

ON MUTATION AND KHOVANOV HOMOLOGY

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It is conjectured that the Khovanov homology of a knot is invariant under mutation. In this paper, we review the spanning tree complex for Khovanov homology, and reformulate this conjecture using a matroid obtained from the Tait graph (checkerboard graph) G of a knot diagram K . The spanning trees of G provide a filtration and a spectral sequence that converges to the reduced Khovanov homology of K . We show that the E_2 -term of this spectral sequence is a matroid invariant and hence invariant under mutation.

Keywords: Khovanov homology; Conway mutation; matroid.

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1. Introduction

For any diagram of an oriented link L , Khovanov [7] constructed bigraded abelian groups $H^{i,j}(L)$, whose bigraded Euler characteristic gives the Jones polynomial $V_L(t)$:

$$\chi(H^{*,*}) = \sum_{i,j} (-1)^i q^j \text{rank}(H^{i,j}) = (q + q^{-1})V_L(q^2)$$

For knots (or links with a marked component), Khovanov also defined reduced homology groups $\tilde{H}^{i,j}(L)$ whose bigraded Euler characteristic is $q^{-1}V_L(q^2)$ [8].

Since the introduction of Khovanov homology in [7], the theory has been developed and generalized far beyond the Jones polynomial (see e.g. [9] and references therein), and beyond classical links to objects like graphs and ribbon graphs (see e.g. [3, 6, 10]).

However, just as the original Jones polynomial eludes a topological interpretation in terms of the knot complement, classical Khovanov homology remains mysterious. The following questions are open for this invariant:

- Does any non-trivial knot have trivial Khovanov homology?
- Which knots have “thin” Khovanov homology (supported on two diagonals)?
- What is the Khovanov homology of (p, q) -torus knots?
- Is Khovanov homology invariant under Conway mutation of knots?

Our purpose here is to present ideas and results that we hope will be useful to tackle the last question. It is conjectured that the Khovanov homology of a knot is invariant under mutation (see [1, 20], and see [21] for a recent proof over $\mathbb{Z}/2\mathbb{Z}$).

Spanning trees and Khovanov homology

There is a 1-1 correspondence between connected link diagrams D and connected planar graphs G with signed edges. G , called the *Tait graph* of D , is obtained by checkerboard coloring complementary regions of D , assigning a vertex to every shaded region, an edge to every crossing and a \pm sign to every edge as follows:



The signs are all equal if and only if D is alternating.

In Section 2 below, we express Khovanov homology using generators that correspond to spanning trees of G . More accurately, with a fixed edge order on G , the construction relies on *activity words* $W(T)$ for each spanning tree T . Before diving into notation, it seems worthwhile to motivate this approach.

We give three motivating reasons to consider Khovanov homology using the spanning trees of the Tait graph. First, Thistlethwaite [16] gave an expansion of the Jones polynomial $V_L(t)$ in terms of spanning trees of any Tait graph $G(L)$. Every spanning tree contributes a monomial to the Jones polynomial. For non-alternating knots, these monomials may cancel with each other, but for alternating knots, such cancelations do not occur. Thus, for alternating knots, the number of spanning trees is exactly the L^1 -norm of Jones coefficients, and the span of $V_L(t)$ is maximal, equal to the crossing number. The bigraded spanning tree complex described below provides an explicit distribution of spanning trees, which is at most $(k + 1)$ -thick for links that become alternating after k crossing changes (see [2]). It also provides a tool to study particular Jones coefficients. For example, if we change a crossing in an alternating knot diagram D , the span and L^1 -norm of Jones coefficients strictly decrease. In the spanning tree complex, we can see how the gradings change to make certain spanning trees cancel in the Euler characteristic.

A second reason is given by the important and closely related example of knot Floer homology. The two knot homology theories are compared in detail in [15]. The

complex for knot Floer homology in [14] has generators that correspond to spanning trees, but no combinatorial differential is known. The more recent complex in [11] is completely combinatorial but has far more generators, so it is quasi-isomorphic (and possibly retracts) to a combinatorial complex generated by spanning trees. The situation for Khovanov homology is similar: The Khovanov complex retracts to a complex generated by spanning trees of G (Theorem 2.2), but it remains an open question whether the differential on the spanning tree complex can be expressed entirely in terms of the combinatorics (activity words) of spanning trees.

The third reason is discussed in Section 3, where we show that the conjectured dependence of the differential on activity words is closely related to the mutation invariance of Khovanov homology. This appears to be a promising approach to prove that Khovanov homology is invariant under component-preserving mutation of links.

In Section 2, we review the construction of the spanning tree complex $\mathcal{C}(K)$ given in [2], the spanning tree filtration and the associated spectral sequence that converges to $\tilde{H}(K)$. In Sections 2.3 and 2.4, we prove new results that show direct incidences and the E_2 -term of this spectral sequence are determined by activity words. Material in Section 2.5 also has not been previously published.

In Section 3, we show that the mutation invariance of any knot invariant can be expressed in terms of the colored cycle matroid $M(K)$, obtained from the Tait graph G of a knot diagram K . In particular, the reduced Khovanov homology $\tilde{H}(K)$ is invariant under mutation if and only if the spanning tree complex $\mathcal{C}(K)$ is determined by $M(K)$ up to quasi-isomorphism. As a partial step, the E_2 -term mentioned above is determined by $M(K)$ and hence invariant under mutation. In Section 3.3, we discuss an approach to prove mutation-invariance of Khovanov homology.

2. Spanning Tree Complex

In [2], for any connected link diagram D , we defined the spanning tree complex $\mathcal{C}(D) = \{\mathcal{C}_v^u(D), \partial\}$, whose generators correspond to spanning trees T of G . In this section, we review the main ideas and related notation, which will be used later.

2.1. Activity words and twisted unknots

Fix an order on the edges of G . For every spanning tree T of G , each edge $e \in G$ has an activity with respect to T , as follows. If $e \in T$, $\text{cut}(T, e)$ is the set of edges that connect $T \setminus e$. If $f \notin T$, $\text{cyc}(T, f)$ is the set of edges in the unique cycle of $T \cup f$. Note $f \in \text{cut}(T, e)$ if and only if $e \in \text{cyc}(T, f)$. An edge $e \in T$ (resp. $e \notin T$) is *live* if it is the lowest edge in its cut (resp. cycle), and otherwise it is *dead*.

For any spanning tree T of G , the *activity word* $W(T)$ gives the activity of each edge of G with respect to T . The letters of $W(T)$ are as follows: L , D , ℓ , d denote a positive edge that is live in T , dead in T , live in $G - T$, dead in $G - T$, respectively;

\bar{L} , \bar{D} , $\bar{\ell}$, \bar{d} denote activities for a negative edge. Note that T is given by the capital letters of $W(T)$.









