume that the other vertices are at $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$ (Figure 4.5).

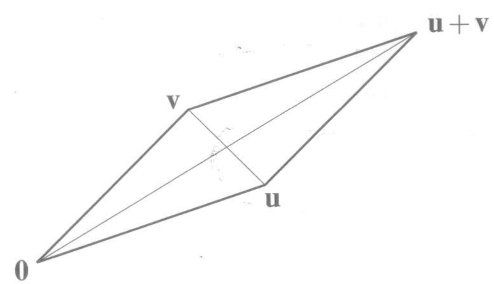

Figure 4.5: Diagonals of a parallelogram

Then the midpoint of the diagonal from $\mathbf{0}$ to $\mathbf{u}+\mathbf{v}$ is $\frac{1}{2}(\mathbf{u}+\mathbf{v})$. And, by the result just proved, this is also the midpoint of the other diagonal—the line segment between $\mathbf{u}$ and $\mathbf{v}$. □

The vector average of two or more points is physically significant because it is the *barycenter* or *center of mass* of the system obtained by placing equal masses at the given points. The geometric name for this vector average point is the *centroid*.

In the case of a triangle, the centroid has an alternative geometric description, given by the following classical theorem about *medians*: the lines from the vertices of a triangle to the midpoints of the respective opposite sides.

Concurrence of medians. *The medians of any triangle pass through the same point, the centroid of the triangle.*

To prove this theorem, suppose that the vertices of the triangle are $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Then the median from $\mathbf{u}$ goes to the midpoint $\frac{1}{2}(\mathbf{v}+\mathbf{w})$, and so on, as shown in Figure 4.6.

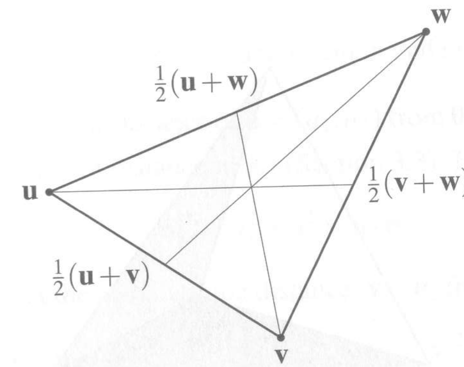

Figure 4.6: The medians of a triangle

Looking at this figure, it seems likely that the medians meet at the point 2/3 of the way from $\mathbf{u}$ to $\frac{1}{2}(\mathbf{v}+\mathbf{w})$, that is, at the point

$$\mathbf{u}+\frac{2}{3}\left(\frac{1}{2}(\mathbf{v}+\mathbf{w})-\mathbf{u}\right) = \mathbf{u}+\frac{1}{3}(\mathbf{v}+\mathbf{w})-\frac{2}{3}\mathbf{u} = \frac{1}{3}(\mathbf{u}+\mathbf{v}+\mathbf{w}).$$

Voilà! This is the centroid, and a similar argument shows that it lies 2/3 of the way between $\mathbf{v}$ and $\frac{1}{2}(\mathbf{u}+\mathbf{w})$ and 2/3 of the way between $\mathbf{w}$ and $\frac{1}{2}(\mathbf{u}+\mathbf{v})$. That is, the centroid is the common point of all three medians. □

You can of course check by calculation that $\frac{1}{3}(\mathbf{u}+\mathbf{v}+\mathbf{w})$ lies 2/3 of the way between $\mathbf{v}$ and $\frac{1}{2}(\mathbf{u}+\mathbf{w})$ and also 2/3 of the way between $\mathbf{w}$ and $\frac{1}{2}(\mathbf{u}+\mathbf{v})$. But the smart thing is not to *do* the calculation but to *predict the result*. We know that calculating the point 2/3 of the way between $\mathbf{u}$ and $\frac{1}{2}(\mathbf{v}+\mathbf{w})$ gives

$$\frac{1}{3}(\mathbf{u}+\mathbf{v}+\mathbf{w}),$$

a result that is unchanged when we permute the letters $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. The other two calculations are the same, except for the ordering of the letters $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Hence, they lead to the same result.

