PERGAMON

Topology 42 (2003) 83–101

TOPOLOGY

www.elsevier.com/locate/top

Vassiliev invariants and the cubical knot complex

Ilya Kofman[a,*], Xiao-Song Lin[b]

[a]*Department of Mathematics, Columbia University, New York, NY 10027, USA*
[b]*Department of Mathematics, University of California, Riverside, Riverside, CA 92521, USA*

Received 1 December 2000; received in revised form 6 August 2001; accepted 14 November 2001

Abstract

We construct a cubical CW-complex $CK(M)$ whose rational cohomology algebra contains Vassiliev invariants of knots in the 3-manifold M. We construct $\overline{CK}(\mathbf{R}^3)$ by attaching cells to $CK(\mathbf{R}^3)$ for every degenerate 1-singular and 2-singular knot, and we show that $\pi_1(\overline{CK}(\mathbf{R}^3)) = 1$ and $\pi_2(\overline{CK}(\mathbf{R}^3)) = \mathbf{Z}$. We give conditions for Vassiliev invariants to be nontrivial in cohomology. In particular, for $\mathbf{R}^3$ we show that v_2 uniquely generates $H^2(CK, D)$, where D is the subcomplex of degenerate singular knots. More generally, we show that any Vassiliev invariant coming from the Conway polynomial is nontrivial in cohomology. The cup product in $H^*(CK)$ provides a new graded commutative algebra of Vassiliev invariants evaluated on ordered singular knots. We show how the cup product arises naturally from a cocommutative differential graded Hopf algebra of ordered chord diagrams. © 2002 Elsevier Science Ltd. All rights reserved.

MSC: Primary 57M27; Secondary 55U15

Keywords: Vassiliev invariants; Cubical CW-complex; Hopf algebra

1. Introduction

In [15], Vassiliev constructed new knot invariants by a spectral sequence approximation of $H^0(\mathscr{M}\backslash\Sigma;\mathbb{Q})$, where $\mathscr{M}$ is the space of all closed curves in $\mathbb{R}^3$, and Σ is the space of singular curves. In [9,8], Vassiliev invariants were extended to knots in certain 3-manifolds satisfying extra conditions, and in [16] Vassiliev generalized his theory to all 3-manifolds. A natural question is whether one could realize each Vassiliev invariant as a genuine cohomology class of a space instrinsically associated with the underlying 3-manifold M. For this purpose, we construct a cubical CW-complex $CK(M)$, resembling Vassiliev's space σ (a resolution of $\underline{\sigma}$), whose rational cohomology algebra contains Vassiliev invariants.

* Corresponding author. Tel.: +1-212-854-3210; fax: +1-212-854-8962.
E-mail addresses: ikofman@math.columbia.edu (I. Kofman), xl@math.ucr.edu (X.-S. Lin).

0040-9383/03/$ - see front matter © 2002 Elsevier Science Ltd. All rights reserved.
PII: S0040-9383(02)00013-7

Starting with an algebraic chain complex of knots with ordered double points or base point in any oriented 3-manifold M, we geometrically realize the chain complex as the CW-complex $CK(M)$ (Section 2). A similar chain complex used to generalize skein modules [3] does not seem geometrically realizable.

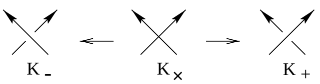
For $\mathbb{R}^3$, we give conditions for Vassiliev invariants to be nontrivial in cohomology, and a disparity emerges between even and odd orders (Section 4). Vassiliev invariants from the Conway polynomial—the best understood invariants of strictly even order—are all nontrivial in cohomology. For example, v_2^n is nontrivial in $H^{2n}(CK(\mathbb{R}^3), D; \mathbb{Q})$, where D is the subcomplex of degenerate singular knots. In particular, v_2 uniquely generates $H^2(CK(\mathbb{R}^3), D)$. Possibly, all nontrivial cohomology of CK/D arises from Vassiliev invariants of even order.

The cup product in $H^*(CK)$ provides a new graded commutative algebra of Vassiliev invariants evaluated on ordered singular knots. In Section 5, we show how the cup product arises naturally from a cocommutative differential graded Hopf algebra of ordered chord diagrams.

As a topological consequence, by a theorem of Quillen [14], Vassiliev invariants under the cup product are the rational cohomology ring of a simply connected pointed topological space. In Section 3, we construct $\overline{CK}(\mathbb{R}^3)$ which seems very close to such a space for even Vassiliev invariants. We attach cells to $CK(\mathbb{R}^3)$ for every degenerate 1-singular and 2-singular knot, and show that $\pi_1(\overline{CK}(\mathbb{R}^3)) = 1$ and $\pi_2(\overline{CK}(\mathbb{R}^3)) = \mathbb{Z}$ by "almost general position" arguments [9].

2. Constructing the knot complexes $CK(M)$

Let M be an oriented 3-manifold. Let X_n be the set of equivalence classes of oriented knots with n double points in M, equivalent up to rigid-vertex isotopy. Let $K_{\times\cdots\times}$ denote an element of X_n. As M is oriented, any double point can be resolved in two canonical ways:



We call a singular crossing *degenerate* if its positive and negative resolutions determine the same knot type. A *degenerate knot* is a singular knot with at least one degenerate crossing. A singular crossing is *nugatory* if a separating S^2 intersects the knot in only that point. Such a crossing is degenerate, and it seems to be unknown whether there exist degenerate crossings which are not nugatory.

Let X_n^0 be the set of equivalence classes of knots in X_n with double points ordered from 1 to n. For $K_{1\ldots n} \in X_n^0$ and any $\sigma \in S_n$, let $K_{\sigma(1\ldots n)}$ denote the knot obtained by permuting the ordering of $K_{1\ldots n}$. In $X_n^0, K_{1\ldots n} = K_{\sigma(1\ldots n)}$ if there exists an orientation-preserving rigid-vertex isotopy which maps $K_{1\ldots n}$ to $K_{\sigma(1\ldots n)}$, matching double points with the same labels. Let $C_n(X^0)$ denote the free abelian group with coefficients in $\mathbb{Q}$ generated by $K_{1\ldots n} \in X_n^0$.

Let X_n^b be the set of equivalence classes of knots in X_n with a base point b apart from all the double points. $(K, b) = (K', b')$ if there exists an orientation-preserving rigid-vertex isotopy which maps (K, b) to (K', b'), with the base point apart from the double points during the isotopy. As knots are oriented, we can order double points from 1 to n around the knot starting from the base point. Whenever possible, we will suppress notation for base points and denote based n-singular knots with this natural ordering by $K_{1\dots n}$. Let $C_n(X^b)$ denote the free abelian group with coefficients in $\mathbb{Q}$ generated by $K_{1\dots n} \in X_n^b$.

Let $C_n(X)$ be either $C_n(X^0)$ or $C_n(X^b)$. Define $\mathrm{d}_i : C_n(X) \to C_{n-1}(X)$ by $\mathrm{d}_i K_{1\dots n} = K_{\times\cdots\underset{i}{+}\cdots\times} - K_{\times\cdots\underset{i}{-}\cdots\times}$, such that each resolution has its ordering induced from $K_{1\dots n}$. Define the boundary operator ∂: $C_n(X) \to C_{n-1}(X)$ by $\partial K_{1\dots n} = \sum_{i=1}^{n} (-1)^{i+1} \mathrm{d}_i K_{1\dots n}$. The boundary operator is well defined with respect to the equivalence relations for X_n^0 and X_n^b because any order-preserving isotopy from $K_{1\dots n}$ to $K_{\sigma(1\dots n)}$ induces an order-preserving isotopy between their ith resolutions with the induced ordering. For any knot in X_n^0 or X_n^b, if $1 \leqslant i < j \leqslant n, \mathrm{d}_i \mathrm{d}_j = \mathrm{d}_{j-1} \mathrm{d}_i$, so $\partial^2 = 0$.

Let $C_n(D)$ denote the respective subgroup of $C_n(X)$ generated by degenerate n-singular knots, with $C_0(D) = 0$. If $K_{1\dots n}$ is degenerate, then for some $i, \mathrm{d}_i K_{1\dots n} = 0$ and $\partial K_{1\dots n} \in C_{n-1}(D)$. Thus, we obtain relative chain complexes $C_n(X,D) = C_n(X)/C_n(D)$.

Theorem 1. *There exist CW-complexes CK_b and CK_0 such that for $X = X^b$ and X^0, respectively, $C_*^{CW}(CK) \cong C_*(X)$.*

Proof. We construct CK_b and CK_0 by attaching n-cubes in one–one correspondence with n-singular knots $K_{1\dots n} \in X_n^b$ or X_n^0, respectively. For $n = 0$, each vertex corresponds to an oriented knot $K \in X_0$. Henceforth, CK will denote either CK_0 or CK_b, and all unspecified statements are valid for both complexes.

Let $CK^{(n)}$ denote the n-skeleton of CK. $CK^{(1)}$ can be constructed immediately from the formula $\partial K_\times = K_+ - K_-$. Let $I = [-1,1]$. For any $K_\times$, the attaching map $f_{K_\times} : \partial I \to CK^{(0)}$ is given by $f(\pm 1) = K_\pm$. Therefore, any knots which differ by a single crossing change are connected by an edge oriented from K_- to K_+. An edge joins distinct vertices if and only if $K_\times$ is nondegenerate. Only knots in the same conjugacy class of $\pi_1(M)$ are obtained by crossing changes, so components of $CK(M)$ correspond bijectively with conjugacy classes of $\pi_1(M)$.

