

## THE WEIL-PETERSSON GEOMETRY ON THE THICK PART OF THE MODULI SPACE OF RIEMANN SURFACES

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ABSTRACT. In the thick part of the moduli space of Riemann surface, we show the sectional curvature of the Weil-Petersson metric is bounded independent of the genus of the surface.

### 1. INTRODUCTION

Results on asymptotic geometry of the Weil-Petersson metric on Teichmüller space are presented in this paper. However, there is *no curve pinching* in the setting of this paper. We restrict ourselves on the thick part of the moduli space, where there is a positive lower bound on the least hyperbolic length of any short closed curves on the surface. We obtain boundedness of the Riemannian curvatures on the thick part of the moduli space with respect to the genus  $g$ , by considering **surfaces with large genera**.

The motivation of this paper is the recent consideration of articles [12] and [13] of Maryam Mirzakhani, where she studied the hyperbolic lengths of simple closed geodesics on a Riemann surface and calculated Weil-Petersson volume of the moduli space of Riemann surfaces. The Weil-Petersson volume grows exponentially in  $g$  (see also [23]), in particular, a rational multiple of  $\pi^{6g-6+2n}$ , where  $n$  is the number of punctures on the surface. Thus it is natural to ask how the curvatures vary in terms of  $g$ . We find, in theorem 1.2, that the absolute value of the Weil-Petersson sectional curvature is bounded with respect to  $g$ , in the thick part of the moduli space.

Let  $\Sigma$  be a smooth, oriented, closed Riemann surface of genus  $g$ , and we always assume  $g > 1$ . The study of conformal structures, or equivalently hyperbolic metrics, on a compact Riemann surface naturally leads to the classical Teichmüller theory.

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Teichmüller space  $\mathcal{T}_g$  is the space of conformal structures on  $\Sigma$ , modulo an equivalent relationship, where two conformal structures  $\sigma$  and  $\rho$  are equivalent if there is a biholomorphic map between  $(\Sigma, \sigma)$  and  $(\Sigma, \rho)$ , in the homotopy class of the identity. Riemann's moduli space  $\mathcal{M}_g$  of Riemann surfaces is obtained as the quotient of Teichmüller space by the mapping class group.

Teichmüller space  $\mathcal{T}_g$  is known ([1]) to be a complex manifold of complex dimension  $3g-3$ , and the cotangent space at  $\Sigma$  is identified with  $QD(\Sigma)$ , the space of holomorphic quadratic differentials. One also identifies the tangent space with  $HB(\Sigma)$ , the space of harmonic Beltrami differentials.

We recall that the Weil-Petersson cometric on Teichmüller space is defined on  $QD(\Sigma)$  by the  $L^2$ -norm:

$$(1) \quad \|\phi\|_{WP}^2 = \int_{\Sigma} \frac{|\phi|^2}{\sigma} dzd\bar{z},$$

where  $\sigma|dz|^2$  is the hyperbolic metric on  $\Sigma$ . By duality, we obtain a Riemannian metric on the tangent space of  $\mathcal{T}_g$ : for  $\mu \in HB(\Sigma)$ , we have

$$(2) \quad \|\mu\|_{WP}^2 = \int_{\Sigma} |\mu|^2 \sigma dzd\bar{z} = \int_{\Sigma} |\mu|^2 dA.$$

The differential geometry of this metric on Teichmüller space has been extensively studied. Ahlfors showed that the Ricci curvature, the holomorphic sectional curvature and scalar curvatures are negative ([1], [2]). Royden showed the holomorphic sectional curvature is bounded away from zero, and conjectured an explicit bound  $\frac{-1}{2\pi(g-1)}$  ([15]), later proved by Wolpert ([24]).

Throughout this paper, we denote  $l_0(\sigma)$  as the **systole**, the length of the shortest closed geodesic, of the surface  $\Sigma$ . As an important invariant of hyperbolic metrics, the systole is related to several important aspects of the Weil-Petersson geometry, such as the injectivity radius and curvatures near the compactification divisor, as seen in [25], [26], [7], [8].