Thistlethwaite assigned a monomial $\mu(T)$ to each T as follows:

$$L^p D^q \ell^r d^s \bar{L}^x \bar{D}^y \bar{\ell}^z \bar{d}^w \Rightarrow \mu(T) = (-1)^{p+r+x+z} A^{-3p+q+3r-s+3x-y-3z+w}.$$

Theorem 2.1 ([16]). *Let G be the Tait graph of any connected link diagram D with any order on its edges. Let $\langle D \rangle$ denote the Kauffman bracket polynomial of D . Summing over all spanning trees T of G , $\langle D \rangle = \sum_T \mu(T)$.*

The activity word $W(T)$ contains much more information than just $\mu(T)$. A *twisted unknot* U is a diagram of the unknot obtained from the round unknot using only Reidemeister I moves. $W(T)$ determines a twisted unknot $U(T)$ by changing the crossings of D according to Table 1 for dead edges, and leaving the crossings unchanged for live edges (Lemma 1 [2]). In Table 1, the sign of the crossing in $U(T)$ is indicated for unsmoothed crossings, and Kauffman state markers are indicated for smoothed crossings.

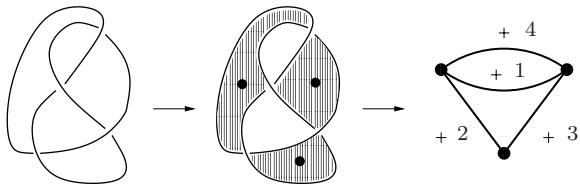
Table 1. Activity word for a spanning tree determines a twisted unknot

L	D	ℓ	d	\bar{L}	\bar{D}	$\bar{\ell}$	\bar{d}
-	A	+	B	+	B	-	A
							

We can also consider each $U(T)$ as a partial smoothing of D determined by $W(T)$. In fact, there exists a skein resolution tree for D whose leaves are exactly all the partial resolutions $U(T)$, for each spanning tree T of G (Theorem 2 [2]). Let $\sigma(U) = \#A\text{-smoothings} - \#B\text{-smoothings}$, and let $w(U)$ be the writhe. If U corresponds to T , then $\mu(T) = A^{\sigma(U)}(-A)^{3w(U)}$ is exactly the monomial above Theorem 2.1. As Louis Kauffman pointed out, this is how humans would compute $\langle D \rangle$: Instead of smoothing all the way to the final Kauffman states, a human would stop upon reaching any twisted unknot U , and use the formula $\mu(T)$. We illustrate all of this for the figure-eight knot diagram in Figure 1.

For any connected link diagram D , we choose the checkerboard coloring such that its Tait graph G has more positive edges than negative edges, and in case of equality that the unbounded region is unshaded. In [2], we defined the spanning tree complex $\mathcal{C}(D) = \{\mathcal{C}_v^u(D), \partial\}$, whose generators correspond to spanning trees T of G . The u and v -grading are determined by $W(T)$ as follows:

$$u(T) = \#L - \#\ell - \#\bar{L} + \#\bar{\ell} \quad \text{and} \quad v(T) = \#L + \#D.$$



Spanning trees	 T_1	 T_2	 T_3	 T_4	 T_5
Activities	$LLdd$	$LdDd$	ℓDDd	ℓLdD	$\ell \ell DD$
Weights	A^{-8}	$-A^{-4}$	$-A^4$	1	A^8

$\langle D \rangle = A^{-8} - A^{-4} + 1 - A^4 + A^8 \qquad V_D(t) = t^{-2} - t^{-1} + 1 - t + t^2$

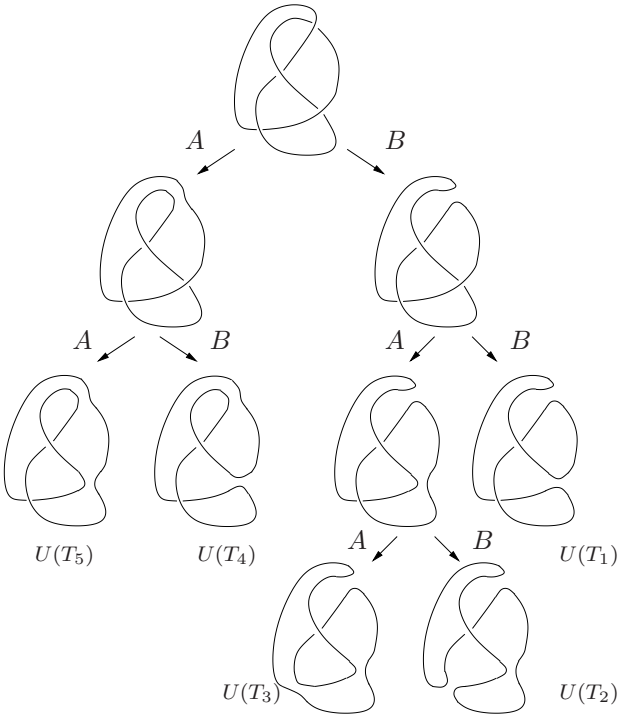


Fig. 1. Spanning trees and twisted unknots for figure-8 knot

Theorem 2.2 ([2]). *For any connected link diagram D , there exist spanning tree complexes $\mathcal{C}(D) = \{\mathcal{C}_v^u(D), \partial\}$ and $\mathcal{UC}(D) = \{\mathcal{UC}_v^u(D), \partial\}$ with ∂ of bi-degree $(-1, -1)$ that are deformation retracts of the reduced and unreduced Khovanov complexes, respectively.*

The differential in $\mathcal{C}(D)$ is defined indirectly. As discussed in detail below, for each T the Khovanov complex $\tilde{\mathcal{C}}(U(T))$ is contractible, and we proceed by a sequence of collapses of each $\tilde{\mathcal{C}}(U(T))$ to a single generator $Z(T)$. The differential on spanning trees is the one induced by all such collapses.

Note that $u(T) = -w(U(T))$. Interestingly, $v(T)$ has appeared in several guises elsewhere: (1) Rasmussen's δ -grading (Definition 4.4 [15]) satisfies $\delta = 2v + k$, where k is a constant that depends only on D . (2) A connected link diagram determines a *ribbon graph*, which is a graph embedded in a surface such that its complement is a union of 2-cells. The genus g of the ribbon graph is the genus of the minimal such surface. Each spanning tree of G corresponds to a ribbon graph with one complementary 2-cell, whose genus satisfies $g + v = k'$, where k' is another constant that depends only on D (Theorem 2.1 [3]).

2.2. Fundamental cycle of a twisted unknot

We review the main ideas underlying Theorem 2.2, which will be used in the next section.