Exercises

4.3.1 Show that a square with vertices $\mathbf{t}, \mathbf{u}, \mathbf{v}, \mathbf{w}$ has center $\frac{1}{4}(\mathbf{t}+\mathbf{u}+\mathbf{v}+\mathbf{w})$.

The theorem about concurrence of medians generalizes beautifully to three dimensions, where the figure corresponding to a triangle is a *tetrahedron*: a solid with four vertices joined by six lines that bound the tetrahedron's four triangular faces (Figure 4.7).

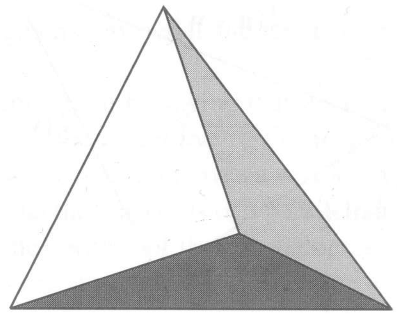

Figure 4.7: A tetrahedron

4.3.2 Suppose that the tetrahedron has vertices $\mathbf{t}$, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Show that the centroid of the face opposite to $\mathbf{t}$ is $\frac{1}{3}(\mathbf{u}+\mathbf{v}+\mathbf{w})$, and write down the centroids of the other three faces.

4.3.3 Now consider each line joining a vertex to the centroid of the opposite face. In particular, show that the point 3/4 of the way from $\mathbf{t}$ to the centroid of the opposite face is $\frac{1}{4}(\mathbf{t}+\mathbf{u}+\mathbf{v}+\mathbf{w})$—the centroid of the tetrahedron.

4.3.4 Explain why the point $\frac{1}{4}(\mathbf{t}+\mathbf{u}+\mathbf{v}+\mathbf{w})$ lies on the other three lines from a vertex to the centroid of the opposite face.

4.3.5 Deduce that the four lines from vertex to centroid of opposite face meet at the centroid of the tetrahedron.

4.4 The inner product

If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are vectors in $\mathbb{R}^2$, we define their *inner product* $\mathbf{u}\cdot\mathbf{v}$ to be $u_1v_1 + u_2v_2$. Thus, the inner product of two vectors is not another vector, but a real number or "scalar." For this reason, $\mathbf{u}\cdot\mathbf{v}$ is also called the *scalar product* of $\mathbf{u}$ and $\mathbf{v}$.

It is easy to check, from the definition, that the inner product has the algebraic properties

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u},$$
$$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w},$$
$$(a\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(a\mathbf{v}) = a(\mathbf{u}\cdot\mathbf{v}),$$

which immediately give information about length and angle:

- The length $|\mathbf{u}|$ is the distance of $\mathbf{u} = (u_1, u_2)$ from $\mathbf{0}$, which is $\sqrt{u_1^2+u_2^2}$ by the definition of distance in $\mathbb{R}^2$ (Section 3.3). Hence,

$$|\mathbf{u}|^2 = u_1^2 + u_2^2 = \mathbf{u}\cdot\mathbf{u}.$$

It follows that the square of the distance $|\mathbf{v}-\mathbf{u}|$ from $\mathbf{u}$ to $\mathbf{v}$ is

$$|\mathbf{v}-\mathbf{u}|^2 = (\mathbf{v}-\mathbf{u})\cdot(\mathbf{v}-\mathbf{u}) = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\mathbf{u}\cdot\mathbf{v}.$$

- Vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\mathbf{u}\cdot\mathbf{v} = 0$. Because $\mathbf{u}$ has slope u_2/u_1 and $\mathbf{v}$ has slope v_2/v_1, and we know from Section 3.5 that they are perpendicular if and only the product of their slopes is -1. That means

$$\frac{u_2}{u_1} = -\frac{v_1}{v_2} \quad \text{and hence} \quad u_2v_2 = -u_1v_1,$$

multiplying both sides by u_1v_2. This equation holds if and only if

$$0 = u_1v_1 + u_2v_2 = \mathbf{u}\cdot\mathbf{v}.$$

We will see in the next section how to extract more information about angle from the inner product. The formula above for $|\mathbf{v}-\mathbf{u}|^2$ turns out to be the "cosine rule" or "law of cosines" from high-school trigonometry. But even the criterion for perpendicularity gives a simple proof of a far-from-obvious theorem:

Concurrence of altitudes. *In any triangle, the perpendiculars from the vertices to opposite sides (the* altitudes*) have a common point.*

To prove this theorem, take $\mathbf{0}$ at the intersection of two altitudes, say those through the vertices $\mathbf{u}$ and $\mathbf{v}$ (Figure 4.8). Then it remains to show that the line from $\mathbf{0}$ to the third vertex $\mathbf{w}$ is perpendicular to the side $\mathbf{v}-\mathbf{u}$.

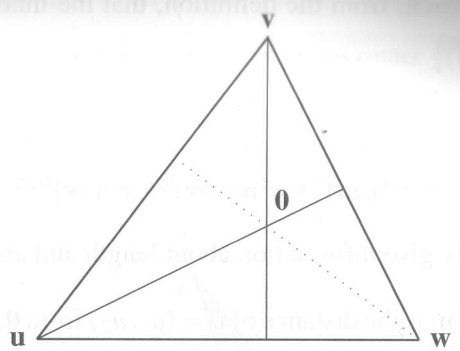

Figure 4.8: Altitudes of a triangle

Because $\mathbf{u}$ is perpendicular to the opposite side $\mathbf{w}-\mathbf{v}$, we have

$$\mathbf{u}\cdot(\mathbf{w}-\mathbf{v})=\mathbf{0},\quad \text{that is,}\quad \mathbf{u}\cdot\mathbf{w}-\mathbf{u}\cdot\mathbf{v}=\mathbf{0}.$$

Because $\mathbf{v}$ is perpendicular to the opposite side $\mathbf{u}-\mathbf{w}$, we have

$$\mathbf{v}\cdot(\mathbf{u}-\mathbf{w})=\mathbf{0},\quad \text{that is,}\quad \mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{w}=\mathbf{0}.$$

Adding these two equations, and bearing in mind that $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$, we get

$$\mathbf{u}\cdot\mathbf{w}-\mathbf{v}\cdot\mathbf{w}=\mathbf{0},\quad \text{that is,}\quad \mathbf{w}\cdot(\mathbf{v}-\mathbf{u})=\mathbf{0}.$$

Thus, $\mathbf{w}$ is perpendicular to $\mathbf{v}-\mathbf{u}$, as required. □

Exercises

The inner product criterion for directions to be perpendicular, namely that their inner product is zero, gives a neat way to prove the theorem in Exercise 2.2.2 about the diagonals of a rhombus.

4.4.1 Suppose that a parallelogram has vertices at $\mathbf{0}$, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$. Show that its diagonals have directions $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$.

4.4.2 Deduce from Exercise 4.4.1 that the inner product of these directions is $|\mathbf{u}|^2-|\mathbf{v}|^2$, and explain why this is zero for a rhombus.

The inner product also gives a concise way to show that the equidistant line of two points is the perpendicular bisector of the line connecting them (thus proving more than we did in Section 3.3).

4.4.3 The condition for $\mathbf{w}$ to be equidistant from $\mathbf{u}$ and $\mathbf{v}$ is

$$(\mathbf{w}-\mathbf{u})\cdot(\mathbf{w}-\mathbf{u})=(\mathbf{w}-\mathbf{v})\cdot(\mathbf{w}-\mathbf{v}).$$

Explain why, and show that this condition is equivalent to

$$|\mathbf{u}|^2-2\mathbf{w}\cdot\mathbf{u}=|\mathbf{v}|^2-2\mathbf{w}\cdot\mathbf{v}.$$

4.4.4 Show that the condition found in Exercise 4.4.3 is equivalent to

$$\left(\mathbf{w}-\frac{\mathbf{u}+\mathbf{v}}{2}\right)\cdot(\mathbf{u}-\mathbf{v})=0,$$

and explain why this says that $\mathbf{w}$ is on the perpendicular bisector of the line from $\mathbf{u}$ to $\mathbf{v}$.