In this paper, we assume there is a positive lower bound  $r_0$  on the injectivity radius, denoted by  $inj_{\sigma}(\Sigma)$ , of  $\Sigma$ , therefore the systole  $l_0(\sigma)$  also has a positive lower bound. We call **the thick part** of the moduli space as the compact subset of the moduli space where  $inj_{\sigma}(\Sigma) > r_0 > 0$ , i.e., *the thick part of moduli space consists of fat surfaces*.

Our first result is concerning the Weil-Petersson holomorphic sectional curvature on a large part of the moduli space:

**Theorem 1.1.** *There exists a positive constant  $C_1$ , independent of  $g$ , such that the Weil-Petersson holomorphic sectional curvature  $K_h$  satisfies that*

$$-C_1 < K_h < \frac{-1}{2\pi(g-1)},$$

*on the thick part of the moduli space.*

Tromba and Wolpert proved that the Weil-Petersson sectional curvatures are negative ([19], [24]). For the asymptotics of the sectional curvatures, much of the consideration has been on the Weil-Petersson geometry near the compactification divisor. It is shown that, there are directions (near the compactification divisor) in which the sectional curvatures have no negative upper bound ([7]), nor lower bound ([8], [18]), for fixed  $g$ .

When the surface is away from the compactification divisor, from a compactness argument, all curvatures are bounded and the bounds are in terms of  $g$  and  $l_0(\sigma)$ . In this paper, we find the curvatures are bounded, independent of the genus  $g$ , i.e.,

**Theorem 1.2.** *On the thick part of the moduli space, there is a positive constant  $C_2$ , independent of the genus  $g$ , such that the sectional curvature  $K$  of the Weil-Petersson metric satisfies that*

$$-C_2 < K < 0.$$

As an easy application, we also find that the following estimates of the Ricci curvature and the scalar curvature:

**Theorem 1.3.** *On the thick part of the moduli space, there are positive constants  $C_3, C_4$ , independent of the genus  $g$ , such that the Ricci curvature of the Weil-Petersson metric satisfies that*

$$-C_3g < Ric(d_{WP}) < \frac{-1}{2\pi(g-1)}.$$

and the scalar curvature  $s$  satisfies that

$$-C_4g^2 < s < -\frac{9g-6}{4\pi}.$$

The main technique involved is the analysis of harmonic maps between hyperbolic surfaces. The influence of Eells-Sampson's pioneer work ([5]) is reflected in many perspectives of Teichmüller theory, for example, see [9], [4], [20], [21], [11], and many others. One often finds that many functions associated to harmonic maps satisfy some geometric differential equations, where one can apply techniques in geometric analysis.

Here is the outline of the paper. In §2, we give a quick exposition of the background and introduce our notions. We prove our main theorems in §3.

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## 2. BACKGROUND

In this paper, let  $\Sigma$  be a fixed, compact, smooth, oriented surface of genus  $g > 1$ . We also denote  $\sigma|dz|^2$  as a hyperbolic metric on  $\Sigma$ , for conformal coordinates  $z$ . On  $(\Sigma, \sigma|dz|^2)$ , we denote the Laplacian as

$$\Delta = \frac{4}{\sigma} \frac{\partial^2}{\partial z \partial \bar{z}},$$

with nonpositive eigenvalues.

Teichmüller space is a complex manifold with the mapping class group acting by biholomorphisms, of complex dimension  $3g - 3$ , the number of independent closed curves in any pair-of-pants decomposition of the surface.

The Weil-Petersson metric is invariant under the action of the mapping class group, hence it descends to a metric on the moduli space. It is shown that the Weil-Petersson metric is Kählerian ([1]), with negative sectional curvature ([19], [24]). The Weil-Petersson Riemannian curvature tensor is given as the Tromba-Wolpert formula ([19], [24]):