Let D be a connected link diagram with a basepoint P away from the crossings of D . In the version of Khovanov homology in [19, 18], generators of the reduced Khovanov complex $\tilde{\mathcal{C}}(D)$ are given by *enhanced Kauffman states* of D . A Kauffman state s is a choice of smoothings of all crossings of D , and enhancements are \pm signs on every loop of s . The reduced Khovanov complex consists only of enhanced states for which every loop that contains P has a positive enhancement. Enhanced states are incident in $\tilde{\mathcal{C}}(D)$ if and only if exactly one A marker can be changed to a B marker, such that loops unaffected by the marker change keep their enhancements, and the changed loops are enhanced to increase the enhancement signature by one.

For any twisted unknot U , $\tilde{\mathcal{C}}(U)$ is contractible, with the same homology as that of the positively enhanced round unknot \bigcirc^+ . Starting from the round unknot, by a sequence of positive and negative twists, we can obtain any U . For every such twist, Figure 2 indicates how to obtain a linear combination of maximally disconnected enhanced states. We define the *fundamental cycle* $Z_U \in \tilde{\mathcal{C}}(U)$ to be the linear combination of enhanced states of U given by iterating the local changes in Figure 2. Let $f_U : \tilde{\mathcal{C}}(\bigcirc) \rightarrow \tilde{\mathcal{C}}(U)$ be defined by $f_U(\bigcirc^+) = Z_U$.

On the other hand, there exists a sequence of elementary collapses $r_U : \tilde{\mathcal{C}}(U) \rightarrow \tilde{\mathcal{C}}(\bigcirc)$, such that $r_U \circ f_U = \text{id}$ and $f_U \circ r_U \simeq \text{id}$. Essentially, this follows from invariance of Khovanov homology under the first Reidemeister move [7].

We can summarize this discussion as follows: Each spanning-tree generator $T \in \mathcal{C}(D)$ corresponds to a contractible Khovanov subcomplex $\tilde{\mathcal{C}}(U(T))$, for which the fixed point of the retraction is the fundamental cycle $Z_{U(T)}$. The basepoint P

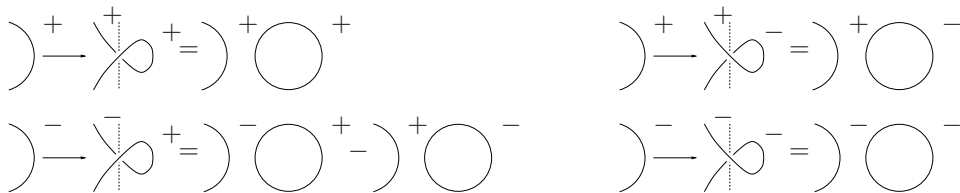


Fig. 2. How to obtain the fundamental cycle of a twisted unknot

determines a *unique* fixed point for this retraction: (U, P) is given by a sequence of first Reidemeister moves from (\bigcirc, P) , which determines Z_U uniquely.

Let $\iota : \tilde{\mathcal{C}}(U(T)) \rightarrow \tilde{\mathcal{C}}(D)$ be the inclusion of enhanced states given by appropriately shifting the gradings. The image $Z(T) = \iota(Z_{U(T)}) \in \tilde{\mathcal{C}}(D)$ is called the *fundamental cycle* of T . Note that $Z(T)$ is not generally a cycle in $\tilde{\mathcal{C}}(D)$, even though $Z_{U(T)}$ is a cycle in $\tilde{\mathcal{C}}(U(T))$. In the proof of Theorem 2.2, the map from $\mathcal{C}(D) \rightarrow \tilde{\mathcal{C}}(D)$ given by $T \rightarrow Z(T)$ induces an isomorphism on homology.

Up to linear combinations of enhancements, $Z(T)$ is just a single Kauffman state: the maximally disconnected state of $U(T)$, obtained by replacing every positive or negative twist in $U(T)$ by an A or B marker, respectively. So from Table 1 we obtain the markers for $Z(T)$ from the activity word $W(T)$:

$$\begin{array}{cccccccc} L & D & \ell & d & \bar{L} & \bar{D} & \bar{\ell} & \bar{d} \\ \hline B & A & A & B & A & B & B & A \end{array} \quad (2.1)$$

It also follows that distinct enhanced states $s, s' \in \tilde{\mathcal{C}}(U(T))$ differ only at markers that are live in $W(T)$. If $i \neq j$, the enhanced states $s_i \in \tilde{\mathcal{C}}(U(T_i))$ and $s_j \in \tilde{\mathcal{C}}(U(T_j))$ differ in at least one marker that is dead in both $W(T_i)$ and $W(T_j)$.

Finally, it is straightforward to extend these ideas to unreduced Khovanov homology. Using the gradings in [19], $\tilde{\mathcal{C}}(\bigcirc) \cong \mathbb{Z}^{(0,-1)}$ and $\mathcal{C}(\bigcirc) \cong \mathbb{Z}^{0,1} \oplus \mathbb{Z}^{0,-1}$. Hence, for every T , there are two fundamental cycles for $U(T)$, and two corresponding generators in $\mathcal{UC}(D)$: T^+ in grading $(u(T), v(T))$, and T^- in grading $(u(T) + 2, v(T) + 1)$. With the activity word $W(T)$ and the basepoint P , we can associate a unique generator in $\mathcal{C}(D)$ to each of T^+ and T^- by using the same rules in Figure 2 to obtain $Z_{U(T)}^\pm \in \mathcal{C}(D)$, starting with \bigcirc^+ for T^+ and \bigcirc^- for T^- .

2.3. Activity words and the differential on the spanning tree complex

The proof of Theorem 2.2 does not provide an intrinsic description of the differential on the spanning tree complex $\mathcal{C}(D)$ without reference to enhanced states. The main result of this section is that the simplest kind of incidence in $\mathcal{C}(D)$ is determined by activity words.

For a complex (C, ∂) over \mathbb{Z} with graded basis $\{e_i\}$, let $\langle \cdot, \cdot \rangle$ denote the inner product defined by $\langle e_i, e_j \rangle = \delta_{ij}$. We say x is *incident* to y in (C, ∂) if $\langle \partial x, y \rangle \neq 0$ and their *incidence number* is $\langle \partial x, y \rangle$.

Let T_1, T_2 be spanning trees with fundamental cycles $Z_1, Z_2 \in \tilde{C}(D)$. We define T_1 and T_2 to be *directly incident* if $\langle \partial Z_1, Z_2 \rangle \neq 0$ in $\tilde{C}(D)$. In this case, $\langle \partial Z_1, Z_2 \rangle = (-1)^\beta$, where β is the number of B -markers after the A -marker that is changed. By Lemma 2.1 below, if T_1 and T_2 are directly incident, then they are incident in $\mathcal{C}(D)$ and $\langle \partial T_1, T_2 \rangle = \langle \partial Z_1, Z_2 \rangle = \pm 1$. However, T_1 and T_2 may be incident in $\mathcal{C}(D)$ even though $\langle \partial Z_1, Z_2 \rangle = 0$, which is discussed in Section 2.5.