Having established that the line equidistant from $\mathbf{u}$ and $\mathbf{v}$ is the perpendicular bisector, we conclude that the perpendicular bisectors of the sides of a triangle are concurrent—because this is obviously true of the equidistant lines of its vertices.

4.5 Inner product and cosine

The inner product of vectors $\mathbf{u}$ and $\mathbf{v}$ depends not only on their lengths $|\mathbf{u}|$ and $|\mathbf{v}|$ but also on the angle θ between them. The simplest way to express its dependence on angle is with the help of the *cosine* function. We write the cosine as a function of angle θ, $\cos\theta$. But, as usual, we avoid measuring angles and instead define $\cos\theta$ as the ratio of sides of a right-angled triangle. For simplicity, we assume that the triangle has vertices $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$ as shown in Figure 4.9.

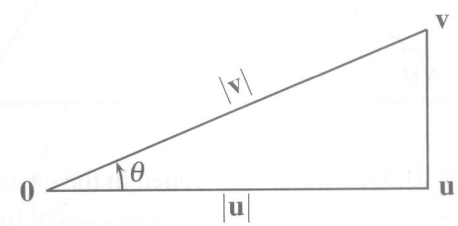

Figure 4.9: Cosine as a ratio of lengths

Then the side $\mathbf{v}$ is the hypotenuse, θ is the angle between the side $\mathbf{u}$ and the hypotenuse, and its cosine is defined by

$$\cos\theta=\frac{|\mathbf{u}|}{|\mathbf{v}|}.$$

We can now use the inner product criterion for perpendicularity to derive the following formula for inner product.

Inner product formula. *If θ is the angle between vectors* **u** *and* **v**, *then*

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta.$$

This formula follows because the side $\mathbf{v}-\mathbf{u}$ of the triangle is perpendicular to side **u**; hence,

$$0 = \mathbf{u}\cdot(\mathbf{v}-\mathbf{u}) = \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{u}.$$

Therefore, $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2 = |\mathbf{u}||\mathbf{v}|\frac{|\mathbf{u}|}{|\mathbf{v}|} = |\mathbf{u}||\mathbf{v}|\cos\theta.$ □

This formula gives a convenient way to calculate the angle (or at least its cosine) between any two lines, because we know from Section 4.4 how to calculate $|\mathbf{u}|$ and $|\mathbf{v}|$. It also gives us the "cosine rule" of trigonometry directly from the calculation of $(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})$.

Cosine rule. *In any triangle, with sides* **u**, **v**, *and* $\mathbf{u}-\mathbf{v}$, *and angle* θ *opposite to the side* $\mathbf{u}-\mathbf{v}$,

$$|\mathbf{u}-\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta.$$

Figure 4.10 shows the triangle and the relevant sides and angle, but the proof is a purely algebraic consequence of the inner product formula.

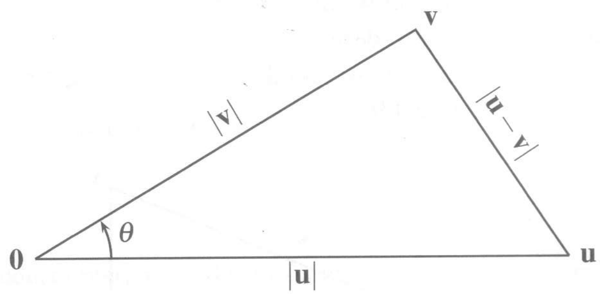

Figure 4.10: Quantities mentioned in the cosine rule

The algebra is simply the following:

$$\begin{aligned}|\mathbf{u}-\mathbf{v}|^2 &= (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\\ &= \mathbf{u}\cdot\mathbf{u} - 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}\\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\mathbf{u}\cdot\mathbf{v}\\ &= |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta.\end{aligned}$$ □