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \int_{\Sigma} D(\mu_{\alpha}\bar{\mu}_{\beta})\mu_{\gamma}\bar{\mu}_{\delta}dA + \int_{\Sigma} D(\mu_{\alpha}\bar{\mu}_{\delta})\mu_{\gamma}\bar{\mu}_{\beta}dA,$$

where  $\Delta$  is the Laplacian for the hyperbolic metric  $\sigma$  on the surface, and  $D = -2(\Delta - 2)^{-1}$  is a self-adjoint, compact operator. Those  $\mu$ 's in the formula are tangent vectors, i.e., harmonic Beltrami differentials. There is a natural pairing between  $QD(\Sigma) = \{\phi(z)dz^2\}$  and  $HB(\Sigma) = \{\mu(z)\frac{d\bar{z}}{dz}\}$ :

$$\langle \phi dz^2, \mu \frac{d\bar{z}}{dz} \rangle = \operatorname{Re} \int_{\Sigma} \phi \mu dz d\bar{z}.$$

A remarkable property of the Weil-Petersson metric is that it is incomplete ([10], [22]). This is caused by pinching off at least one short closed geodesic on the surface. The Weil-Petersson completion modulo the mapping class group is topologically the Deligne-Mumford compactification of the moduli space. The compactification divisor, thus consists of a union of lower dimensional Teichmüller spaces, each such space consists of noded Riemann surfaces, obtained by pinching nontrivial short closed geodesics on the surface ([3], [10]). Therefore the compactification divisor can be described via the systole  $l_0(\sigma)$  as the set  $\{l_0(\sigma) = 0\}$ .

Since we will analyze harmonic maps between compact hyperbolic surfaces, we recall some fundamental facts here. For a Lipschitz map  $w : (\Sigma, \sigma|dz|^2) \rightarrow (\Sigma, \rho|dw|^2)$ , where  $\sigma|dz|^2$  and  $\rho|dw|^2$  are hyperbolic metrics on  $\Sigma$ , and  $z$  and  $w$  are conformal coordinates on  $\Sigma$ , one follows Sampson ([16]) to define

$$\mathcal{H}(z) = \frac{\rho(w(z))}{\sigma(z)} |w_z|^2, \mathcal{L}(z) = \frac{\rho(w(z))}{\sigma(z)} |w_{\bar{z}}|^2.$$

We call  $\mathcal{H}(z)$  the *holomorphic energy density*, and  $\mathcal{L}(z)$  the *anti-holomorphic energy density*. Then the energy density function of  $w$  is simply  $e(w) = \mathcal{H} + \mathcal{L}$ , and the total energy is then given by

$$E(w, \sigma, \rho) = \int_{\Sigma} e \sigma |dz|^2.$$

We also note that the Jacobian determinant relative to the  $\sigma$  metric is therefore given by  $J(z) = \mathcal{H}(z) - \mathcal{L}(z)$ .

The map  $w$  is called *harmonic* if it is a critical point of this energy functional, i.e., it satisfies Euler-Lagrange equation:

$$w_{z\bar{z}} + \frac{\rho_w}{\rho} w_z w_{z\bar{z}} = 0.$$

The  $(2, 0)$  part of the pullback  $w^*\rho$  is the so-called *Hopf differential*:

$$\phi(z)dz^2 = (w^*\rho)^{(2,0)} = \rho(w(z))w_z\bar{w}_z dz^2.$$

It is routine to check that  $w$  is harmonic if and only if  $\phi dz^2 \in QD(\Sigma)$ , and  $w$  is conformal if and only if  $\phi = 0$ .

In our situation, there is a unique harmonic map  $w : (\Sigma, \sigma) \rightarrow (\Sigma, \rho)$  in the homotopy class of the identity, moreover, this map  $w$  is a diffeomorphism with positive Jacobian  $J$ , and  $\mathcal{H} > 0$  ([5], [6], [16], [17]).

A key observation to link the harmonic maps to Teichmüller theory is that one obtains a map from Teichmüller space to  $QD(\Sigma)$ , for some fixed hyperbolic metric  $\sigma$ . More specifically, this map sends any hyperbolic metric on  $\Sigma$  to a holomorphic quadratic differential associated to the unique harmonic map in the homotopy class of the identity. This map is a diffeomorphism ([16], [20]).