Theorem 2.3. *Spanning trees T_1 and T_2 are directly incident if and only if the activity words $W(T_1)$ and $W(T_2)$ differ by changing exactly two (not necessarily adjacent) letters in one of the following four ways:*

$$L \bar{d} \rightarrow d \bar{D}$$

$$\bar{d} D \rightarrow \bar{L} d$$

$$\bar{\ell} D \rightarrow \bar{D} d$$

$$D \bar{d} \rightarrow \ell \bar{D}$$

In particular, T_2 is obtained from T_1 by replacing one positive edge $e \in T_1$ with one negative edge f , such that $f \in \text{cut}(T_1, e)$, and no other edges change activity.

Proof. First, we show that if $W(T_1)$ (on the left) changes in one of the four ways to $W(T_2)$, then T_1 and T_2 are directly incident. Let Z_1 and Z_2 be fundamental cycles of T_1 and T_2 . In all four cases, by (2.1) exactly one A marker of Z_1 is changed to a B marker to get Z_2 , and $(u(T_2), v(T_2)) = (u(T_1) - 1, v(T_1) - 1)$. Changing indices according to equations (2) in [2], it follows by results in [19] that at least one summand of each of Z_1 and Z_2 are incident in $\tilde{C}(D)$.

We claim that $\langle \partial Z_1, Z_2 \rangle \neq 0$. If these are single enhanced states, then we are done. For linear combinations of enhanced states, we must show that incidences among summands do not cancel. A fundamental cycle $Z(T)$ can have more than one summand only if $U(T)$ is smoothed at a crossing c , resulting in a linear combination of enhanced states, as shown in Figure 2. Since c is a crossing of $U(T)$, c is live in $W(T)$. In all four cases, the marker that changes from A to B is dead in both $W(T_1)$ and $W(T_2)$, so the marker at c cannot change. All summands of $Z(T)$ have the same markers, so the sign of every summand is determined by its enhancements. Since the sign of the Khovanov differential depends only on the markers, cancellations cannot occur among terms in $\langle \partial Z_1, Z_2 \rangle$. Since at least some summands of Z_1 and Z_2 are incident and do not cancel, T_1 and T_2 are directly incident.

Conversely, suppose T_1 and T_2 are directly incident. We claim there is exactly one pair of edges e_i, e_j such that $T_2 = (T_1 \setminus e_i) \cup e_j$, and only e_i and e_j change activities.

If a marker does not change, then by (2.1), since edge signs do not change, the activity of the corresponding edge can change as follows:

$$L \leftrightarrow d, \quad D \leftrightarrow \ell, \quad \bar{L} \leftrightarrow \bar{d}, \quad \bar{D} \leftrightarrow \bar{\ell} \quad (2.2)$$

Therefore, without a marker change, the activity of an edge changes if and only if the edge is removed from the tree or inserted into the tree.

From any spanning tree T , we can obtain any other spanning tree by switching pairs of edges $e_i \in T$, $e_j \notin T$, such that $e_j \in \text{cut}(T, e_i)$. Consider switching one such pair of edges for which neither marker changes.

Suppose the markers of e_i and e_j are fixed, and suppose for spanning trees T, T' , we have $T' = (T \setminus e_i) \cup e_j$. In every case in (2.2), e_i and e_j are both live in either T or T' . However, $e_j \in \text{cut}(T, e_i)$ and $e_i \in \text{cut}(T', e_j)$, so only one of e_i or e_j can be live (the lower-ordered edge). This contradiction implies that if neither marker changes, then the activities cannot change, and in particular, this pair of edges cannot be switched.

Since T_1 and T_2 are directly incident, exactly one marker changes. By the argument above, there is exactly one pair of edges e_i, e_j such that $T_2 = (T_1 \setminus e_i) \cup e_j$, and only the activities of e_i and e_j change. Moreover, only the lower-ordered edge can be live in either T_1 or T_2 . Since $v(T_2) = v(T_1) - 1$, e_i must be positive, and e_j negative. Since $u(T_2) = u(T_1) - 1$, if both edges are dead on the right (i.e., with respect to T_2), one edge on the left must be L or $\bar{\ell}$; if both edges are dead on the left, one edge on the right must be \bar{L} or ℓ . These four cases are the ones given in the theorem, and all can occur. \square

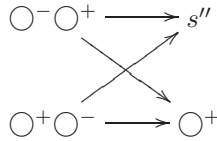
Lemma 2.1. *Let T_1, T_2 be spanning trees with fundamental cycles $Z_1, Z_2 \in \tilde{C}(D)$. If $\langle \partial Z_1, Z_2 \rangle \neq 0$ then in $\mathcal{C}(D)$, $\langle \partial T_1, T_2 \rangle = \langle \partial Z_1, Z_2 \rangle$.*

Proof. If x is incident to y in $\tilde{C}(D)$, we denote this by $x \rightarrow y$ below. Let $U_i = U(T_i)$. We claim that the differential $Z_1 \rightarrow Z_2$ remains after all elementary collapses of twisted unknots, as in Lemma 4 of [2]. It suffices to show that the incidences shown in the diagram below are impossible for any enhanced states s', s'' that are distinct from Z_1, Z_2 . This is the only way for the differential $Z_1 \rightarrow Z_2$ to be removed by elementary collapse.

$$\begin{array}{ccc} s' & \longrightarrow & s'' \\ & \searrow \quad \nearrow & \\ Z_1 & \longrightarrow & Z_2 \end{array}$$

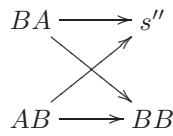
Case 1: $s' \in \tilde{C}(U_1) \subset \tilde{C}(D)$. If $i \neq j$, any incidence between enhanced states in $\tilde{C}(U_i)$ and $\tilde{C}(U_j)$ must occur at a marker that is dead in both $W(T_i)$ and $W(T_j)$. Thus, both s' and Z_1 differ from Z_2 on a dead marker, hence they have the same live markers. Since both are in $\tilde{C}(U_1)$, they have the same dead markers too. Therefore,

s' and Z_1 just differ by the following enhancements:

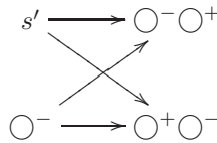


Now, for both s' and Z_1 to be incident to s'' , the same marker must change. This implies that $s'' = Z_2$ since both have the same markers and the same enhancements, which is a contradiction.

Case 2: $s' \notin \tilde{C}(U_1) \subset \tilde{C}(D)$. Suppose $Z_1 \rightarrow Z_2$ at marker 1, and $s' \rightarrow Z_2$ at marker 2, which must be distinct markers. Therefore at markers 1 and 2, we have



Because Z_1 and s' are both incident to s'' , this implies that s'' must have the same markers as Z_2 . Therefore, for Z_1 to be incident to both s'' and Z_2 , the enhancements must be as follows:



Now, for both s'' and Z_2 to be incident to s' , the same marker must change. So marker 1 = marker 2, which is a contradiction. \square

2.4. Spanning tree filtration and spectral sequence

The activity word $W(T)$ determines a partial smoothing $U(T)$. Live edges, denoted below by $*$, correspond to crossings of the twisted unknot, which are not smoothed.