Having constructed $CK^{(1)}$, we proceed inductively. Let $I^n = \{\vec{x} \in \mathbb{R}^n : -1 \leqslant x_i \leqslant 1\}$. Assuming $CK^{(n-1)}$ is constructed, for every $K_{1\dots n}$ we attach an n-cube by $f_{K_{1\dots n}} : \delta I^n \to CK^{(n-1)}$. Suppose $p + q = n$. We label the p-faces of I^n with p-singular knots obtained by resolving q singularities as follows (see Fig. 1):

$$K_{\times\cdots\varepsilon_1\cdots\varepsilon_q\cdots\times} \leftrightarrow \{(x_1,\dots,\varepsilon_1,\dots,\varepsilon_q,\dots,x_p) \colon \varepsilon_i \in \pm 1, -1 \leqslant x_j \leqslant 1\}.$$

By assumption, for every $(n-1)$-singular resolution K of $K_{1\dots n}$, there is a characteristic map $g_K : I^{n-1} \to CK^{(n-1)}$. For $\vec{x} \in K_{\times\cdots\varepsilon_i\cdots\times} \subset \partial I^n$, we define

$$f_{K_{1\dots n}}(x_1,\dots,\varepsilon_i,\dots,x_n) = g_{K_{\times\cdots\varepsilon_i\cdots\times}}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n).$$

To complete the induction, we verify $f_{K_{1\dots n}}$ is well defined on intersections:

$$\begin{aligned} f_{K_{1\dots n}}(x_1,\dots,\varepsilon_i,\dots,\varepsilon_j,\dots,x_n) &= g_{K_{\times\cdots\varepsilon_i\cdots\times}}(x_1,\dots,\widehat{x_i},\dots,\varepsilon_j,\dots,x_n) \\ &= g_{K_{\times\cdots\varepsilon_i\dots\varepsilon_j\cdots\times}}(x_1,\dots,\widehat{x_i},\dots,\widehat{x_j},\dots,x_n) \\ &= g_{K_{\times\cdots\varepsilon_j\cdots\times}}(x_1,\dots,\varepsilon_i,\dots,\widehat{x_j},\dots,x_n). \end{aligned}$$

Let $\Phi : C_*(X) \to C_*^{CW}(CK)$ by $\Phi(K_{1\dots n}) = g_{K_{1\dots n}}(I^n)$, $\forall K_{1\dots n} \in X_n^b$ or X_n^0. Each n-cell is oriented by $g_{K_{1\dots n}}$ from the standard orientation of $I^n \subset \mathbb{R}^n$. Since Φ maps generators bijectively, it remains to

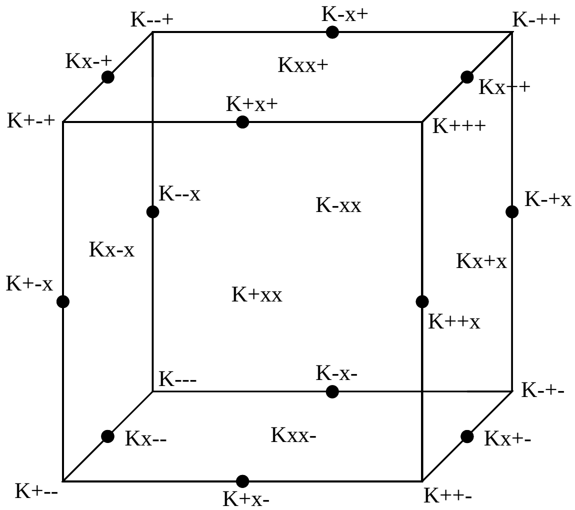


Fig. 1. Labelings on the 3-cube K_{123}.

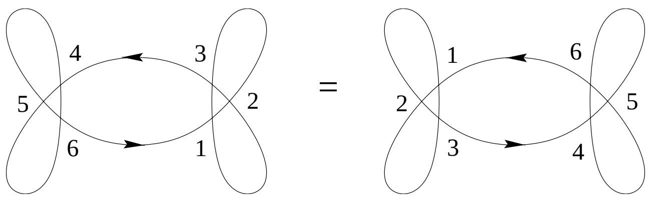
show $\Phi \circ \partial = \partial \circ \Phi$. Let $[\sigma{:}\tau]$ denote the incidence number. By construction, $[\Phi(K_{1\ldots n}){:}\tau] \neq 0$ only if there exists some resolution $K_{\times\cdots\varepsilon_i\cdots\times}$ of $K_{1\ldots n}$ such that $\Phi(K_{\times\cdots\varepsilon_i\cdots\times}) = \tau$

$$
\begin{aligned}
\partial\Phi(K_{1\ldots n}) &= \sum_{\tau} [\Phi(K_{1\ldots n}){:}\ \tau]\tau \\
&= \sum_{i=1}^{n} \sum_{\varepsilon\in\pm 1} [\Phi(K_{1\ldots n}){:}\ \Phi(K_{\times\cdots\varepsilon_i\cdots\times})]\Phi(K_{\times\cdots\varepsilon_i\ldots\times}) \\
&= \sum_{i=1}^{n} \sum_{\varepsilon\in\pm 1} [g_{K_{1\ldots n}}(I^n){:}\ g_{K_{\times\cdots\varepsilon_i\cdots\times}}(I^{n-1})]g_{K_{\times\cdots\varepsilon_i\cdots\times}}(I^{n-1}) \\
&= \sum_{i=1}^{n} \sum_{\varepsilon\in\pm 1} ((-1)^{i+1}\varepsilon_i)g_{K_{\times\cdots\varepsilon_i\cdots\times}}(I^{n-1}) \\
&= \sum_{i=1}^{n} (-1)^{i+1} \left(\sum_{\varepsilon\in\pm 1} \varepsilon_i g_{K_{\times\cdots\varepsilon_i\cdots\times}}(I^{n-1}) \right) \\
&= \sum_{i=1}^{n} (-1)^{i+1}\Phi(\mathrm{d}_i K_{1\ldots n}) = \Phi\partial K_{1\ldots n}. \qquad \square
\end{aligned}
$$

CK is not locally finite. For example, infinitely many knots with unknotting number 1 are connected by an edge to the unknot. Since a CW-complex is metrizable if and only if it is locally finite, the metric topology on *CK* induced from the *Gordian metric* on knots [12], where every edge has length =1, is strictly weaker than the CW topology, although the two are homotopy equivalent.

Remark 2. $C_n(X^b)$ and $C_n(X^0)$ have an interesting common quotient, $C_n(X^\omega)$, generated by elements $K_{1\dots n}$, and relations $K_{1\dots n} \sim_\omega \mathrm{sign}(\sigma) K_{\sigma(1\dots n)}$ for any $\sigma \in S_n$, $n \geqslant 2$. Modulo this relation, $\partial K_{\sigma(1\dots n)} = \mathrm{sign}(\sigma)\partial K_{1\dots n}$, so the natural projections to $C_n(X^\omega)$ are chain maps. The geometric realization of $C_*(X^\omega)$ would seem to be a natural orientable knot complex. Indeed, for CK_0, if $\sigma \in A_n$, $f_{K_{\sigma(1\dots n)}} \simeq f_{K_{1\dots n}} : \partial I^n \to CK_0^{(n-1)}$. However, the construction above fails because some knots are not orientable in this sense.

For $n \geqslant 2$, the projection $(X_n^0/A_n) \to X_n$ is either $2-1$ or $1-1$. *Nonorientable* knots have exactly one preimage, and are killed by $\sim_\omega$. Otherwise, $K_{\times\cdots\times}$ is *orientable* and has a generator in $C_n(X^\omega)$ for each coset of $K_{1\dots n}$ mod A_n. Infinite families of non-orientable knots exist. For example, $K_{1\dots 6}$ shown below is non-orientable since $K_{1\dots 6} = K_{\sigma(1\dots 6)}$, where $\sigma(123456) = (456123)$ is an odd permutation.



3. Homotopy of $\overline{CK}(\mathbb{R}^3)$

Let $\overline{CK}$ be CK with a 2-cell attached to every 1-cell corresponding to a degenerate 1-singular knot, and a 3-cell attached to every 2-sphere corresponding to a degenerate 2-singular knot. To be precise, for $i = 1, 2$, let D_i denote the corresponding degenerate subset of X_i^b or X_i^0. Let $A = A_1 \cup A_2$, where $A_1 = \cup_{\alpha_1 \in D_1} D^2_{\alpha_1}$ and $A_2 = \cup_{\alpha_2 \in D_2} D^3_{\alpha_2}$. Then $\overline{CK} = CK \cup_f A$, where $f_{\alpha_1} : \partial D^2_{\alpha_1} \to g_{\alpha_1}(I)\ \forall \alpha_1 \in D_1$, and $f_{\alpha_2} : \partial D^3_{\alpha_2} \to g_{\alpha_2}(I^2)\ \forall \alpha_2 \in D_2$. The map f_{α_2} will be explained below.

Let $\Phi : S^1 \times D^2 \to M$ be a family of piecewise linear maps $\{\phi_x : S^1 \to M | x \in D^2\}$, and let $\zeta_\Phi = \mathrm{closure}\{x \in D^2 | \phi_x \text{ is not an embedding}\}$.