### 3. PROOF OF MAIN THEOREMS

Let  $\mu = \mu(z)\frac{d\bar{z}}{dz}$  be a unit Weil-Petersson normed harmonic Beltrami differential, thus  $\int_{\Sigma} |\mu|^2 dA = 1$ . The Weil-Petersson holomorphic sectional curvature in the direction of  $\mu$  is given by

$$K_h = -2 \int_{\Sigma} D(|\mu|^2) |\mu|^2 dA.$$

Its upper bound, in terms of the genus, is known, proved by Wolpert:

**Lemma 3.1.** ([25])  $K_h < -\frac{1}{2\pi(g-1)}$ .

**Remark 3.2.** *This upper bound holds without restriction on the systole. Since all sectional curvatures are negative, so the Ricci curvature and scalar curvature are bounded from above by  $-\frac{1}{2\pi(g-1)}$ , and  $-\frac{3(3g-2)}{4\pi}$ , respectively ([25]).*

We now show the following pointwise estimate on  $|\mu(z)|$ , where the tangent vector  $\mu(z)\frac{d\bar{z}}{dz}$  is normalized to have unit Weil-Petersson norm, and we shall apply this estimate to prove our main theorems.

**Theorem 3.3.** *For  $\mu(z)\frac{d\bar{z}}{dz} \in HB(\Sigma)$  with  $\|\mu\|_{WP} = 1$ , there exists a positive constant  $h_0$ , independent of  $g$ , such that  $|\mu(z)| \leq h_0$ , for all  $z \in \Sigma$ , where the surface  $\Sigma$  is in the thick part of the moduli space.*

*Proof.* Recall that  $\mu(z)\frac{d\bar{z}}{dz}$  is a harmonic Beltrami differential, hence is a symmetric tensor given as  $\bar{\phi}(ds^2)^{-1}$  for  $\phi$  a holomorphic quadratic differential with at most simple poles at the cusps and  $ds^2$  the hyperbolic metric tensor ([23]). Since the surface  $\Sigma$  lies in the thick part of moduli space, hence no cusps and this holomorphic quadratic differential  $\phi = \phi(z)dz^2$  has no poles.

Note that  $\phi \in QD(\Sigma)$ , as stated in the previous section, by a theorem of Wolf ([20]), there exists a hyperbolic metric  $\rho$  on surface  $\Sigma$ , and a unique harmonic map  $w : (\Sigma, \sigma) \rightarrow (\Sigma, \rho)$ , such that  $\phi$  is the Hopf differential associated to this harmonic map  $w$ , i.e.,  $\phi(z)dz^2 = \rho(w(z))w_z\bar{w}_zdz^2$ .

Much of our study will be analyzing this harmonic map  $w$ . Note that even though  $inj_\sigma(\Sigma) > r_0 > 0$ , the metric  $\rho$  might not lie in the same thick part of the moduli space. We recall that the holomorphic and anti-holomorphic energy density functions of  $w$  are defined as  $\mathcal{H}(z) = \frac{\rho(w(z))}{\sigma(z)}|w_z|^2$ , and  $\mathcal{L}(z) = \frac{\rho(w(z))}{\sigma(z)}|w_{\bar{z}}|^2$ , respectively. Since the map  $w$  is a diffeomorphism with positive Jacobian determinant, we have  $\mathcal{H}(z) > \mathcal{L}(z) \geq 0$ .

We also find that  $\mathcal{H}\mathcal{L} = \frac{|\phi|^2}{\sigma^2} = |\mu|^2$ , so the zeros of  $\mathcal{L}$  are the zeros of  $|\mu|$ , or equivalently, the zeros of  $\phi$ .

Let  $\nu$  be the Beltrami differential of the map  $w$ , defined by  $\nu = \frac{w_{\bar{z}}d\bar{z}}{w_zdz}$ . It measures the failure of  $w$  to be conformal, and since  $J > 0$ , we have  $|\nu| < 1$ .

One easily finds that  $|\nu|^2 = \frac{\mathcal{L}}{\mathcal{H}}$ , therefore,

$$|\mu| = \sqrt{\mathcal{H}\mathcal{L}} = \mathcal{H}|\nu| < \mathcal{H}.$$

Thus it suffices to estimate  $\mathcal{H}(z)$  to bound  $|\mu|$  pointwisely.