Let D be any connected link diagram with n ordered crossings. For any spanning trees T, T' of G , let (x_1, \dots, x_n) and (y_1, \dots, y_n) be the corresponding partial smoothings of D . We define a relation $T > T'$, or equivalently, $(x_1, \dots, x_n) > (y_1, \dots, y_n)$ if for each i , $y_i = A$ implies $x_i = A$ or $*$, and there exists i such that $x_i = A$ and $y_i = B$. The transitive closure of this relation gives a partial order, also denoted by $>$ (Proposition 1 [2]). We define $\mathcal{P}(D)$ to be the poset of spanning trees of G with this partial order. Note that $\mathcal{P}(D)$ always has a unique maximal tree and unique minimal tree, whose partial smoothings contain the all- A and all- B Kauffman states, respectively.

For example, for the figure-8 knot from Figure 1,

T_1	T_2	T_3	T_4	T_5
$**BB$	$*BAB$	$*AAB$	$**BA$	$**AA$

We get two sequences: $T_5 > T_3 > T_2 > T_1$ and $T_5 > T_4 > T_1$. The maximal and minimal trees correspond respectively to the left-most and right-most unknots in Figure 1.

The poset $\mathcal{P}(D)$ provides a partially ordered filtration of $\tilde{C}(D)$ indexed by $\mathcal{P}(D)$: Let $\tilde{U}_i = \iota(\tilde{C}(U(T_i))) \subset \tilde{C}(D)$. Let $\psi : \mathcal{P}(D) \rightarrow \tilde{C}(D)$ defined by $\psi(T) = +_{T \geq T_i} \tilde{U}_i$. From a partially ordered filtration, we can get a decreasing linearly ordered filtration $F^p \tilde{C}(D)$ by taking all trees of order at least p from all maximal descending ordered sequences in $\mathcal{P}(D)$. For example, the figure-8 knot from Figure 1 has the following filtration $F^p \tilde{C}(D)$,

$$F^1 = \psi(T_5) = \tilde{C}(D), \quad F^2 = \psi(T_3) + \psi(T_4), \quad F^3 = \psi(T_2), \quad F^4 = \psi(T_1)$$

Theorem 2.4 ([2]). *For any knot diagram D , there is a spectral sequence $E_r^{*,*}$ that converges to the reduced Khovanov homology $\tilde{H}^{*,*}(D; \mathbb{Z})$, such that as groups $E_1^{*,*} \cong C_*(D)$, and the spectral sequence collapses for $r \leq c(D)$, where $c(D)$ is the number of crossings.*

The associated graded module consists of submodules of $\tilde{C}(D)$ in bijection with spanning trees:

$$E_0^{p,*} = F^p \tilde{C}(D) / F^{p+1} \tilde{C}(D) = \oplus_i \tilde{U}_i \quad (2.3)$$

Corollary 2.1. *For any knot diagram D , the E_2 -term of the spectral sequence in Theorem 2.4 is determined by the set of activity words for all spanning trees in $G(D)$.*

Proof. Let D be any knot diagram. It follows from the filtration that for any p , if $\tilde{U}_1, \tilde{U}_2 \subset E_0^{p,*}$, then T_1 and T_2 are not comparable in P . Hence, $d_0 : E_0^{p,q} \rightarrow E_0^{p,q+1}$ satisfies $d_0(\tilde{U}_k) \subset \tilde{U}_k$ for every k . This implies that (2.3) is a direct sum of complexes.

Each complex \tilde{U}_k has homology generator corresponding to a spanning tree, so E_1 is isomorphic as a group to the spanning tree complex:

$$E_1^{*,*} = H^*(F^p / F^{p+1}, d_0) = \oplus_k H^*(\tilde{U}_k) \cong \mathcal{C}(D)$$

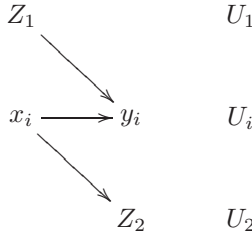
Let $d_1 : E_1^{p,q} \rightarrow E_1^{p+1,q}$. If $T_1 > T_2$ are directly incident and one filtration level apart then $Z_1 \in F^p \tilde{C}(D)$, $Z_2 \in F^{p+1} \tilde{C}(D) \in E_1^{*,*}$, and $\langle \partial(Z_1), Z_2 \rangle \neq 0$. Thus, $\langle d_1(Z_1), Z_2 \rangle \neq 0$. Conversely, if $\langle d_1(Z_1), Z_2 \rangle \neq 0$ then T_1 and T_2 are one filtration level apart and hence directly incident. The partial order and filtration are determined by activity words, and by Theorem 2.3 direct incidences are determined by activity words. Therefore, the E_2 -term of the spectral sequence is determined by activity words. \square

2.5. Higher order incidences

To construct $\mathcal{C}(D)$ as well as $\mathcal{UC}(D)$, we proceed by a sequence of elementary collapses of each \tilde{U}_i to its fundamental cycle $Z(T_i)$, starting from the minimal tree

and ascending in the partial order whenever trees are comparable. Any elementary collapse in \tilde{U}_i does not change incidence numbers in \tilde{U}_j for any $j \neq i$ (Lemma 5 [2]), so we can sequentially collapse each \tilde{U}_i .

Differentials are induced from each collapse, so T_1 and T_2 may be incident in $\mathcal{C}(D)$ without being directly incident; i.e., $\langle \partial Z_1, Z_2 \rangle = 0$ but $\langle d_r Z_1, Z_2 \rangle \neq 0$ for some $r > 1$. In general, $\langle \partial T_1, T_2 \rangle$ is the sum of induced incidence numbers given by all *ladders* from Z_1 to Z_2 . Before giving definitions, here is an example of a 2-incidence ($i \neq 1, 2$ and $x_i, y_i \neq Z_i$), which becomes an incidence after collapsing U_i :



Definition 2.1. Let $T_1 > T_2$ be spanning trees with fundamental cycles $Z_1, Z_2 \in \tilde{\mathcal{C}}(D)$. T_1 and T_2 are 1-incident if they are directly incident. For $k > 1$, T_1 and T_2 are k -incident if there exist $x_i, y_i \in \tilde{U}_{j_i} - Z_{j_i}$ for $1 \leq i \leq k-1$, such that if $x_0 = Z_1$ and $y_k = Z_2$ then

$$\langle \partial x_i, y_{i+1} \rangle \neq 0 \text{ for } 0 \leq i \leq k-1, \quad \langle \partial x_i, y_i \rangle \neq 0 \text{ for } 1 \leq i \leq k-1$$

Such a sequence of ordered pairs of enhanced states will be called a *ladder of enhanced states* from Z_1 to Z_2 .