Theorem 3 (Lin [9, Proposition 2.1]). *A map $\Phi : S^1 \times D^2 \to M$ can be perturbed so that ζ_Φ is a one-dimensional subpolyhedron of D^2 which intersects ∂D^2 in only finitely many boundary vertices. Moreover,*

(1) *if $x, x' \in D^2$ belong to the same component of $(D^2 \backslash \zeta_\Phi)$ or $(\zeta_\Phi \backslash \{\text{interior vertices}\})$, then ϕ_x and $\phi_{x'}$ are ambient isotopic*;
(2) *interior vertices of ζ_Φ are of valence* 4 *or* 1;
(3) *if $x \in \zeta_\Phi$ lies in an edge or is a boundary vertex, ϕ_x has exactly one transverse double point*;
(4) *if $x \in \zeta_\Phi$ is an interior* 4*-valent vertex, ϕ_x has exactly two transverse double points*;
(5) *if $x \in \zeta_\Phi$ is an interior* 1*-valent vertex, ϕ_x is an embedding ambient isotopic to nearby embeddings.*

The resulting map $\Phi : S^1 \times D^2 \to M$ is said to be in almost general position. *Moreover, if $\Phi | S^1 \times \partial D^2$ is already in almost general position, then the perturbation can be made relative to ∂D^2.*

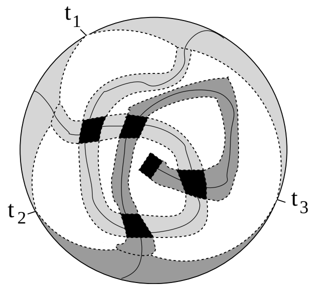


Fig. 2. η_Φ subdivided into sets R_α.

Corollary 4. *If* $\pi_1(M)=\pi_2(M)=1$, *then* $\pi_1(\overline{CK}(M))=1$.

Proof. Any loop γ in $\overline{CK}$ is homotopic to a loop in the 1-skeleton. We will define a map $\Psi:D^2\to\overline{CK}$, such that $\Psi|\partial D^2=\gamma$. We can assume that $T=\{t_i\in S^1:\gamma(t_i)\in\overline{CK}^{(0)}\}$ is finite, and otherwise $\gamma(\theta)\in\overline{CK}^{(1)}$. We can define $h_i:\{\theta\in S^1:t_i\leqslant\theta\leqslant t_{i+1}\}\to I$ and $\Phi:S^1\times S^1\to M$, such that $\gamma(\theta)=g_{\Phi(S^1,\theta_i)}(h_i(\theta))$, where $\Phi(S^1,\theta)$ is a 1-singular knot for finitely many $\theta_i\in S^1$, and is nonsingular otherwise. Therefore, $\Phi|_{S^1\times S^1}$ is in almost general position.

Because $\pi_1(M)=\pi_2(M)=1$, Φ extends to $S^1\times D^2$ rel $S^1\times\partial D^2$. By Theorem 3, we can perturb Φ to be in almost general position relative to ∂D^2, such that ζ_Φ is a graph in D^2 parametrizing immersions which are 1-singular, except that each 4-valent vertex is 2-singular, and each 1-valent vertex is a cusp. For CK_b, Φ can be perturbed both to be in almost general position and to preserve base points.

Consider the following pairs $\{(R_\alpha,h_\alpha)\}$ such that $D^2=\cup R_\alpha$. Let V be the set of interior 1-valent and 4-valent vertices of ζ_Φ. Let η_Φ be the union of a neighborhood of $\partial D^2\backslash T$ and thin bands containing ζ_Φ which are isotopic to a tubular neighborhood of $\zeta_\Phi\cap int(D^2)$ in $int(D^2)$, and such that $\partial(D^2\backslash\eta_\Phi)\cap\partial D^2=T$. Let $\{R_v\colon v\in V\}$ be the intersection or self-intersection of bands at every 4-valent vertex, and a disjoint rectangle at every 1-valent vertex. Then $\eta_\Phi=\{R_v\colon v\in V\}\cup\{R_\tau\colon \tau$ is a component of $\zeta_\Phi\backslash V\}$. Let $h_\tau:R_\tau\to I$ be any continuous map such that $h_\tau|\zeta_\Phi=0, h_\tau(x)=\pm1$ for $x\in\partial R_\tau$ with $\Phi(S^1,x)=K_\pm$, and $h_\tau(x)=h_i(x)\ \forall x\in\partial R_\tau\cap\partial D^2$. Let $h_v:R_v\to I^2$ be any continuous map extending $h_\tau(x)$ on each coordinate $\forall x\in\partial R_v\cap\partial R_\tau$. If α is a component of $D^2\backslash\eta_\Phi$, let $h_\alpha=0$. (see Fig. 2).

We now define $\Psi:D^2\to\overline{CK}$. For $x\in R_\alpha$, $\Psi(x)=g_{\Phi(S^1,x_\alpha)}(h_\alpha(x))$ for some $x_\alpha\in R_\alpha$ such that $\Phi(S^1,x_\alpha)\in X_n$ with n maximal for R_α. □

Theorem 5. $\pi_2(\overline{CK}(\mathbb{R}^3))=\mathbb{Z}$.

Proof. Let $\gamma:S^2\to\overline{CK}$. Up to homotopy, $\gamma(S^2)\subset\overline{CK}^{(2)}$ and is incident to finitely many 2-cells B_i. We can define h_i: $\{x\in S^2|\ \gamma(x)\in B_i\}\to I^2$ and $\Phi:S^1\times S^2\to\mathbb{R}^3$ such that $\gamma(x)=g_{\Phi(S^1,x_i)}(h_i(x))$, where $\Phi(S^1,x)$ is a 2-singular knot for finitely many $x_i\in S^2$, and is 1-singular or nonsingular otherwise. We regard Φ as a 2-parameter family of maps $S^1\to\mathbb{R}^3$ which determines a 2-sphere in $CK^{(2)}$. $\zeta_\Phi=\text{closure}\{x\in S^2|\ \Phi(\cdot,x):S^1\to\mathbb{R}^3\text{is not an embedding}\}$ is a graph of valence 1 or 4 on S^2 with 1-valent vertices corresponding to cusps, and 4-valent vertices to 2-singular knots. Unlike the case for S^3, there is no obstruction to extend Φ to a map $\tilde{\Phi}:S^1\times D^3\to\mathbb{R}^3$.

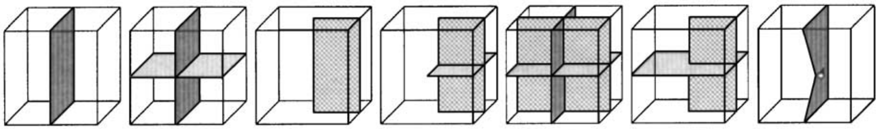


Fig. 3. 2-Complexes which appear locally in $\zeta_{\tilde{\Phi}}$.

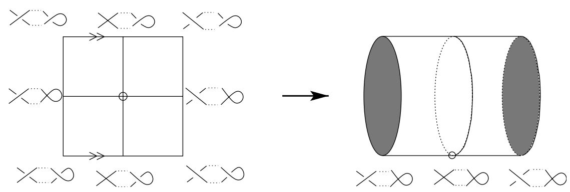


Fig. 4. 2-Sphere which appears in (6).

We can extend the argument in Theorem 3 from a 2-parameter to a 3-parameter family of maps $S^1 \to \mathbb{R}^3$. The partitions $3=1+1+1$, $3=2+1$, $3=1+2$, $3=3+0$ give the possible ways to group the parameters. Namely, deform three different small segments of S^1 by one parameter; deform two small segments by 2 parameters and 1 parameter, or vice versa; or just deform one small segment by several parameters. Deformations at different segments may intersect each other, resulting in the local singular sets shown in Fig. 3. Therefore, we can perturb $\tilde{\Phi}$ relative to ∂D^3 such that

(i) $\zeta_{\tilde{\Phi}} = \text{closure}\{x \in D^3 | \ \tilde{\Phi}(\cdot, x): S^1 \to \mathbb{R}^3 \text{is not an embedding}\}$ is a complex of dimension 2 in D^3 with $\zeta_{\tilde{\Phi}}|_{\partial D^3} = \zeta_{\Phi}$, and
(ii) locally, $\zeta_{\tilde{\Phi}}$ is one of the seven 2-complexes shown in Fig. 3.

Following the proof of Corollary 4, define a subdivision of D^3 to be $\{(R_\alpha, h_\alpha)\}$ such that $D^3 = \cup R_\alpha$ and $h_\alpha : R_\alpha \to I^3$. We say that a nonempty 2-complex $\zeta_{\tilde{\Phi}}$ in D^3 *can be realized in* $\overline{CK}$ if there exists a map $\Psi : D^3 \to \overline{CK}$ such that $\Psi|_{\partial D^3} = \gamma$, and a finite subdivision of D^3 such that for $x \in R_\alpha$, $\Psi(x) = g_{\tilde{\Phi}(S^1, x_\alpha)}(h_\alpha(x))$ for some $x_\alpha \in R_\alpha$ such that $\tilde{\Phi}(S^1, x_\alpha) \in X_n$ with n maximal for R_α.

The 2-complexes (1) and (2) can be realized in CK. By extending the proof of Corollary 4, (3) can be realized in $CK \cup_f A_1$, and hence in $\overline{CK}$.

(4) and (6) correspond to degenerate 2-singular knots. For any such $K, g_K(I^2)$ contains a 2-sphere, as shown in Fig. 4. In our construction of $\overline{CK}$, f_{α_2} attaches a 3-cell to every such 2-sphere, so that (4) and (6) can be realized in $\overline{CK}$. This resembles some kind of "compactification" of CK with respect to its 2-skeleton, as suggested in Fig. 5.