Let  $z_0 \in \Sigma$  such that  $\mathcal{H}(z_0) = \max_{z \in \Sigma} \mathcal{H}(z)$ . We follow a calculation of Schoen-Yau to define a local one-form  $\theta = \sqrt{\sigma(z)}dz$ , and find ([17]):

$$|w_\theta|^2 = \frac{\rho}{\sigma}|w_z|^2 = \mathcal{H}(z),$$

and

$$(3) \quad \Delta|w_\theta|^2 = 4|w_{\theta\theta}|^2 + 2J|w_\theta|^2 - 2|w_\theta|^2.$$

We rewrite this as

$$(4) \quad \Delta\mathcal{H} = 4|w_{\theta\theta}|^2 + 2J\mathcal{H} - 2\mathcal{H} > -2\mathcal{H}.$$

Therefore  $\mathcal{H}$  is a subsolution to an elliptic equation  $(\Delta + 2)f = 0$ .

Recalling that  $inj_\sigma(\Sigma) > r_0 > 0$ , we embed a hyperbolic ball  $B_{z_0}(\frac{r_0}{2})$  into  $\Sigma$ , centered at  $z_0$  with radius  $\frac{r_0}{2}$ . Morrey's theorem ([14], theorem 5.3.1) on subsolutions of elliptic differential equations guarantees that there is a constant  $C(r_0)$ , such that,

$$\mathcal{H}(z_0) = \sup_{B_{z_0}(\frac{r_0}{4})} \mathcal{H}(z) \leq C(r_0) \int_{B_{z_0}(\frac{r_0}{2})} \mathcal{H}(z) \sigma dz d\bar{z}.$$

Another consequence of formula (3) is the Bochner identity (see [17]), as now  $\log\mathcal{H}$  is well defined:

$$(5) \quad \Delta \log\mathcal{H} = 2\mathcal{H} - 2\mathcal{L} - 2.$$

The minimal principle implies  $\mathcal{H}(z) \geq 1$  for all  $z \in \Sigma$ , and we find

$$\int_\Sigma \mathcal{L}dA \leq \int_\Sigma \mathcal{H}\mathcal{L}dA = \int_\Sigma |\mu|^2dA = \|\mu\|_{WP} = 1.$$

It is not hard to see that we can actually bound the total energy of this harmonic map  $w$  from above, in terms of the genus. More precisely, we recall that  $w$  is a diffeomorphism, so

$$\int_{\Sigma} (\mathcal{H} - \mathcal{L}) dA = \int_{\Sigma} J dA = \text{Area}(w(\Sigma)) = 4\pi(g - 1),$$

therefore the total energy satisfies

$$\begin{aligned} E(w) &= \int_{\Sigma} e(w) dA = \int_{\Sigma} (\mathcal{H} + \mathcal{L}) dA \\ &= \int_{\Sigma} (\mathcal{H} - \mathcal{L}) dA + 2 \int_{\Sigma} \mathcal{L} dA \\ &\leq \text{Area}(\Sigma, \sigma) + 2 \\ &= 4\pi(g - 1) + 2. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{B_{z_0}(\frac{r_0}{2})} \mathcal{H}(z) \sigma dz d\bar{z} &< \int_{B_{z_0}(\frac{r_0}{2})} (\mathcal{H}(z) + \mathcal{L}(z)) \sigma dz d\bar{z} \\ &= E(w) - \int_{\Sigma \setminus B_{z_0}(\frac{r_0}{2})} e(z) \sigma dz d\bar{z} \\ &\leq 4\pi(g - 1) + 2 - (4\pi(g - 1) - A_1(B_{z_0}(\frac{r_0}{2}))) \\ &= A_1(B_{z_0}(\frac{r_0}{2})) + 2. \end{aligned}$$

where  $A_1(B_{z_0}(\frac{r_0}{2}))$  is the hyperbolic area of the ball  $B_{z_0}(\frac{r_0}{2})$ .

We set  $h_0 = C(r_0)(A_1(B_{z_0}(\frac{r_0}{2})) + 2)$ , then  $h_0$  is independent of the genus  $g$ , as it is obtained from a local estimate in a geodesic ball. Therefore,

$$|\mu(z)| < \mathcal{H}(z) \leq \mathcal{H}(z_0) < h_0,$$

for all  $z \in \Sigma$ . □

**Remark 3.4.** *The assumption of the surface lying in the thick part of the moduli space is essential to this argument, since we used an estimate in an embedded geodesic ball.*

As an application of this estimate, we can prove theorem 1.1 easily.