If T_1 and T_2 are k -incident, then collapsing along $\{\tilde{U}_{j_i} \mid 1 \leq i \leq k-1, j \neq 1, 2\}$, as in Definition 2.1, the incidence number between Z_1 and $y_k = Z_2$ induced from this ladder is $(-1)^{k-1} \langle \partial Z_1, y_1 \rangle \prod_{i=1}^{k-1} \langle \partial x_i, y_i \rangle \langle \partial x_i, y_{i+1} \rangle$. Moreover, since each enhanced state belongs to a unique \tilde{U}_i , for every ladder from Z_1 to Z_2 , such a collapse implies the following:

- (1) $T_1 > T_{j_1} \geq \dots \geq T_{j_{k-1}} > T_2$ and these relations are transitive.
- (2) Exactly k A -markers of Z_1 are changed to B -markers of Z_2 .
- (3) Exactly $(k-1)$ B -markers of Z_1 are changed to A -markers of Z_2 .
- (4) For each j_i , x_i, y_i are resolutions of \tilde{U}_{j_i} , with a differential $x_i \rightarrow y_i$ by changing a marker that is live in $W(T_{j_i})$.
- (5) If at level i and level $i+1$ ($i \neq 1, k-1$), the same spanning tree T occurs, then there exists a differential $x_i \rightarrow y_{i+1}$ by changing a marker that is live in $W(T)$.
- (6) If at level i and level $i+1$, distinct spanning trees occur, then there exists a differential $x_i \rightarrow y_{i+1}$ by changing a marker that is dead in both $W(T_{j_i})$ and $W(T_{j_{i+1}})$.

We will say that a sequence of (possibly repeated) activity words $\{W(T_{j_i})\}$ along with the following extra data is an *admissible activity sequence* if

- (1) The first and last activity words correspond to spanning trees T_1 and T_2 , whose bigradings permit a nonzero differential.
- (2) The sequence satisfies the partial order: $T_1 > T_{j_i} \geq T_{j_{i+1}} > T_2$ for all i .
- (3) A sequence of ordered pairs of markers indicates how to change the live markers in each $W(T_{j_i})$.

Each ladder of enhanced states gives rise to a unique admissible activity sequence. The converse is an open question that is fundamental to understanding the differential on the spanning tree complex:

Question 2.1. Which admissible activity sequences correspond to ladders of enhanced states?

Theorem 2.3 answers Question 2.1 in the simplest case.

As discussed in Section 2.2, given $W(T)$ and basepoint P , we can compute $Z(T) \in \tilde{C}(D)$ from $U(T)$. In particular, $W(T_1)$ and $W(T_2)$ completely determine Z_1 and Z_2 . But starting with Z_1 , and just specifying allowed marker changes may not be sufficient to produce a ladder (or possibly a linear combination of ladders) of enhanced states to Z_2 . The difficulty inherent in Question 2.1 is whether the enhancements on the states “take care of themselves,” or whether the enhancements can obstruct the existence of a ladder, given a sequence of allowed marker changes from Z_1 to Z_2 .

For the unreduced spanning tree complex $\mathcal{UC}(D)$, an admissible sequence must also include the signs of the spanning tree generators: $\{\pm W(T_{j_i})\}$. With the signed activity word $\pm W(T)$ and the basepoint P , we can compute Z^\pm for each generator T^\pm . Because any enhancement is allowed on the state with P , it seems less likely, given an admissible signed activity sequence, that enhancements can obstruct a ladder.

In Section 3.3, we show how this is directly related to the mutation-invariance of Khovanov homology.

3. Mutation and Matroids

A *marked 2-tangle* is a 2-tangle contained in a round ball such that its four endpoints are equally spaced around the equator of the boundary sphere, called a *Conway sphere*. Let L be a link that contains a marked 2-tangle τ . A *mutation* of L is the following operation: Remove the Conway sphere containing τ , rotate it by π about one of its three coordinate axes, and glue it back to form the link L' .

The same operation can be described for any planar diagram D of L . The projection of the Conway sphere is a *Conway circle* that meets D in four points, which are the endpoints of the marked 2-tangle diagram contained in the disc. A mutation of D is then given by one of the three corresponding involutions of the disc. Diagrams D and D' are called *mutants* if D' can be obtained from D by a sequence of mutations.

3.1. Tait graphs and mutation

There are two choices for the checkerboard coloring of D , and the resulting Tait graphs are the planar duals of each other. The projection of D is the medial graph of G , and the signs on G determine the crossings of D . This determines a one-one correspondence between checkerboard-colored link diagrams and planar embeddings of signed graphs. In order to study mutation using Tait graphs, we define two moves on graphs:

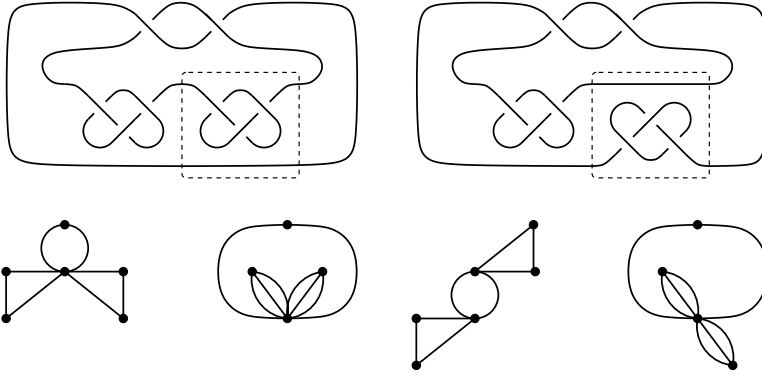


Fig. 3. Connect sum for links and their Tait graphs

1-flip Let v_1 and v_2 be vertices of disjoint graphs G_1 and G_2 . A *vertex identification* is $G = G_1 \sqcup G_2 / v_1 \sim v_2$. If v is a cut-vertex of G , i.e. $G - v$ is disconnected, a *vertex splitting* at v of G is the inverse operation of vertex identification. A 1-flip of G is a vertex splitting followed by a vertex identification.

2-flip For $i \in \{1, 2\}$, let u_i, v_i be vertices of disjoint graphs G_i such that $G = G_1 \sqcup G_2 / (u_1, v_1) \sim (u_2, v_2)$. A 2-flip of is the identification $G_1 \sqcup G_2 / (u_1, v_1) \sim (v_2, u_2)$.

We extend both these moves to signed graphs by requiring that the signs on the corresponding edges are preserved.

For a link diagram D , a 1-flip corresponds to breaking a connect sum and reconnecting at a different place. Since the connect sum operation is well-defined for knots, 1-flips do not change the knot type. However, 1-flips may change link type; see Figure 3. We will consider only component-preserving link mutation later.