The basic model for the boundary of (7) is shown in Fig. 6. The identifications on this 2-cell follow from the isotopy shown in Fig. 9, so its image in CK is a 2-sphere. Also shown in Fig. 6 is the corresponding graph ζ_Φ on the 2-sphere. In the basic model, we show only fragments of a knot diagram because changing other crossings can be realized by a homotopy of 2-spheres in CK. Let K be any 2-singular knot $\Phi(S^1, x)$, such that x is the 4-valent vertex in the boundary of (7). If K' is obtained from K by changing any nonsingular crossing, then we can find a 3-cell $g_{K_{123}}(I^3)$ in CK with the front face of I^3 mapped by $g_K(I^2)$ and back face mapped by $g_{K'}(I^2)$. We may continue

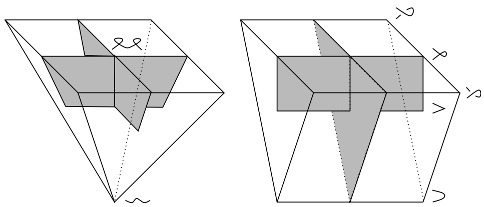


Fig. 5. Another view of (4) and (6).

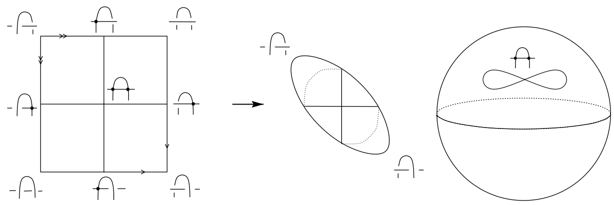


Fig. 6. Basic model for the boundary of (7) and related graph $\zeta_\Phi \subset S^2$.

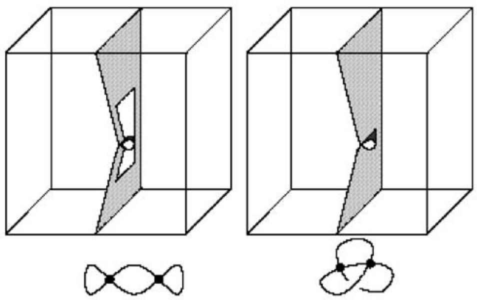
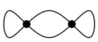
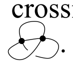
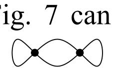
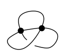
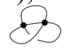


Fig. 7. Two extensions of Fig. 6 to a 2-complex with associated knots.

changing crossings, gluing each 3-cell front to back, and keeping the identification of the sides of I^3. The resulting 3-cell is a homotopy in CK from the top 2-sphere $g_K(I^2)$ to the bottom 2-sphere, which can be chosen to be any representative of our basic model.

Since, up to homotopy, we can change crossings, the basic model can be realized as a 2-sphere in CK in two different ways:  or . Up to crossing changes, these are the two distinct classes of 2-singular knots. They correspond to the two possible extensions of the boundary of (7) to locally different 2-complexes, as shown in Fig. 7. The first 2-complex in Fig. 7 can be realized in $\overline{CK}$ because it is essentially like (4). Thus, the 2-sphere corresponding to  is trivial in $\pi_2(\overline{CK})$.

However, the 2-sphere corresponding to  is nontrivial in $\pi_2(\overline{CK})$, and any choice of ordering or base point is equivalent up to homotopy. The latter claim is that  is unique up to crossing changes in X_2^b and in X_2^0, which can be shown by an explicit isotopy, but is obvious by its chord diagram. We can prove nontriviality from the results of Section 4. For $\Phi : S^1 \times I^2 \to \mathbb{R}^3$ shown in

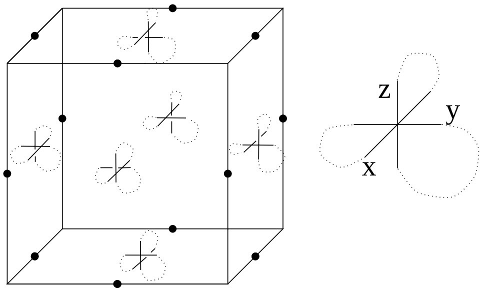
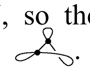
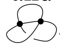
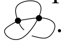


Fig. 8. $3=2+1$ model for (5).

Fig. 6, let $\gamma : S^2 \to \overline{CK}$ be this element in $\pi_2(\overline{CK})$. By the Hurewicz theorem, Corollary 4 implies that $[\gamma]$ is nontrivial in homotopy whenever it is nontrivial in homology. By the proof of Theorem 9, $v_2([\gamma]) \neq 0$ proves our claim.

In the only remaining case, the triple point in (5) is an immersion with 3 parameters to obtain 2-singular knots. The partitions $3=1+1+1$, $3=2+1$, $3=1+2$ give the possible ways to group the parameters. The first case is three separate double points, such that each double point can be independently perturbed to obtain a 2-singular knot, which can be realized in CK.

The other two cases, equivalent by symmetry, arise from a triple point on a singular knot. The model, shown in Fig. 8, is to perturb the z-strand by 2 parameters, and the x-strand by 1 parameter to obtain 2-singular knots. (Fig. 36 in [15] is similar.) The boundary of the cube can be mapped to a 2-sphere in CK by any realization of the six 2-singular diagram fragments shown. As above, changing crossings can be realized by a homotopy along 3-cells in CK, so the boundary of the cube is homotopic to the boundary of the cube formed by 3  and 3 . Since the latter is trivial in homotopy, the boundary of the cube is a multiple of . Therefore, $\pi_2(\overline{CK}) \cong \mathbb{Z}$, generated by . □

4. Vassiliev invariants and cohomology of CK

Any knot invariant $\phi : X_0 \to \mathbb{Q}$ can be extended to singular knots inductively by the following skein relation:

$$\phi(K_{\times}) = \phi(K_{+}) - \phi(K_{-}), \quad \phi(K_{\times\cdots\times\cdots\times}) = \phi(K_{\times\cdots+\cdots\times}) - \phi(K_{\times\cdots-\cdots\times}). \tag{1}$$

Definition 6. A knot invariant is of *finite type* or a *Vassiliev invariant* if there exists $n \in \mathbf{N}$ such that its extension to singular knots vanishes on knots with more than n double points. The smallest such n is the *order* of the invariant. Let $\mathbb{V}_n(M)$ be the vector space of invariants of knots in M of order $\leqslant n$.

Any invariant $\phi : X_n \to \mathbb{Q}$ extends to a cochain $\phi \in C^n(CK)$ by forgetting the ordering or base point: $\phi(K_{1\ldots n}) = \phi(K_{\times\cdots\times})$. In particular, $\mathbb{V}_N|_{C_n}$ is a subspace of the cochain complex $\{C^n(CK, D; \mathbb{Q}), \delta^n\}$ in this way. For any $v, w \in \mathbb{V}_N$ if $v|_{C_n} = w|_{C_n}$ then on the space of knots, $v = w$ up to invariants of order $n-1$. Therefore, $\mathbb{V}_N/\mathbb{V}_{n-1} \hookrightarrow C^n(CK,D)$ is an embedding. We only consider rational coefficients, and adopt the standard notation, $H^n(CK,D)=Z^n(CK,D)/B^n(CK,D)$.

Proposition 7. *For any* $\phi\in\mathbb{V}_n$, $\delta^n\phi=0$. *i.e.*, $\mathbb{V}_n/\mathbb{V}_{n-1}\hookrightarrow Z^n(CK)$.

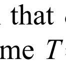
Proof. For any $K_{1\ldots n+1}, \phi(\mathrm{d}_i K_{1\ldots n+1})=0\ \forall i$, so $\phi(\partial K_{1\ldots n+1})=0$. □

From the exact sequence of the pair (CK,D), we obtain

$$\begin{array}{ccccccc} H^{n-1}(D;\mathbb{Q}) & \longrightarrow & H^n(CK,D;\mathbb{Q}) & \xrightarrow{\alpha} & H^n(CK;\mathbb{Q}) & \xrightarrow{\beta} & H^n(D;\mathbb{Q}) \\ & & \nwarrow{\scriptstyle \exists\gamma} & & \uparrow & \nearrow{\scriptstyle 0} & \\ & & & & \mathbb{V}_n|_{C_n} & & \end{array} \tag{2}$$

Since $\mathbb{V}_n|_{C_n}\subset \ker\beta = \operatorname{Im}\alpha$, there is an induced map $\gamma:\mathbb{V}_*\to H^*(CK,D)$.

Theorem 8. *For* $\phi\in\mathbb{V}_2(\mathbb{R}^3)$, $0\neq[\phi]\in H^2(CK(\mathbb{R}^3))$.

Proof. For any nondegenerate $T\in X_2(\mathbb{R}^3)$, $\mathbb{V}_2(\mathbb{R}^3)$ is generated by ϕ such that $\phi(T)=1$. If $T'\in X_2$ is obtained from T by crossing changes, then $\phi(T')=\phi(T)$, so we may assume $T=$. By Proposition 7, $\delta\phi=0$. Suppose by contradiction $\phi=\delta\psi$ for some $\psi\in C^1(CK)$. For any corresponding element $T\in X_2^0(\mathbb{R}^3)$ or $X_2^b(\mathbb{R}^3)$, $1=\phi(T)=\delta\psi(T)=\psi(\partial(T))=\psi(\mathrm{d}_1T-\mathrm{d}_2T)=\psi(0)=0\Rightarrow\Leftarrow$. □

Recall $\overline{CK}=CK\cup_f A$. Extend $v\in\mathbb{V}_N$ to $C^*(\overline{CK})$ by $v|_\sigma=v|_{\partial\sigma}=0,\ \forall\sigma\in A$.