A nice way to close this circle of ideas is to consider the special case in which **u** and **v** are the sides of a right-angled triangle and $\mathbf{u}-\mathbf{v}$ is the hypotenuse. In this case, **u** is perpendicular to **v**, so $\mathbf{u}\cdot\mathbf{v} = 0$, and the cosine rule becomes

$$\text{hypotenuse}^2 = |\mathbf{u}-\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$$

—which is the Pythagorean theorem. This result should not be a surprise, however, because we have already seen how the Pythagorean theorem is built into the definition of distance in $\mathbb{R}^2$ and hence into the inner product.

Exercises

The Pythagorean theorem can also be proved directly, by choosing **0** at the right angle of a right-angled triangle whose other two vertices are **u** and **v**.

4.5.1 Show that $|\mathbf{v}-\mathbf{u}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$ under these conditions, and explain why this is the Pythagorean theorem.

While on the subject of right-angled triangles, we mention a useful formula for studying them.

4.5.2 Show that $(\mathbf{v}+\mathbf{u})\cdot(\mathbf{v}-\mathbf{u}) = |\mathbf{v}|^2 - |\mathbf{u}|^2$.

This formula gives a neat proof of the theorem from Section 2.7 about the angle in a semicircle. Take a circle with center **0** and a diameter with ends **u** and $-\mathbf{u}$ as shown in Figure 4.11. Also, let **v** be any other point on the circle.

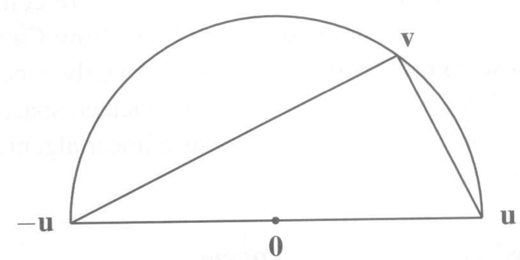

Figure 4.11: Points on a semicircle

4.5.3 Show that the sides of the triangle meeting at **v** have directions $\mathbf{v}+\mathbf{u}$ and $\mathbf{v}-\mathbf{u}$ and hence show that they are perpendicular.

4.6 The triangle inequality

In vector geometry, the triangle inequality $|\mathbf{u}+\mathbf{v}| \le |\mathbf{u}|+|\mathbf{v}|$ of Exercises 3.3.1 to 3.3.3 is usually derived from the fact that

$$|\mathbf{u}\cdot\mathbf{v}| \le |\mathbf{u}||\mathbf{v}|.$$

This result, known as the *Cauchy–Schwarz inequality*, follows easily from the formula in the previous section. The inner product formula says

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta,$$

and therefore,

$$\begin{aligned} |\mathbf{u}\cdot\mathbf{v}| &\le |\mathbf{u}||\mathbf{v}||\cos\theta| \\ &\le |\mathbf{u}||\mathbf{v}| \quad \text{because} \quad |\cos\theta| \le \mathbf{1}. \end{aligned}$$

Now, to get the triangle inequality, it suffices to show that $|\mathbf{u}+\mathbf{v}|^2 \le (|\mathbf{u}|+|\mathbf{v}|)^2$, which we do as follows:

$$\begin{aligned} |\mathbf{u}+\mathbf{v}|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \\ &= |\mathbf{u}|^2 + \mathbf{2u}\cdot\mathbf{v} + |\mathbf{v}|^2 \quad \text{because } \mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2 \text{ and } \mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2 \\ &\le |\mathbf{u}|^2 + \mathbf{2}|\mathbf{u}||\mathbf{v}| + |\mathbf{v}|^2 \quad \text{by Cauchy–Schwarz} \\ &= (|\mathbf{u}|+|\mathbf{v}|)^2 \end{aligned}$$

