COMPLEX LENGTH OF SHORT CURVES AND MINIMAL FIBRATIONS OF HYPERBOLIC 3-MANIFOLDS FIBERING OVER THE CIRCLE

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Abstract. We investigate the maximal solid tubes around short simple closed geodesics in hyperbolic three-manifolds and how the complex length of curves relates to closed least area incompressible minimal surfaces. As applications, we prove the existence of closed hyperbolic three-manifolds fibering over the circle which are not foliated by closed imcompressible minimal surfaces isotopic to the fiber. We also show the existence of quasi-Fuchsian manifolds containing arbitrarily many embedded closed incompressible minimal surfaces. Our strategy is to prove main theorems under natural geometric conditions on the complex length of closed curves on a fibered hyperbolic three-manifold, then by computer programs, we find explicit examples where these conditions are satisfied.

1. Introduction

1.1. Motivating Questions. As fundamental objects in differential geometry, minimal hypersurfaces in Euclidean space and other Riemannian manifolds have been extensively investigated ever since the "Plateau Problem" in the 1930s. We are particularly interested in the 3-dimensional case and this paper is part of a larger goal to understand closed incompressible minimal surfaces in several different classes of hyperbolic three-manifolds, their connections to Teichmüller theory, and the "moduli spaces" of these minimal surfaces (see [**GHW10, HL12, HW17**]).

Throughout the paper, we denote by S an oriented closed surface of genus $g \geq 2$, and we denote $\mathcal{M}_\psi$ or $\mathcal{M}$ a *mapping torus or fibered hyperbolic three-manifold* with monodromy ψ, which is an oriented closed hyperbolic three-manifold that fibers over the circle *with fiber* S if ψ is pseudo-Anosov. We set up the following additional notations for the paper:

(i) $\mathscr{M}$: a *quasi-Fuchsian* manifold which is diffeomorphic to $S \times \mathbb{R}$;
(ii) $\mathcal{T}(S)$: Teichmüller space of the surface S;
(iii) $\mathcal{QF}(S)$: the quasi-Fuchsian space of S;

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(iv) $\mathrm{AH}(S)$: the algebraic deformation space of Kleinian surface groups associated to the surface S;
(v) $\mathcal{L} = \ell + \sqrt{-1}\theta$: the complex length of a simple closed geodesic γ in the hyperbolic three-manifold, where ℓ is the real length, and θ is the twisting angle. We always assume that $\ell > 0$ and $\theta \in [-\pi, \pi)$.
(vi) $\mathbb{T}(\gamma)$ is the maximal solid tube around a simple closed geodesic γ in a hyperbolic three-manifold, whose radius is denoted by r_0 (see Definition 3.1).

We will study maximal solid tubes in metrically complete hyperbolic three-manifolds (without parabolics). These tubes play fundamental roles in the quest of determining complete (or closed) hyperbolic three-manifolds of small volume (see for instance [**Mey87, Ago02, ACS06**]). Understanding how closed incompressible least area minimal surfaces interact with deep tubes enables us to pursue some natural questions in hyperbolic geometry. Our work is motivated by some beautiful conjectures/open problems in the field. It is well-known that any quasi-Fuchsian manifold admits at least one closed, embedded, and incompressible minimal surface. The following question, probably due to Hass-Thurston (see [**GW07**]) and Uhlenbeck [**Uhl83**]), addresses the multiplicity question:

Question 1.1. *For any integer $N > 0$, and any closed surface S of genus $g \geq 2$, does there exist a quasi-Fuchsian group $G \cong \pi_1(S)$ such that the resulting quasi-Fuchsian manifold $\mathscr{M} = \mathbb{H}^3/G$ contains at least N distinct, immersed, closed, incompressible minimal surfaces, all diffeomorphic to S?*

Note that Anderson ([**And83**]) constructed a quasi-Fuchsian manifold containing at least two incompressible minimal surfaces, and we ([**HW15**]) have constructed, given any prescribed positive integer N, a quasi-Fuchsian manifold (*whose genus depends on N*) containing at least N distinct, embedded, closed, incompressible, (locally least area) minimal surfaces.