2-flips correspond to mutation for link diagrams. Figure 4 shows the Kinoshita-Terasaka and Conway mutants along with their Tait graphs (unsigned edges are positive). The graphs in the second row come from the checkerboard coloring with the unbounded region shaded, and the graphs in the third row from the other checkerboard coloring.

Some mutations change only the planar embedding of G but not G itself, so not all types of mutation can be realized as 2-flips. For example the graphs in the third

row of Figure 4 are not related by 2-flips. To address this, we define the following two moves on planar embeddings of G that preserve the graph itself.

planar 1-flip A planar 1-flip replaces a 1-connected component of a planar embedding with its rotation by π about an axis in the plane which intersects the cut vertex.

planar 2-flip A planar 2-flip replaces a 2-connected component of a planar embedding with its rotation by π around the axis determined by the 2-connecting vertices.

We similarly extend both these moves to embeddings of signed graphs by requiring that the signs on the corresponding edges are preserved.

Any two planar embeddings of a signed graph are related by a sequence of planar 1-flips and planar 2-flips (see [12]). As before, these moves correspond to reconnecting connect sums and mutations of link diagrams, respectively. Although, 1-flips can also correspond to mutation in link diagrams whose Tait graphs have a cut vertex; for example, see Figure 3 and [20].

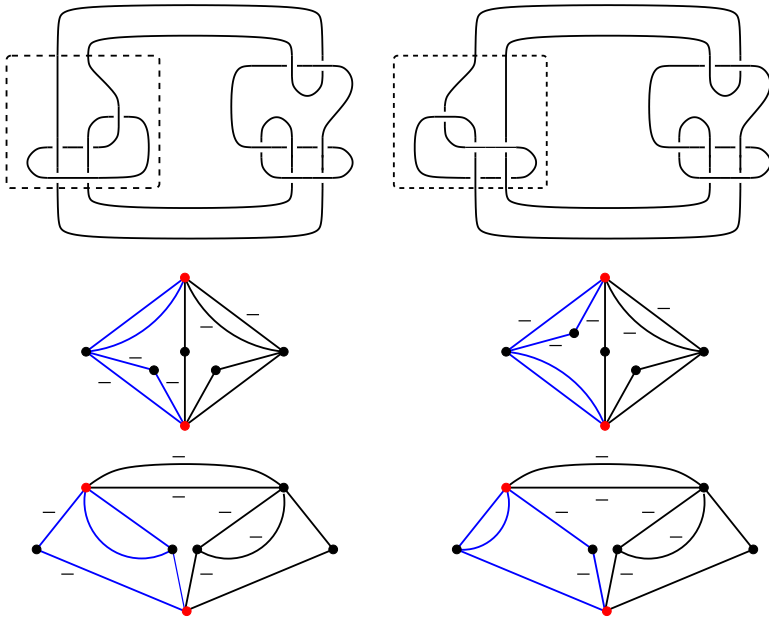


Fig. 4. Kinoshita-Terasaka and Conway mutants and their Tait graphs

A graph G is said to be *2-isomorphic* to a graph H if G can be obtained from H by any sequence of vertex identifications, vertex splittings, or 2-flips. Hence, a connected graph G is 2-isomorphic to a connected graph H if G can be obtained from H by any sequence of 1-flips and 2-flips. In particular, isomorphic graphs are

2-isomorphic. We require 2-isomorphisms of signed graphs to preserve signs on the edges.

Prop 3.1. Let D and D' be connected link diagrams with checkerboard colorings chosen so that their unbounded regions are both shaded or both unshaded. Let G and G' be their respective Tait graphs. Then D and D' are mutants if and only if G and G' are 2-isomorphic.

Proof. For any Tait graph, any type of mutation corresponds to either a 1-flip (possibly a planar 1-flip), a 2-flip or a planar 2-flip, and all of these can be realized by mutation. As mentioned above, any two planar embeddings of a graph are related by a sequence of planar 1-flips and planar 2-flips. Specifying the coloring of the unbounded region distinguishes a Tait graph from its planar dual. \square

Thus, in order to study mutation via Tait graphs, we need to study invariants of 2-isomorphism classes of signed graphs. As we discuss below, these naturally come from matroids.

3.2. Matroids

We recall some ideas from the theory of matroids (see [13]). A matroid M is a finite set of elements, together with a family of subsets, called *independent sets*, such that

- (1) The empty set is independent,
- (2) Every subset of an independent set is independent,
- (3) For every subset A of M , all maximal independent sets contained in A have the same number of elements.

A maximal independent set in M is called a *basis* for M , and any two bases of M have the same number of elements, which is the rank of M .

For example, let E be the set of edges of a graph G , and let \mathcal{I} be the collection of subsets of edges that do not contain a cycle. Then (E, \mathcal{I}) is a matroid $M(G)$, called the *graphic matroid* of G . For a connected graph G , the bases of $M(G)$ are the spanning trees of G .

For any connected link diagram D with a checkerboard coloring and Tait graph G , let the *colored graphic matroid* $M(D)$ be the graphic matroid $M(G)$ with edges colored by $\{\pm 1\}$ as in the Tait graph, according to the crossings of D .

Whitney [22] determined precisely when two graphs have isomorphic graphic matroids. This fundamental result, which motivated matroid theory, is called the 2-isomorphism theorem (for background see [13]). If we require that any isomorphism of colored graphic matroids be color-preserving, then the 2-isomorphism theorem extends to signed graphs (see e.g., [17]):

Theorem 3.1. For signed graphs G and H with no isolated vertices, their colored graphic matroids are isomorphic if and only if G and H are 2-isomorphic.

Theorem 3.1 and Proposition 3.1 imply the following:

Corollary 3.1. *Let D and D' be connected link diagrams with checkerboard colorings chosen so that their unbounded regions are both shaded or both unshaded. Let $M(D)$ and $M(D')$ be their respective colored graphic matroids. Then D and D' are mutants if and only if $M(D) \cong M(D')$.*

Consequently, any knot invariant φ is invariant under mutation if and only if for any knot diagram K , $\varphi(K)$ is an invariant of the colored graphic matroid $M(K)$. For any matroid M , activities can be defined with respect to its basis, just as we did for the graphic matroid $M(G)$ using spanning trees. We will use that activity words are determined by $M(K)$, essentially due to Crapo [4].

For example, by Theorem 2.1, the Jones polynomial $V_K(t)$ has an expansion using activity words. Therefore, the Jones polynomial is an invariant of $M(K)$, and hence invariant under mutation. Below, we extend this idea to Khovanov homology.