□

The reason for the fuss about the Cauchy–Schwarz inequality is that it holds in spaces more complicated than $\mathbb{R}^2$, with more complicated inner products. Because the triangle inequality follows from Cauchy–Schwarz, it too holds in these complicated spaces. We are mainly concerned with the geometry of the plane, so we do not need complicated spaces. However, it is worth saying a few words about $\mathbb{R}^n$, because linear algebra works just as well there as it does in $\mathbb{R}^2$.

Higher dimensional Euclidean spaces

$\mathbb{R}^n$ is the set of ordered n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers $x_1, x_2, \ldots, x_n$. These ordered n-tuples are called *n-dimensional vectors*. If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^n$, then we define the *vector sum* $\mathbf{u}+\mathbf{v}$ by

$$\mathbf{u}+\mathbf{v} = (u_1+v_1, u_2+v_2, \ldots, u_n+v_n),$$

and the *scalar multiple* $a\mathbf{u}$ for a real number a by

$$a\mathbf{u} = (au_1, au_2, \ldots, au_n).$$

It is easy to check that $\mathbb{R}^n$ has the properties enumerated at the beginning of Section 4.1. Hence, $\mathbb{R}^n$ is a real vector space under the vector sum and scalar multiplication operations just described.

$\mathbb{R}^n$ becomes a *Euclidean space* when we give it the extra structure of an inner product with the properties enumerated in Section 4.4. These properties hold if we define the inner product $\mathbf{u}\cdot\mathbf{v}$ by

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n,$$

as is easy to check. This inner product enables us to define *distance* in $\mathbb{R}^n$ by the formula

$$|\mathbf{u}|^2 = \mathbf{u}\cdot\mathbf{u}$$

which gives the distance $|\mathbf{u}|$ of $\mathbf{u}$ from the origin. This result is compatible with the concept of distance in $\mathbb{R}^2$ or $\mathbb{R}^3$ given by the Pythagorean theorem. For example, the distance of (u_1, u_2, u_3) from $\mathbf{0}$ in $\mathbb{R}^3$ is

$$|\mathbf{u}| = \sqrt{u_1^2+u_2^2+u_3^2},$$

as Figure 4.12 shows.

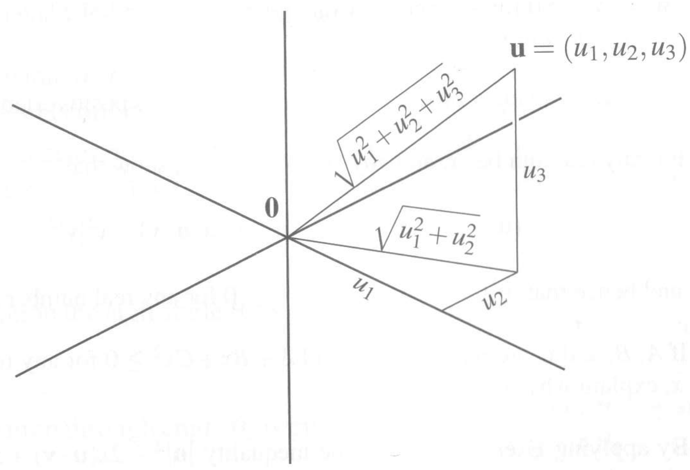

Figure 4.12: Distance in $\mathbb{R}^3$

- $\sqrt{u_1^2+u_2^2}$ is the distance from $\mathbf{0}$ of $(u_1,u_2,0)$ (the hypotenuse of a right-angled triangle with sides u_1 and u_2),

- $\sqrt{u_1^2+u_2^2+u_3^2}$ is the distance from $\mathbf{0}$ of (u_1,u_2,u_3) (the hypotenuse of a right-angled triangle with sides $\sqrt{u_1^2+u_2^2}$ and u_3).