Ever since Thurston's geometrization theorem for fibered three-manifolds (see for instance [**Thu82**]), their geometry is extremely important in hyperbolic three-manifold theory. We will also investigate closed minimal surfaces in closed hyperbolic three-manifolds that fiber over the circle.

Definition 1.2. *We call a C^2-fibration* ***minimal*** *or* ***geometrically taut*** *on an oriented closed hyperbolic three-manifold $\mathcal{M}$ that fibers over the circle with fiber S if each leaf is a closed incompressible minimal surface, which is homeomorphic to the fiber S.*

By a celebrated theorem of Sullivan ([**Sul79**]), any closed Riemannian manifold with *taut foliation* (a codimensional one C^2-foliation such that there is a closed loop transversal to each leaf) admits a minimal foliation with respect to some Riemannian metric. The existence of a minimal fibration structure has tremendous applications in Riemannian geometry. A famous question in this direction is the following (see for instance [**Has05, Rub07**]):

Question 1.3. *Does there exist a fibered hyperbolic three-manifold which admits a minimal foliation?*

These questions are intricately related, see for instance [**And83**], where Anderson further conjectured that any closed hyperbolic three-manifold does not admit a local parameter family of closed minimal surfaces, in particular, does not admit a foliation of closed minimal surfaces. These questions have had profound impact in the theory of hyperbolic three-manifolds, as well as many other fields. In this work, we address problems related to these questions.

1.2. **Main results.** In this paper, we analyze the relationship between the complex length of simple closed geodesics in a metrically complete hyperbolic three-manifold (essentially just inside solid tubes) and closed least area minimal surfaces in such hyperbolic three-manifolds. In one dimension lower, when a simple closed geodesic γ is short enough, any closed geodesic disjoint from γ can not go too deep inside the collar neighborhood of γ. Intuitively our argument is similar in spirit, but we need to involve the complex length (real length and twist angle) to prevent a closed incompressible least area minimal surface going too deep into a maximal solid tube. As a consequence of this relationship, we prove statements regarding the multiplicity of closed incompressible minimal surfaces in quasi-Fuchsian manifolds, and the (non)existence of minimal fibrations on certain oriented closed hyperbolic three-manifolds that fiber the circle.

Before we state our main results, we define some constants that will appear in the main statements which play an essential role in our argument. These constants are unified through the following function:

Definition 1.4. *We define the function* $\mathbb{W}(x) : [1,\infty) \to (0,1)$ *as follows:*

$$\mathbb{W}(x) = \frac{\sqrt{3}}{4\pi}\left[\cosh^{-1}\left(\frac{1}{1+\sqrt{1+(8x^2-8x+1)^2}}+1\right)\right]^2 . \tag{1.1}$$

It is elementary to verify that $\mathbb{W}(x)$ is a decreasing function of $x \in [1,\infty)$, and $\lim_{x\to\infty} \mathbb{W}(x) = 0$. The maximum value is $\mathbb{W}(1) \approx 0.107071$, a fundamental constant in hyperbolic three-manifold theory: Meyerhoff's constant.

Now we define the following "Otal's constant", depending only on the genus $g \geq 2$ of S:

$$\varepsilon_{\text{Otal}}(g) = \mathbb{W}(g) = \frac{\sqrt{3}}{4\pi}\left[\cosh^{-1}\left(\frac{1}{1+\sqrt{1+(8g^2-8g+1)^2}}+1\right)\right]^2 . \tag{1.2}$$

Otal ([**Ota95, Ota03**]) showed that when a curve (i.e., simple closed geodesic) is sufficiently short, it is unknotted in a natural sense, and we always have $0 < \varepsilon_{\text{Otal}}(g) \le \varepsilon_{\text{Otal}}(2) = \mathbb{W}(2) \approx 0.01515$. We prove the following theorem on the multiplicity of closed minimal surfaces in quasi-Fuchsian manifolds:

Theorem 1.5. *If an oriented closed hyperbolic three-manifold $\mathcal{M}$ that fibers over the circle with fiber S contains a simple closed geodesic whose complex length $\mathcal{L} = \ell + \sqrt{-1}\,\theta$ satisfies:*

(i) $\ell < \varepsilon_{\text{Otal}}(g)$;
(ii)

$$\frac{|\theta|}{\sqrt{\ell}} > \sqrt[4]{3\pi^2} \approx 2.33268 , \tag{1.3}$$

then for any positive integer N, there exists a quasi-Fuchsian manifold $\mathscr{M} \cong S \times \mathbb{R}$ which contains at least N embedded closed incompressible least area minimal surface.

The techniques developed in [**HW15**] do not extend to the case of arbitrary genus. Theorem 1.5 states that for *ANY* genus $g \ge 2$, assuming the above two conditions on the complex length of some short curve on an oriented closed hyperbolic three-manifold $\mathcal{M}$ that fibers over the circle with fiber S, then one can find a quasi-Fuchsian manifold $\mathscr{M}$ which contains arbitrarily many embedded closed incompressible minimal surfaces. For different integer N's, the quasi-Fuchsian manifolds obtained from this scheme are possibly different. This result is also an improvement from [**Wan12**]. Via computer programs, in Appendix §5.2, explicit examples of fibered hyperbolic three-manifolds which satisfy the conditions in Theorem 1.5 are produced. Higher genera cases follow from standard finite coverings of lower genera examples. Therefore, we obtain an affirmative answer to Question 1.1.

It is well-known ([**Thu80**] or [**Thu98**, Corollary 4.3]) that $\lim_{\ell\to 0}\theta = 0$ (we provide a proof in the Appendix of this paper), but their quantitative nature for short curves is notoriously difficult to control. Minsky ([**Min99**, Lemma 6.4]) obtained a uniform upper bound for any simple closed geodesic in a Kleinian surface group with complex length $\mathcal{L} = \ell + \sqrt{-1}\,\theta$:

$$\frac{|\theta|}{\sqrt{\ell}} < \sqrt{\frac{2\pi}{C_1}} , \tag{1.4}$$

where C_1 is a positive constant depending only on g. See Corollary 5.2 in the appendix for an explicit bound.

Next we define a universal constant

$$\varepsilon_0 = \frac{\sqrt{3}}{4\pi}\left[\cosh^{-1}\left(\frac{1}{1+\sqrt{1+(7+4\sqrt{3})^2}}+1\right)\right]^2 \approx 0.01822\ . \tag{1.5}$$

Using above $\mathbb{W}$-function notation (1.1), one verifies that $\varepsilon_0 = \mathbb{W}\left(\frac{2+\sqrt{3}}{2}\right)$ and $\mathbb{W}(g) < \varepsilon_0$ for all $g \geq 2$.

In this paper, whenever we mention a foliation or fibration on a mapping torus, we always assume it is C^2, and each leaf is a closed surface diffeomorphic to the surface S which is used to defined the mapping torus, as we apply results of Sullivan ([**Sul79**]), Harvey-Lawson ([**HL82**]) and Hass ([**Has86**]) in an essential way for our next result. We prove the following result related to Question 1.3:

Theorem 1.6. *If an oriented closed hyperbolic three-manifold $\mathcal{M}$ that fibers over the circle with fiber S contains a simple closed geodesic whose complex length $\mathcal{L} = \ell + \sqrt{-1}\,\theta$ satisfies:*

(i) $\ell < \varepsilon_{\text{Otal}}(g)$;
(ii) $|\theta|/\sqrt{\ell} > \sqrt[4]{3\pi^2} \approx 2.33268$,

then $\mathcal{M}$ does not admit a minimal fibration.