*Proof.* (of theorem 1.1) Recall that the operator  $D = -2(\Delta - 2)^{-1}$  is self-adjoint, and  $\|\mu\|_{WP} = 1$ :

$$|K_h| = 2 \int_{\Sigma} D(|\mu|^2) |\mu|^2 dA < 2h_0^2 \int_{\Sigma} D(|\mu|^2) dA = 2h_0^2.$$

□

We now shift our attention to general Weil-Petersson sectional curvatures and to prove theorem 1.2.

*Proof.* (of theorem 1.2) We recall from the previous section the Riemannian curvature tensor of the Weil-Petersson metric is given by ([19], [24]):

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \int_{\Sigma} D(\mu_{\alpha}\bar{\mu}_{\beta})\mu_{\gamma}\bar{\mu}_{\delta}dA + \int_{\Sigma} D(\mu_{\alpha}\bar{\mu}_{\delta})\mu_{\gamma}\bar{\mu}_{\beta}dA,$$

where  $\mu$ 's in the formula are harmonic Beltrami differentials, and  $D$  again is the operator  $-2(\Delta - 2)^{-1}$ .

To calculate the sectional curvature, we choose two arbitrary orthonormal harmonic Beltrami differentials  $\mu_0$  and  $\mu_1$ . In other words, we have

$$\int_{\Sigma} |\mu_0|^2 dA = \int_{\Sigma} |\mu_1|^2 dA = 1 \text{ and } \int_{\Sigma} \mu_0 \bar{\mu}_1 dA = 0.$$

Then the Gaussian curvature of the plane spanned by  $\mu_0$  and  $\mu_1$  is ([24])

$$\begin{aligned} K(\mu_0, \mu_1) &= \frac{1}{4}(R_{0\bar{1}0\bar{1}} - R_{0\bar{1}1\bar{0}} - R_{1\bar{0}0\bar{1}} + R_{1\bar{0}1\bar{0}}) \\ &= \operatorname{Re}\left(\int_{\Sigma} D(\mu_0\bar{\mu}_1)\mu_0\bar{\mu}_1 dA\right) - \frac{1}{2}\operatorname{Re}\left(\int_{\Sigma} D(\mu_0\bar{\mu}_1)\mu_1\bar{\mu}_0 dA\right) \\ &\quad - \frac{1}{2}\int_{\Sigma} D(|\mu_1|^2)|\mu_0|^2 dA. \end{aligned}$$

From ([24], lemma 4.3) and Hölder inequality, we have

$$\begin{aligned} \left|\operatorname{Re}\left(\int_{\Sigma} D(\mu_0\bar{\mu}_1)\mu_0\bar{\mu}_1 dA\right)\right| &\leq \int_{\Sigma} |D(\mu_0\bar{\mu}_1)||\mu_0\bar{\mu}_1| dA \\ &\leq \int_{\Sigma} \sqrt{D(|\mu_0|^2)}\sqrt{D(|\mu_0|^2)}|\mu_0\bar{\mu}_1| dA \\ &\leq \int_{\Sigma} D(|\mu_0|^2)|\mu_1|^2 dA = \int_{\Sigma} D(|\mu_1|^2)|\mu_0|^2 dA, \end{aligned}$$

and similarly

$$\left|\int_{\Sigma} D(\mu_0\bar{\mu}_1)\mu_1\bar{\mu}_0 dA\right| \leq \int_{\Sigma} D(|\mu_1|^2)|\mu_0|^2 dA.$$

Therefore

$$|K(\mu_0, \mu_1)| \leq 2 \int_{\Sigma} D(|\mu_1|^2)|\mu_0|^2 dA.$$

We apply theorem 3.3 to find that there is a  $h_0 > 0$ , independent of  $g$ , such that  $|\mu_0| < h_0$ . So  $|K| < 2h_0^2 \int_{\Sigma} D(|\mu_1|^2)dA = 2h_0^2$ .  $\square$

It is now straightforward to see that theorem 1.3 holds since all sectional curvatures are negative.

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