All theorems proved in this chapter for vectors in the plane $\mathbb{R}^2$ hold in $\mathbb{R}^n$. This fact is clear if we take the plane in $\mathbb{R}^n$ to consist of vectors of the form $(x_1,x_2,0,\ldots,0)$, because such vectors behave exactly the same as vectors (x_1,x_2) in $\mathbb{R}^2$. But in fact *any* given plane in $\mathbb{R}^n$ behaves the same as the special plane of vectors $(x_1,x_2,0,\ldots,0)$. We skip the details, but it can be proved by constructing an isometry of $\mathbb{R}^n$ mapping the given plane onto the special plane. As in $\mathbb{R}^2$, any isometry is a product of reflections. In $\mathbb{R}^n$, at most $n+1$ reflections are required, and the proof is similar to the one given in Section 3.7.

Exercises

A proof of Cauchy–Schwarz using only general properties of the inner product can be obtained by an algebraic trick with quadratic equations. The general properties involved are the four listed at the beginning of Section 4.4 and the assumption that $\mathbf{w}\cdot\mathbf{w}=|\mathbf{w}|^2\geq 0$ for any vector $\mathbf{w}$ (an inner product with the latter property is called *positive definite*).

4.6.1 The Euclidean inner product for $\mathbb{R}^n$ defined above is positive definite. Why?

4.6.2 For any real number x, and any vectors $\mathbf{u}$ and $\mathbf{v}$, show that

$$(\mathbf{u}+x\mathbf{v})\cdot(\mathbf{u}+x\mathbf{v})=|\mathbf{u}|^2+2x(\mathbf{u}\cdot\mathbf{v})+x^2|\mathbf{v}|^2,$$

and hence that $|\mathbf{u}|^2+2x(\mathbf{u}\cdot\mathbf{v})+x^2|\mathbf{v}|^2\geq 0$ for any real number x.

4.6.3 If A, B, and C are real numbers and $A+Bx+Cx^2\geq 0$ for any real number x, explain why $B^2-4AC\leq 0$.

4.6.4 By applying Exercise 4.6.3 to the inequality $|\mathbf{u}|^2+2x(\mathbf{u}\cdot\mathbf{v})+x^2|\mathbf{v}|^2\geq 0$, show that

$$(\mathbf{u}\cdot\mathbf{v})^2\leq|\mathbf{u}|^2|\mathbf{v}|^2,\quad\text{and hence}\quad|\mathbf{u}\cdot\mathbf{v}|\leq|\mathbf{u}||\mathbf{v}|.$$

4.7 Rotations, matrices, and complex numbers

Rotation matrices

In Section 3.6, we defined a rotation of $\mathbb{R}^2$ as a function $r_{c,s}$, where c and s are two real numbers such that $c^2+s^2=1$. We described $r_{c,s}$ as the function that sends (x,y) to $(cx-sy,sx+cy)$, but it is also described by the *matrix of coefficients* of x and y, namely

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix},\quad\text{where } c=\cos\theta \text{ and } s=\sin\theta.$$

Because most readers will already have seen matrices, it may be useful to translate some previous statements about functions into matrix language, where they may be more familiar. (Readers not yet familiar with matrices will find an introduction in Section 7.2.)

Matrix notation allows us to rewrite $(x,y)\mapsto(cx-sy,sx+cy)$ as

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} cx-sy \\ sx+cy \end{pmatrix}$$

Thus, the function $r_{c,s}$ is applied to the variables x and y by multiplying the column vector $\begin{pmatrix} x \\ y \end{pmatrix}$ on the left by the matrix $\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$. Functions are thereby separated from their variables, so they can be composed without the variables becoming involved—simply by multiplying matrices.

This idea gives proofs of the formulas for $\cos(\theta_1+\theta_2)$ and $\sin(\theta_1+\theta_2)$, similar to Exercises 3.5.3 and 3.5.4, but with the variables x and y filtered out:

- Rotation through angle θ_1 is given by the matrix $\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}$.