3.3. Khovanov homology and matroids

For a given connected link diagram D with basepoint P , we choose the checkerboard coloring such that its Tait graph G has more positive edges than negative edges, and in case of equality that the unbounded region is unshaded. Let $M(D)$ be the colored graphic matroid of D with this coloring. The generators of $\mathcal{C}(D)$, which are the spanning trees of G , are bases of $M(D)$. Since both the u and v -gradings are determined by the activities and signs, the bigrading on $\mathcal{C}(D)$ is determined by $M(D)$.

Generally, Conway mutation, as in Figure 3, may not preserve components. Indeed, two mutant links were shown to have different Khovanov homology in [20], using this connect sum ambiguity for links. From our point of view, such a mutation moves the basepoint P from one component to another, leading to a different $Z_{U(T)}$ for every T , which sometimes changes the homology. To eliminate this ambiguity, we can either consider only knot diagrams, or require that Conway mutation of links be component-preserving. For purposes of exposition, it is easier to just discuss mutation of knot diagrams.

Whenever K and K' are mutant knot diagrams, by Corollary 3.1, $M(K) \cong M(K')$. Therefore, $\mathcal{C}(K) \cong \mathcal{C}(K')$ as bigraded abelian groups. We conjecture that the differential on $\mathcal{C}(K)$ is determined by $M(K)$ in the following way.

Conjecture 3.1. *Let K and K' be knot diagrams such that $M(K) \cong M(K')$. If $T_1, T_2 \in \mathcal{C}(K)$ and $T'_1, T'_2 \in \mathcal{C}(K')$ are generators corresponding to spanning trees,*

$$\langle \partial T_1, T_2 \rangle = \langle \partial T'_1, T'_2 \rangle \quad \text{whenever} \quad W(T_1) = W(T'_1), \quad W(T_2) = W(T'_2)$$

If Conjecture 3.1 holds, then $\mathcal{C}(K) \cong \mathcal{C}(K')$ as bigraded chain complexes for mutant knot diagrams K and K' . This would imply that $\tilde{H}(K)$ is invariant under mutation.

A quasi-isomorphism between chain complexes is a morphism that induces an isomorphism on homology. Any two chain complexes of free abelian groups with isomorphic homology are quasi-isomorphic.^a This implies the following equivalence:

For a knot diagram K , the reduced Khovanov homology $\tilde{H}(K)$ is invariant under mutation if and only if $\mathcal{C}(K)$ is determined up to quasi-isomorphism by $M(K)$.

Corollary 3.2. *For any knot diagram, the E_2 -term of the spectral sequence in Theorem 2.4 is invariant under mutation.*

Proof. Corollary 2.1 implies that $E_2^{*,*}(K)$ is determined by $M(K)$. If K and K' are mutant knot diagrams, $M(K) \cong M(K')$, which implies $E_2^{*,*}(K) \cong E_2^{*,*}(K')$. \square

Theorem 3.1 is at the heart of these results in terms of spanning trees. But in terms of enhanced Kauffman states, mutation appears to be a rather violent operation on the Khovanov complex. It is interesting to relate these two points of view.

Generally, we are given a connected link diagram D with a basepoint P . Conway mutation τ on D induces a mutation on the Kauffman states of D . Conway mutation of an enhanced state S of D may identify arcs with opposite enhancements. To assign enhancements unambiguously for $\tau(S)$, (1) any state disjoint from the Conway circle must keep its enhancement, and (2) all enhancements must be preserved in the part of D that contains the basepoint P . The latter requirement induces enhancements on arcs in the other part of D that intersect the Conway circle. We will refer to this operation, which must preserve the link components, as Conway mutation of (D, P) , denoted by $\tau(D, P)$.

Let $(D', P) = \tau(D, P)$. By Theorem 3.1, spanning trees $T' = \tau(T)$ if and only if $W(T) = W(T')$. As discussed in Section 2.2, with the activity word $W(T)$ and the basepoint P , we can associate a unique generator in $\mathcal{C}(D)$ to each of T^+ and T^- , and for $\tilde{\mathcal{C}}(D)$ just use T^+ . Thus, the activity word $\pm W(T_i)$ for T_i^\pm determines a unique generator Z_i^\pm in each of the respective Khovanov complexes, $\mathcal{C}(D)$ and $\mathcal{C}(D')$, as well as in $\tilde{\mathcal{C}}(D)$ and $\tilde{\mathcal{C}}(D')$. Hence, the maps induced by Conway mutation $\tau : \mathcal{C}(D) \rightarrow \mathcal{C}(D')$ and $\tau_{\mathcal{U}} : \mathcal{UC}(D) \rightarrow \mathcal{UC}(D')$ are isomorphisms of bigraded abelian groups.

This provides an approach to prove mutation-invariance of Khovanov homology. In Section 2.5, we defined an admissible activity sequence, which depends only on $M(D)$. Even without an explicit answer to Question 2.1, these sequences may record the essential information about the differential:

Conjecture 3.2. *Let $(D', P) = \tau(D, P)$. For every ladder of enhanced states in $\tilde{\mathcal{C}}(D)$, the corresponding admissible activity sequence describes a ladder of enhanced states in $\tilde{\mathcal{C}}(D')$.*

^aThis follows from the fact that every chain complex of free abelian groups decomposes as a direct sum of two-step complexes, for which the relation matrix can be diagonalized. We thank Ciprian Manolescu for this comment.

Conjecture 3.2 appears weaker than Conjecture 3.1, but it too implies the mutation-invariance of reduced Khovanov homology!

If Conjecture 3.2 holds then every differential in $\mathcal{C}(D')$ may be computed from some collection of admissible activity sequences. If so, for every ladder in $\tilde{C}(D)$, there is a corresponding ladder in $\tilde{C}(D')$ with the same induced incidence number as in $\tilde{C}(D)$. This would imply that $\tau : \mathcal{C}(D) \rightarrow \mathcal{C}(D')$ is a quasi-isomorphism. In other words, $\mathcal{C}(D)$ is determined up to quasi-isomorphism by $M(D)$.

Because signs on the spanning trees (or their activity words) are not contained in $M(D)$, the unreduced Khovanov complex $\mathcal{UC}(D)$ in general is not determined by $M(D)$. However, the following analogue of Conjecture 3.2 similarly implies the mutation-invariance of unreduced Khovanov homology:

Conjecture 3.3. *Let $(D', P) = \tau(D, P)$. For every ladder of enhanced states in $C(D)$, the corresponding admissible signed activity sequence describes a ladder of enhanced states in $C(D')$.*

Remark 3.1. In [1], an attempt to prove mutation-invariance of Khovanov homology was outlined using “re-embedding universality.” However, as explained there, re-embedding universality implies invariance under cabled mutation, which Khovanov homology does not satisfy [5]. We can explain the non-invariance of Khovanov homology under cabled mutation by the fact that cabled mutation corresponds to an n -flip for $n > 2$. Under this operation, Corollary 3.1 does not hold. In fact, the 14-crossing example in [5] has spanning trees whose activity words change after 2-cabled mutation. This lends some support to our approach.

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