Theorem 9. $\mathbb{V}_2(\mathbb{R}^3)|_{C_2}\cong H^2(\overline{CK}(\mathbb{R}^3))$.

Proof. Since CK is a CW-subcomplex of $\overline{CK}$,

$$\tilde{H}^*(\overline{CK},CK)\cong\tilde{H}^*(\overline{CK}/CK)\cong\tilde{H}^*\left(\bigvee_{\alpha\in D_1}S^2_\alpha\bigvee_{\beta\in D_2}S^3_\beta\right).$$

From the exact sequence of the pair $(\overline{CK},CK)$, we obtain

$$\begin{array}{ccccccc} \tilde{H}^2(\bigvee S^2_\alpha\bigvee S^3_\beta) & \longrightarrow & \tilde{H}^2(\overline{CK}) & \xrightarrow{\alpha} & \tilde{H}^2(CK) & \xrightarrow{\delta} & \tilde{H}^3(\bigvee S^2_\alpha\bigvee S^3_\beta) \\ & & \nwarrow{\scriptstyle \exists\tilde{\gamma}} & & \uparrow & \nearrow{\scriptstyle 0} & \\ & & & & \mathbb{V}_2|_{C_2} & & \end{array} \tag{3}$$

Since $v|_\sigma=0,\ \forall\sigma\in A, \mathbb{V}_2|_{C_2}\subset\ker\delta=\operatorname{Im}\alpha$. Thus, there is an induced map $\tilde{\gamma}:\mathbb{V}_2|_{C_2}\to\tilde{H}^2(\overline{CK})$, which is nontrivial by Theorem 8. By the Hurewicz theorem, Corollary 4 and Theorem 5 imply that $H^2(\overline{CK}(\mathbb{R}^3);\mathbb{Z})=\mathbb{Z}$. Therefore, $\mathbb{V}_2$ uniquely generates $H^2(\overline{CK}(\mathbb{R}^3))$. □

Many arguments for Vassiliev invariants rely on augmented Reidemeister moves on projections of links and graphs. Such arguments are available only in $\mathbb{R}^3$ or S^3, so Theorem 8 holds only for $\mathbb{R}^3$ or

S^3. In [9], these concepts were extended to more general 3-manifolds by studying Map$(S^1 \times D^2, M)$ in almost general position. In particular, Theorem 10 generalizes an important result proved by Stanford for $\mathbb{R}^3$, and extended further by Kalfagianni (Theorem 4.1 [8]).

We say ϕ is *differentiable* if $\phi(\mathrm{d}_i K_{1\dots n}) = \phi(\mathrm{d}_j K_{1\dots n})\ \forall i,j$. In fact, if we forget the ordering, $\phi : X_n \to \mathbb{Q}$ extends to X_{n+1} satisfying the skein relation (1) if and only if ϕ is differentiable. We say $\phi : X_n \to \mathbb{Q}$ is *integrable* if $\exists \psi : X_{n-1} \to \mathbb{Q}$ such that, for any ordering, $\phi(K_{1\dots n}) = \psi \circ \mathrm{d}_i(K_{1\dots n})\ \forall i$. In this case, we write $\int \phi = \psi$. We will say $\phi \in C^n(CK)$ is integrable if ϕ is an integrable invariant of X_n, invariant under changes of ordering or base point.

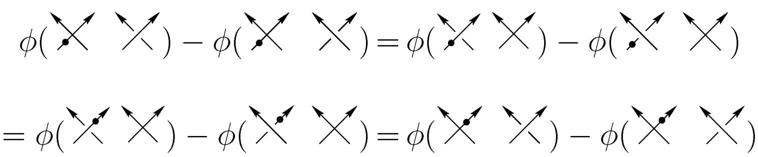
Theorem 10 (Lin [9]). *Suppose $\pi_1(M) = \pi_2(M) = 1$. Then $\phi : X_n(M) \to \mathbb{Q}$ is integrable if and only if it satisfies*: (i) *the* 1-*term relation*, (ii) *the* 4-*term relation*, (iii) *ϕ is differentiable*.

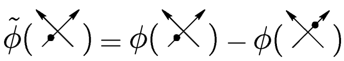
Corollary 11. *If $\pi_1(M) = \pi_2(M) = 1$, then $H^1(CK(M), D) = 0$.*

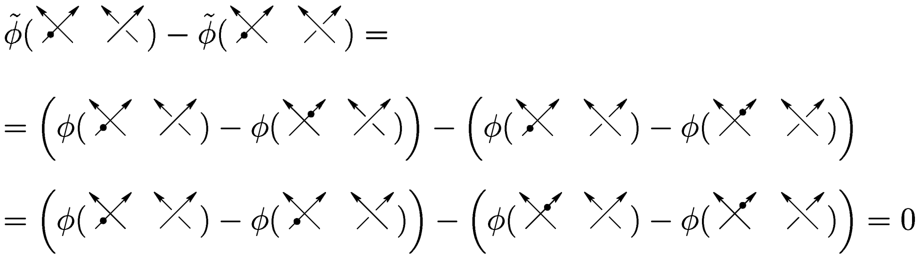
Proof. If $\phi \in Z^1(CK, D)$, we can find $\psi \in C^0(CK)$ such that $\phi = \delta\psi$. (i) $\phi|_{D_1} = 0$, so ϕ satisfies the 1-term relation. (ii) The 4-term relation is trivially satisfied for X_1. (iii) $\delta\phi = 0$ implies $\phi(\mathrm{d}_1 K_{12}) = \phi(\mathrm{d}_2 K_{12})$ so ϕ is differentiable. For CK_0, Theorem 10 implies that ϕ is integrable, so $\exists\psi$ with $\phi(K_\times) = \psi(\mathrm{d}K_\times) = \delta\psi(K_\times)$. For CK_b, we must also show that ϕ is invariant under changes of base point, so the result follows from the following lemma. □

Lemma 12. *Suppose $\pi_1(M) = 1$. If $\phi \in C^1(CK_b(M))$ is differentiable, then ϕ is invariant under changes of base point.*

Proof. Since ϕ is differentiable, the following identity holds:



Define . Clearly, ϕ is invariant under changes of base point if and only if $\tilde{\phi} = 0$.



As $\tilde{\phi}$ is invariant under crossing changes and $\pi_1(M) = 1$, for all $K \in X_1^b$, $\tilde{\phi}(K) =$

□

Lemma 13. *If $\pi_1(M) = \pi_2(M) = 1$, then the natural restriction map $\rho : H^1(CK) \to H^1(D)$ is an isomorphism.*

Proof. From the exact sequence of the pair (CK, D) and Corollary 11, it suffices to show that ρ is surjective.

$$Z^1(CK) = \{\phi \in C^1(CK)\colon\ \delta\phi = 0\} = \{\phi \in C^1(CK)\colon\ \phi \text{ is differentiable}\},$$

$$B^1(CK) = \{\phi \in C^1(CK)\colon\ \phi = \delta\psi\} = \{\phi \in C^1(CK)\colon\ \phi \text{ is integrable}\}.$$

By Theorem 10, $H^1(CK) = \{\phi \in Z^1(CK)\colon\ \phi$ does not satisfy 1-term relations$\}$. Let D_n denote degenerate n-singular knots. For $K_{12} \in D_2$, $\partial K_{12} = (-1)^{i+1} \mathrm{d}_i K_{12}$ for $i = 1$ or 2, so if $\phi|_{\partial D_2} = 0$ then ϕ is invariant under crossing changes, and therefore constant.

$$Z^1(D) = \{\phi \in C^1(D)\colon\ \phi|_{\partial D_2} = 0\} = \{\phi \in C^1(D)\colon\ \phi \text{ is constant}\},$$

$$B^1(D) = \{\phi \in C^1(D)\colon\ \phi = \delta\psi|_{D_1} = 0\}.$$

Extend any map in $H^1(D) = Z^1(D)$ to a constant map in $Z^1(CK)$. □

Theorem 14. *For $M = \mathbb{R}^3$, $\mathbb{V}_2|_{C_2} \cong H^2(CK, D)$.*

Proof. From Lemma 13, we obtain

$$\begin{array}{ccccccccc} H^1(CK) & \xrightarrow{\cong} & H^1(D) & \xrightarrow{0} & H^2(CK,D) & \longrightarrow & H^2(CK) & \longrightarrow & H^2(D) \\ & & & & \nwarrow{\scriptstyle \exists\gamma} & & \uparrow & \nearrow{\scriptstyle 0} & \\ & & & & & & \mathbb{V}_2|_{C_2} & & \end{array}$$

By Corollary 4, $H^1(\overline{CK}) = 0$. By Theorem 9 we obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(D \cup_f A) & \xrightarrow{\alpha} & H^2(\overline{CK}, D \cup_f A) & \longrightarrow & H^2(\overline{CK}) & \xrightarrow{0} & H^2(D \cup_f A) \\ & & & & \nwarrow{\scriptstyle \exists\tilde{\gamma}} & & \uparrow{\scriptstyle \cong} & \nearrow{\scriptstyle 0} & \\ & & & & & & \mathbb{V}_2|_{C_2} & & \end{array}$$

We can relate these diagrams by the following commutative diagram, where the verticals are restriction maps, and the isomorphism is by excision

$$\begin{array}{ccc} H^1(D) & \xrightarrow{0} & H^2(CK, D) \\ \uparrow & & \uparrow{\scriptstyle \cong} \\ H^1(D \cup_f A) & \xrightarrow{\alpha} & H^2(\overline{CK}, D \cup_f A) \end{array}$$

Thus, $\alpha = 0$, and consequently $\tilde{\gamma} : \mathbb{V}_2 \xrightarrow{\cong} H^2(\overline{CK}, D \cup_f A) \cong H^2(CK, D)$. □

In higher dimensional cohomology, we consider even and odd cases separately.