- Rotation through angle θ_2 is given by the matrix $\begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}$.

- Hence, rotation through $\theta_1+\theta_2$ is given by the product of these two matrices. That is,

4.7.5 Explain why any u with $|u| = 1$ can be written in the form $\cos\theta + i\sin\theta$ for some angle θ, and conclude that multiplication by u rotates the point 1 (hence the whole plane) through angle θ.

It follows, in particular, that multiplication by $i = (0,1)$ sends $(1,0)$ to $(0,1)$ and hence rotates the plane through $\pi/2$. This result in turn implies $i^2 = -1$, because multiplication by i^2 then rotates the plane through π, which is also the effect of multiplication by -1.

4.8 Discussion

Because the geometric content of a vector space with an inner product is much the same as Euclidean geometry, it is interesting to see how many axioms it takes to describe a vector space. Remember from Section 2.9 that it takes 17 Hilbert axioms to describe the Euclidean plane, or 16 if we are willing to drop completeness of the line.

To define a vector space, we began in Section 4.1 with eight axioms for vector addition and scalar multiplication:

$$\begin{aligned} \mathbf{u}+\mathbf{v} &= \mathbf{v}+\mathbf{u} & 1\mathbf{u} &= \mathbf{u} \\ \mathbf{u}+(\mathbf{v}+\mathbf{w}) &= (\mathbf{u}+\mathbf{v})+\mathbf{w} & a(\mathbf{u}+\mathbf{v}) &= a\mathbf{u}+a\mathbf{v} \\ \mathbf{u}+\mathbf{0} &= \mathbf{u} & (a+b)\mathbf{u} &= a\mathbf{u}+b\mathbf{u} \\ \mathbf{u}+(-\mathbf{u}) &= \mathbf{0} & a(b\mathbf{u}) &= (ab)\mathbf{u}. \end{aligned}$$

Then, in Section 4.4, we added three (or four, depending on how you count) axioms for the inner product:

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= \mathbf{v}\cdot\mathbf{u}, \\ \mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) &= \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}, \\ (a\mathbf{u})\cdot\mathbf{v} &= \mathbf{u}\cdot(a\mathbf{v}) = a(\mathbf{u}\cdot\mathbf{v}), \end{aligned}$$

We also need relations among inner product, length, and angle—at a minimum the cosine formula,

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta,$$

so this is 12 or 13 axioms so far.

But we have also assumed that the scalars $a, b, \ldots$ are real numbers, so there remains the problem of writing down axioms for them. At the very least, one needs axioms saying that the scalars satisfy the ordinary rules of calculation, the so-called *field axioms* (this is usual when defining a vector space):

$$\begin{aligned} a+b &= b+a, & ab &= ba & &\text{(commutative laws)} \\ a+(b+c) &= (a+b)+c, & a(bc) &= (ab)c & &\text{(associative laws)} \\ a+0 &= a, & a1 &= a & &\text{(identity laws)} \\ a+(-a) &= 0, & aa^{-1} &= 1 & &\text{(inverse laws)} \\ & & a(b+c) &= ab+ac & &\text{(distributive law)} \end{aligned}$$

Thus, the usual definition of a vector space, with an inner product suitable for Euclidean geometry, takes more than 20 axioms! Admittedly, the field axioms and the vector space axioms are useful in many other parts of mathematics, whereas most of the Hilbert axioms seem meaningful only in geometry. And, by varying the inner product slightly, one can change the geometry of the vector space in interesting ways. For example, one can obtain the geometry of *Minkowski space* used in Einstein's special theory of relativity. To learn more about the vector space approach to geometry, see *Linear Algebra and Geometry, a Second Course* by I. Kaplansky and *Metric Affine Geometry* by E. Snapper and R. J. Troyer.

Still, one can dream of building geometry on a much simpler set of axioms. In Chapter 6, we will realize this dream with *projective geometry*, which we begin studying in Chapter 5.