Theorem 15. *Suppose* $\pi_1(M)=\pi_2(M)=1$. *For any Vassiliev invariant* $\phi\in\mathbb{V}_N$, $[\phi]=0\in H^{2k+1}(CK,D)$ $\forall N,k\geqslant 0$.

Proof. Every Vassiliev invariant $\phi\in\mathbb{V}_N$ is integrable: $\int(\phi|_{C_{2k+1}})=\phi|_{C_{2k}}$, so by Lemma 16, $[\phi]=0\in H^{2k+1}(CK)$ $\forall k\geqslant 0$. Therefore, from (2), $\gamma(\phi)\in\ker\alpha$. Since $\mathbb{V}_*\to H^*(D)$ is always zero, $[\phi]=0\in H^{2k+1}(CK,D)$ $\forall k\geqslant 0$. □

We note that Lemma 16 has appeared earlier (Lemma 7.2 [4]).

Lemma 16. *For n odd, if* $\phi\in C^n(CK)$ *is integrable, then* $[\phi]=0\in H^n(CK)$.

Proof. Let $\int\phi=\psi$. As ϕ is integrable, it is differentiable so $\phi\in Z^n(CK)$ because the first alternating sum has an even number of equal terms:

$$\delta\phi(K_{1\dots n+1})=\phi(\partial K_{1\dots n+1})=\sum_{i=1}^{n+1}(-1)^{i+1}\phi(\mathrm{d}_iK_{1\dots n+1})=0,$$

$$\delta\psi(K_{1\dots n})=\sum_{i=1}^{n}(-1)^{i+1}\psi(\mathrm{d}_iK_{1\dots n})=\sum_{i=1}^{n}(-1)^{i+1}\phi(K_{1\dots n})=\phi(K_{1\dots n}).\qquad\square$$

Theorem 17. *Suppose* $\pi_1(M)=\pi_2(M)=1$. *For any* $n=2k$, $\mathbb{V}_n(M)=\{\phi\in Z^n(CK,D)$: ϕ *is integrable*$\}$.

Proof. Every Vassiliev invariant is an integrable cocycle by Proposition 7. Conversely, let ϕ be an integrable cocycle. If ϕ is both integrable and invariant under crossing changes, then ϕ is a weight system. Since $\pi_1(M)=1$, $\mathbb{V}_n(M)$ contains a vector subspace isomorphic to $\mathbb{V}_n(\mathbb{R}^3)$ (Theorem 0.1 [8]). Over $\mathbb{Q}$, any weight system can be integrated all the way to a knot invariant by the Kontsevich integral, so $\phi\in\mathbb{V}_n(M)$. The result follows from Lemma 18.

Lemma 18. *For n even, suppose* $\phi\in C^n(CK)$ *is differentiable. Then* $\delta\phi=0$ *if and only if* ϕ *is invariant under crossing changes.*

Proof. $\delta\phi(K_{1\dots n+1})=\sum_{i=1}^{n+1}(-1)^{i+1}\phi(\mathrm{d}_iK_{1\dots n+1})=\phi(\mathrm{d}_{n+1}K_{1\dots n+1})$ because the alternating sum has an odd number of equal terms, with the first and last terms positive. Therefore, $\delta\phi=0$ if and only if $\phi(K_+)-\phi(K_-)=\phi(\mathrm{d}_{n+1}K_\times)=0$. □

Any knot invariant $\phi\in C^0(CK)$ can be extended to an invariant of singular knots by the skein relation (1). Then ϕ is integrable, but may not be of finite type. The proofs of Theorems 15 and 17 imply that if ϕ is not of finite type, then $[\phi]=0\in H^n(CK,D)$, $\forall n\geqslant 0$. We do not know whether $H^{2k+1}(CK,D)$ is nontrivial for any $k\geqslant 1$.

Corollary 19. *If* $\phi\in\mathbb{V}_N\backslash\mathbb{V}_{N-1}$ *and* $[\phi]\in H^n(CK,D)$ *is nontrivial, then* $N=n=2k$.

Proof. By hypothesis, $\phi\in\mathbb{V}_N|_{C_n}$. As $[\phi]\neq 0, n\leqslant N$, and $n=2k$ by Theorem 15. By Theorem 17, $\phi\in\mathbb{V}_n|_{C_n}$, so $N\leqslant n$. Therefore, $N=n=2k$. □

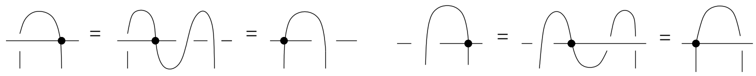
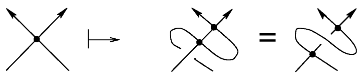


Fig. 9. An isotopy.

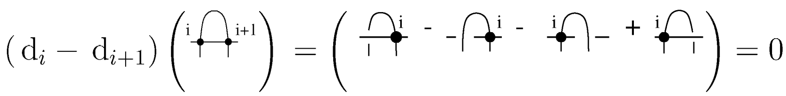
We do not know whether the converse to Corollary 19 holds. Namely, if $\phi \in \mathbb{V}_n \backslash \mathbb{V}_{n-1}$ for even n, is $[\phi] \in H^n(CK, D)$ nontrivial? After considering this problem together, Akira Yasuhara proved the following partial affirmative result for $\mathbb{R}^3$, generalizing Theorem 8.

Theorem 20 (Yasuhara). *For any $n \geqslant 1$, there exist Vassiliev invariants in $\mathbb{V}_{2n}(\mathbb{R}^3)$ which are nontrivial in $H^{2n}(CK(\mathbb{R}^3), D)$.*

Proof. Fix $n \geqslant 1$. For any n-chord diagram D_n, let D_{2n} be obtained by replacing each chord of D_n with an adjacent pair of chords, allowing either $\|$ or $\times$ for each of these pairs. Any n-singular knot which represents D_n can be perturbed to a $2n$-singular knot representing D_{2n} by this local change:



Let $K_{1\ldots 2n}$ represent D_{2n}, such that every perturbed pair of double points is ordered consecutively modulo $2n$ (e.g., an ordering from any base point). By the isotopy in Fig. 9, $\forall i \in \{1, 3, \ldots, 2n-1\}$,

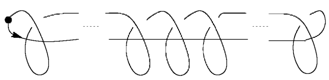


Therefore, $\partial K_{1\ldots 2n} = 0$. Any $v \in \mathbb{V}_{2n}(\mathbb{R}^3)$ which is nonzero on $K_{1\ldots 2n}$ is not a coboundary, and represents a nontrivial cohomology class in $H^{2n}(CK(\mathbb{R}^3))$. By (2), v lifts to a nontrivial class in $H^{2n}(CK(\mathbb{R}^3), D)$. □

Corollary 21. *For any $v \in \mathbb{V}_{2n}(\mathbb{R}^3)$ coming from the Conway polynomial, v is nontrivial in $H^{2n}(CK(\mathbb{R}^3), D)$.*

Proof. Let W_C be the Conway weight system. For any n-chord diagram D, let D' be the perturbed chord diagram obtained by replacing each chord of D by an adjacent pair of intersecting chords. By Theorem 2 of [2], $W_C(D') = W_C(\bigcirc) = 1$, where $\bigcirc$ denotes the chord diagram with zero chords. Let $K_{1\ldots 2n}$ represent D' as in the proof of Theorem 20, so $\partial K_{1\ldots 2n} = 0$. Since $v(K_{1\ldots 2n}) = W_C(D') = 1$, v is not a coboundary. □

For example, if $v_2 \in \mathbb{V}_2(\mathbb{R}^3)$ then v_2^n is nontrivial in $H^{2n}(CK(\mathbb{R}^3), D)$ because v_2^n is nonzero on $K_{1\ldots 2n}$ shown below with $\partial K_{1\ldots 2n} = 0$.



Remark 22. Using some auxiliary geometric object to characterize the automorphisms of a space has a long history. In surface theory, every automorphism of the curve complex is induced by a surface diffeomorphism, and a bijection on vertices is induced if and only if it preserves edges in the curve complex [7,11]. In our case, the group of isotopy classes of diffeomorphisms of M acts naturally on $CK(M)$. For $\mathbb{R}^3$, the only two classes are the identity and reflection, $K_{1\dots n} \mapsto K^*_{1\dots n}$. By analogy with the curve complex, we propose Conjecture 23.

Define $F: CK(\mathbb{R}^3) \to CK(\mathbb{R}^3)$ by $F \circ g_{K_{1\dots n}}(x) = g_{K^*_{1\dots n}}(-x)$ for any $x \in I^n$. For the induced map $F: C_n(X) \to C_n(X)$, $F(K_{1\dots n}) = [F \circ g_{K_{1\dots n}}(I^n)] = (-1)^n[g_{K^*_{1\dots n}}(I^n)] = (-1)^n K^*_{1\dots n}$. Since $(\mathrm{d}_i(K_{1\dots n}))^* = -\mathrm{d}_i K^*_{1\dots n}$, it follows that F is a chain map, and induces F^* on $H^*(CK)$. Whenever $K_{1\dots n}$ is degenerate, so is $K^*_{1\dots n}$, so F induces a map on $H^*(CK, D)$. If $v_n \in \mathbb{V}_n$ is canonical in the sense of [2], $v_n(K^*) = (-1)^n v_n(K)$.

$$v_n \circ F(K_{1\dots m}) = (-1)^m v_n(K^*_{1\dots m}) = (-1)^{n+2m} v_n(K_{1\dots m}) = (-1)^n v_n(K_{1\dots m}).$$

By Corollary 19, if v_n is nontrivial in $H^m(CK, D)$, then $n = m = 2k$. Thus, F^* acts as the identity on the subgroup of Vassiliev invariants in $H^*(CK, D)$. Geometrically, F is a bijection on n-cells of CK. Gillete and Van Buskirk [5] presented a minimal knot diagram with a crossing which can be switched to obtain the mirror image knot. F maps this edge to itself, but with reversed orientation.

Conjecture 23. *If Φ is a self-homeomorphism of $CK(\mathbb{R}^3)$ which is a bijection on vertices, such that both Φ and Φ^{-1} preserve edges, then Φ is homotopic to the identity or to the map F.*

We now consider the cup product in $H^*(CK)$, following Section 9.3 of [6]. Let $H = (h_1 \dots h_p) \subset \{1 \dots n\}$ be any subset (possibly empty) with the natural order on its elements, and let K be the complementary subset with the natural order on its elements. Let $\rho_{(HK)}$ denote the sign of $\sigma \in S_n$, where $\sigma(HK) = (1 \dots n)$. For $\varepsilon = \pm 1$, let $\lambda^{\varepsilon}_H(u_1, \dots, u_p) = (v_1, \dots, v_n)$, where $v_i = \varepsilon$ if $i \notin H$ and $v_{h_r} = u_r$, $r = 1, \dots, p$. Thus, λ^{-1}_H is an isometry of I^p onto a particular back p-face of I^n, and λ^{+1}_H maps onto the parallel front p-face of I^n. Define $\Delta: C_n(CK) \to (C_*(CK) \otimes C_*(CK))_n$ by

$$\Delta(g_{K_{1\dots n}}) = \sum_{H \subset \{1\dots n\}} \rho_{(HK)} g_{K_{1\dots n}} \circ \lambda^{-1}_H \otimes g_{K_{1\dots n}} \circ \lambda^{+1}_K = \sum_{H \subset \{1\dots n\}} \rho_{(HK)} g_{\mathrm{d}^-_K K_{1\dots n}} \otimes g_{\mathrm{d}^+_H K_{1\dots n}}.$$

In our notation, d^+_H takes the positive resolution of double points with labels in H, and similarly for d^-_K. We can simply refer to the singular knots:

$$\Delta(K_{1\dots n}) = \sum_{H \subset \{1\dots n\}} \rho_{(HK)} \mathrm{d}^-_K K_{1\dots n} \otimes \mathrm{d}^+_H K_{1\dots n}. \tag{4}$$

If $u, v \in H^*(CK), u \cup v(K_{1\dots n}) = \sum_{H \subset \{1\dots n\}} \rho_{(HK)} u(\mathrm{d}^-_K K_{1\dots n}) \cdot v(\mathrm{d}^+_H K_{1\dots n})$. In $\mathbb{V}_n$, $u \cdot v(K_{1\dots n})$ is the same expression without $\rho_{(HK)}$ [17]. For example,

$$\begin{aligned} & v_2 \cup v_2(K_{1234}) \\ &= v_2(K_{\times\times--}) v_2(K_{++\times\times}) - v_2(K_{\times-\times-}) v_2(K_{+\times+\times}) + v_2(K_{\times--\times}) v_2(K_{+\times\times+}) \\ &\quad + v_2(K_{-\times\times-}) v_2(K_{\times++\times}) - v_2(K_{-\times-\times}) v_2(K_{\times+\times+}) + v_2(K_{--\times\times}) v_2(K_{\times\times++}). \end{aligned}$$

On p. 210 of [14], Quillen showed that if $A = \oplus_{n \geqslant 0} A_n$ is a graded commutative algebra over $\mathbb{Q}$ with A_n finite dimensional for all n, and $A_0 = \mathbb{Q}, A_1 = 0$, then A is the rational cohomology ring of a simply connected pointed topological space. The algebra of Vassiliev invariants under the cup product, $\mathbb{V} = (\oplus_{n\geqslant 0}\mathbb{V}_n, \cup)$, satisfies these conditions. Therefore, there exists a simply connected pointed space X such that $H^*(X,\mathbb{Q}) = \mathbb{V}$. For even Vassiliev invariants, the simply connected knot complex $\overline{CK}(\mathbb{R}^3)$ seems very close to such a space since by Theorem 9, $\mathbb{V}_2 \cong H^2(\overline{CK})$, and by (3), $H^p(\overline{CK}) \cong H^p(CK)\ \forall p \geqslant 4$.

5. Hopf algebra of ordered chord diagrams

We show the cup product in $H^*(CK_0)$ for Vassiliev invariants arises naturally from a cocommutative differential graded Hopf algebra $\mathscr{A}^0$ of ordered chord diagrams.

Let $\mathfrak{D} = \oplus\mathfrak{D}_n$ denote the free abelian group over $\mathbb{Q}$ generated by chord diagrams with boundary oriented counterclockwise, graded by the number of chords. An *ordered chord diagram* has its chords ordered from 1 to n, equivalent up to rotation. Let $\mathfrak{D}^0$ denote the free abelian group over $\mathbb{Q}$ generated by equivalence classes of ordered chord diagrams.

Let $\mathscr{A}$ be the usual Hopf algebra of chord diagrams modulo the 4T relation [1]. The commutative product $D^1 \cdot D^2$ corresponds to the direct sum of knots and is well defined due to the 4T relation. The cocommutative coproduct is defined as follows: For $D \in \mathfrak{D}_n$, choose any ordering of its chords from 1 to n. Let $H,K \subset \{1\dots n\}$ be complementary subsets (possibly empty). Let D_H denote the chord diagram obtained from D by removing chords with labels in H. Define $\Delta(D) = \sum_{H\subset\{1\dots n\}} D_K \otimes D_H$. If $v \in \mathbb{V}_m$ and $D \in \mathfrak{D}_m$, then $W_m(v)(D) = v(K_D)$ defines a weight system. The following relations are well known (see, e.g., [1,17]). If $v_1 \in \mathbb{V}_p$ and $v_2 \in \mathbb{V}_q$, let $n = p+q$

$$v_1 \cdot v_2(K_{1\dots n}) = \sum_{H\subset\{1\dots n\}} v_1(\mathrm{d}_K^- K_{1\dots n}) \cdot v_2(\mathrm{d}_H^+ K_{1\dots n}),$$

$$W_n(v_1 \cdot v_2) = (W_p(v_1) \otimes W_q(v_2)) \circ \Delta.$$

The fact that the algebra of Vassiliev invariants is a commutative and cocommutative Hopf algebra was obtained via weight systems on chord diagrams [1]. This fact can be proved directly via the dual bialgebra of singular knots modulo the skein relation (1) [13,10]. Take knots as group-like elements, $\Delta(K) = K \otimes K$, and extend Δ to singular knots by the skein relation (1) to obtain the expression analogous to (4): $\Delta(K_{1\dots n}) = \sum_{H\subset\{1\dots n\}} \mathrm{d}_K^- K_{1\dots n} \otimes \mathrm{d}_H^+ K_{1\dots n} \bmod K_\times = K_+ - K_-$.

We now define a bialgebra of ordered chord diagrams which is compatible with the cup product for Vassiliev invariants:

$$v_1 \cup v_2(K_{1\dots n}) = \sum_{H\subset\{1\dots n\}} \rho_{(HK)} v_1(\mathrm{d}_K^- K_{1\dots n}) \cdot v_2(\mathrm{d}_H^+ K_{1\dots n}).$$

The 4T relation is given by four diagrams, which are the same except for one “fixed” chord and one “moving” chord. The *ordered 4T relation* on $\mathfrak{D}^0$ is given by the same expression, where the fixed chord and the moving chord have the same label in all four diagrams (see Fig. 10). Thus, for each 4T relation on $\mathfrak{D}_n$, $n \geqslant 2$, we obtain $n(n-1)$ ordered 4T relations on $\mathfrak{D}^0_n$.

Let $\mathscr{A}^0$ be the quotient of $\mathfrak{D}^0$ by all ordered 4T relations. If $D^1 \in \mathfrak{D}^0_p$ and $D^2 \in \mathfrak{D}^0_q$, the product $D^1 \cup D^2$ is defined to be the chord diagram $D^1 \cdot D^2$ with its ordering given by the same labels for chords from D^1 and by labeling the ith chord of D^2 by $p+i$. By the same argument as in Proposition 4.4 [17], the cup product is well defined on $\mathfrak{D}^0$ modulo the ordered 4T relation.

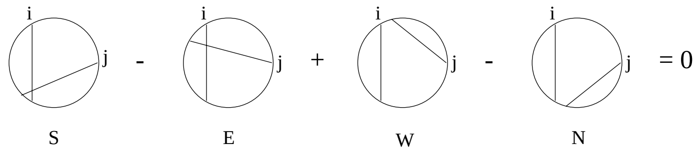


Fig. 10. Ordered 4T relation.

For any $D \in \mathfrak{D}^0$, let D_H be obtained by removing the chords with labels in H, with its ordering induced from D. Let $\rho_{(HK)}$ be defined as in (4). Define $\Delta^0 : \mathfrak{D}_n^0 \to \mathfrak{D}_n^0 \otimes \mathfrak{D}_n^0$ by $\Delta^0(D) = \sum_{H \subset \{1 \dots n\}} \rho_{(HK)} D_K \otimes D_H$. Then Δ^0 descends to a coproduct on $\mathscr{A}_n^0$: Δ^0(ordered 4T) = (ordered 4T) $\otimes \bigcirc + \bigcirc \otimes$ (ordered 4T), where $\bigcirc$ denotes any chord diagram in the ordered 4T relation with chords i and j removed. To show Δ^0 is cocommutative, apply $T(a \otimes b) = (-1)^{|a|\,|b|} b \otimes a$. $T \circ \Delta^0(D) = \sum_{|H|=p} (-1)^{pq} \rho_{(HK)} D_H \otimes D_K = \sum_H \rho_{(KH)} D_H \otimes D_K = \Delta^0(D)$.

Proposition 24. *For* $D^1, D^2 \in \mathscr{A}^0$, $\Delta^0(D^1 \cup D^2) = \Delta^0(D^1) \cup \Delta^0(D^2)$.

Proof. If $D^1 \in \mathscr{A}_p^0$ and $D^2 \in \mathscr{A}_q^0$, let $n = p + q$.

$$\begin{aligned}
&\Delta^0(D^1) \cup \Delta^0(D^2) \\
&= \left(\sum_{H_1 \subset \{1 \dots p\}} \rho_{(H_1 K_1)} D^1_{K_1} \otimes D^1_{H_1} \right) \cup \left(\sum_{H_2 \subset \{1 \dots q\}} \rho_{(H_2 K_2)} D^2_{K_2} \otimes D^2_{H_2} \right) \\
&= \sum_{H_1, H_2} \rho_{(H_1 K_1)} \rho_{(H_2 K_2)} (-1)^{|D^1_{H_1}||D^2_{K_2}|} D^1_{K_1} \cup D^2_{K_2} \otimes D^1_{H_1} \cup D^2_{H_2} \\
&= \sum_{H \subset \{1 \dots n\}} \rho_{(HK)} (D^1 \cup D^2)_K \otimes (D^1 \cup D^2)_H = \Delta^0(D^1 \cup D^2)
\end{aligned}$$

as $H = (H_1, H_2), K = (K_1, K_2)$ implies $\rho_{(H_1 K_1)} \rho_{(H_2 K_2)} (-1)^{|K_1||H_2|} = \rho_{(HK)}$. □

If $v \in \mathbb{V}_n$, $W_n(v)(D_{1\dots n}) = v(K_{1\dots n})$ is a weight system in $(\mathscr{A}_n^0)^*$. Since $W_n : \mathbb{V}_n \to (\mathscr{A}_n^0)^*$ is a graded map, $(W_p \otimes W_q)(v_1 \otimes v_2) = (-1)^{pq} W_p(v_1) \otimes W_q(v_2)$.

Theorem 25. *If* $v_1 \in \mathbb{V}_p$ *and* $v_2 \in \mathbb{V}_q$ *then*

$$W_{p+q}(v_1 \cup v_2) = (-1)^{pq} (W_p(v_1) \otimes W_q(v_2)) \circ \Delta^0.$$

Proof. Let $n = p + q$. For $D_{1\dots n} \in \mathfrak{D}_n^0$, let $K_{1\dots n}$ be any representative knot

$$\begin{aligned}
W_n(v_1 \cup v_2)(D_{1\dots n}) &= v_1 \cup v_2(K_{1\dots n}) \\
&= \sum_{|H|=p} \rho_{(HK)} v_1(\mathrm{d}_K^- K_{1\dots n}) \cdot v_2(\mathrm{d}_H^+ K_{1\dots n})
\end{aligned}$$

$$= \sum_{|H|=p} \rho_{(HK)} W_p(v_1)(D_K) \cdot W_q(v_2)(D_H)$$

$$= (-1)^{pq}(W_p(v_1) \otimes W_q(v_2)) \circ \Delta^0(D_{1\ldots n}). \qquad \square$$

We now define the differential on $\mathscr{A}^0$. For $D_{1\ldots n} \in \mathfrak{D}^0$, let $\mathrm{d}_i D_{1\ldots n} = D_{1\ldots n} \backslash \{i\text{th chord}\}$ with the induced ordering. Define $\partial D_{1\ldots n} = \sum_{i=1}^{n} (-1)^{i+1} \mathrm{d}_i D_{1\ldots n}$.

Proposition 26. ∂ (ordered 4T)=0.

Proof. S, E, W, N in Fig. 10 are the same except for the chords shown. We claim $\partial(S-E+W-N)=0$. For $k \neq i, j$, $\mathrm{d}_k(S-E+W-N) = 0$ by the ordered 4T relation. Therefore,

$$\partial S = (-1)^i \mathrm{d}_i S + (-1)^j \mathrm{d}_j S = (-1)^i \mathrm{d}_i N + (-1)^j \mathrm{d}_j N = \partial N,$$

$$\partial E = (-1)^i \mathrm{d}_i E + (-1)^j \mathrm{d}_j E = (-1)^i \mathrm{d}_i W + (-1)^j \mathrm{d}_j W = \partial W. \qquad \square$$

If $1 \leqslant i < j \leqslant n, \mathrm{d}_i \mathrm{d}_j = \mathrm{d}_{j-1} \mathrm{d}_i$, so $\partial^2 = 0$. Also, ∂ is a derivation with respect to the cup product. By the proof of Theorem 20, given $D \in \mathfrak{D}_n$, if D' is any perturbation of D, then some $K_{D'} \in X_{2n}^0$ represents D', with $\partial K_{D'} = 0$. Now, with the same ordering, $\partial D' = 0$.

Acknowledgements

This work includes part of the first author's Ph.D. dissertation at the University of Maryland, College Park, under the direction of William Goldman and Yongwu Rong (GWU). The first author also thanks Dror Bar-Natan, Józef Przytycki, James Schafer, Adam Sikora, and Akira Yasuhara for helpful discussions, and acknowledges support by the NSF and the UMCP Mathematics Dissertation Fellowship. The second author acknowledges support by the NSF and by the Overseas Youth Cooperation Research Fund of NSFC. This project first took shape in 1998, when the second author gave a course, "Topology of Manifolds," in the IMA graduate summer school. We thank the University of Iowa mathematics department for providing a stimulating environment.

References

[1] D. Bar-Natan, On the Vassiliev knot invariants, Topology 34 (1995) 423–472.
[2] D. Bar-Natan, S. Garoufalidis, On the Melvin–Morton–Rozansky conjecture, Invent. Math. 125 (1) (1996) 103–133.
[3] D. Bullock, C. Frohman, J. Kania-Bartoszynska, Skein homology, Canad. Math. Bull. 41 (1998) 140–144.
[4] R. Fenn, C. Rourke, B. Sanderson, James bundles and applications. http://www.maths.warwick.ac.uk/~cpr, 1996, Warwick preprint.
[5] R.M. Gillette, J.M. Van Buskirk, Knots invariant under switching an essential crossing, Amer. Math. Soc. Abstracts 720-55-10 22 (1) (1975) A226.
[6] P. Hilton, S. Wylie, Homology Theory, Cambridge University Press, Cambridge, 1960.
[7] N.V. Ivanov, Automorphism of complexes of curves and of Teichmüller spaces, Internat. Math. Res. Notices 14 (1997) 651–666.
[8] E. Kalfagianni, Finite type invariants for knots in 3-manifolds, Topology 37 (1997) 673–707.

[9] X.-S. Lin, Finite type link invariants of 3-manifolds, Topology 33 (1994) 45–71.
[10] X.-S. Lin, Finite type invariants of integral homology 3-spheres: a survey, in: V.F.R. Jones, et al. (Eds.), Knot Theory, Institute of Mathematics of the Polish Academy of Sciences, Vol. 42, Banach Center Publications, Warsaw, 1998, pp. 205–220.
[11] F. Luo, Automorphisms of the complex of curves, Topology 39 (2) (2000) 283–298.
[12] H. Murakami, Some metrics on classical knots, Math. Ann. 270 (1985) 35–45.
[13] J.H. Przytycki, Vassiliev-Gusarov skein modules of 3-manifolds and criteria for periodicity of knots, in: K. Johannson (Ed.), Low-Dimensional Topology, International Press, Cambridge, MA, 1994, pp. 143–162.
[14] D. Quillen, Rational homotopy theory, Ann. Math. 90 (1969) 205–295.
[15] V.A. Vassiliev, Complements of discriminants of smooth maps: topology and applications, Transactions of the Mathematics Monographs, Vol. 98, American Mathematical Society, Providence, RI, 1992.
[16] V.A. Vassiliev, On invariants and homology of spaces of knots in arbitrary manifolds, in: B. Feigin and V. Vassiliev (Eds.), Topics in Quantum Groups and Finite-type Invariants, American Mathematical Society Translational Series 2, Vol. 185, American Mathematical Society, Providence, RI, 1998, pp. 155–182.
[17] S. Willerton, Vassiliev knot invariants and the Hopf algebra of chord diagrams, Math. Proc. Cambridge Philos. Soc. 119 (1996) 55–65.