Explicit examples of closed fibered hyperbolic three-manifolds fibering over the circle which satisfy our conditions in Theorem 1.6 are produced also in §5.2. As an immediate corollary, we have:

Corollary 1.7. *There exists some fibered hyperbolic three-manifolds which do not admit any minimal foliation (in the sense of Definition 1.2).*

Remark 1.8. *Recently Hass ([**Has15**]) also obtained results on the question 1.3 using arbitrarily short geodesics on fibered hyperbolic three-manifolds. We are thankful for the correspondence.*

1.3. **Comments on the techniques and constants.** Margulis tubes are fundamental tools in three-manifold theory, but it is usually very difficult to carry out explicit calculations using Margulis tubes of short curves in the study of hyperbolic three-manifolds. We work with *maximal solid tubes* (see [**Mey87**]) instead in this paper since we seek more computable conditions.

Otal's constant $\varepsilon_{\text{Otal}}(g) = \mathbb{W}(g)$ did not directly appear in his work [**Ota95**]. In order to show a sufficiently short geodesic γ is unknotted, he requires that the area of the meridian disk of the Margulis tube of γ is greater than $4\pi(g-1)$. In our argument, we replace the role of Margulis tube by the *maximal solid tube* of γ, and we require, if ℓ (the real length of γ) is less than this "Otal's constant", then the area of the meridian disk of the maximal solid tube of γ is greater than $4\pi(g-1)$ (See Proposition 3.3). The numerical number is calculated following this idea and using Meyerhoff's constant.

The other constant $\varepsilon_0 = \mathbb{W}\left(\frac{2+\sqrt{3}}{2}\right)$ is designed so that a least area minimal surface constructed similar to Calegari-Gabai ([**CG06**]) by the means of shrink-wrapping will be separated from the core curve in the maximal solid tube $\mathbb{T}(\gamma)$ whose complex length satisfies the conditions in Theorem 3.8 (see Lemma 3.10), a main ingredient in the proof of both main Theorems 1.5 and 1.6.

There are two other constants that will appear later. One is ε_1 in the statement of Theorem 3.5. Using our $\mathbb{W}$-function in (1.1), we note here $\varepsilon_1 = \mathbb{W}\left(\frac{3}{2}\right) \approx 0.03347$. The other is Meyerhoff's constant $\varepsilon_2 = \mathbb{W}(1)$ which appears in Theorem 3.2. This is to guarantee the existence of the maximal solid tubes around short curves.

In terms of our technical needs, we need $\ell < \varepsilon_2$ to define maximal solid tubes for short curves, and we need a stricter $\ell < \varepsilon_1$ for a technical reason in a key inequality (3.9) in Theorem 3.5. We need the above mentioned separation between a closed minimal surface and a short curve, established using an even stricter condition $\ell < \varepsilon_0$, to prove Theorem 3.8, and finally we require further $\ell < \varepsilon_{\text{Otal}}(g)$ in the proof of Theorems 1.5 and 1.6 to prevent curves from being knotted or linked. In short, we have the following ordered constants which control the real length of a short geodesic:

$$\varepsilon_{\text{Otal}}(g) \le \varepsilon_{\text{Otal}}(2) < \varepsilon_0 = \mathbb{W}\left(\frac{2+\sqrt{3}}{2}\right) < \varepsilon_1 = \mathbb{W}\left(\frac{3}{2}\right) < \varepsilon_2 = \mathbb{W}(1)\ . \tag{1.6}$$

1.4. Outline of the paper.

The organization of the paper is as follows: we summarize necessary background on Kleinian surface groups, mapping tori, minimal helicoids in $\mathbb{H}^3$ and the maximal solid tube around short curves in hyperbolic three-manifolds in §2. We develop our methods in §3, and use these techniques to prove our main theorems in §4. We include an appendix to include a proof of a Proposition by Thurston on complex length of short curves which provides an upper bound for the ratio $|\theta|/\sqrt{\ell}$. Also in the appendix, we use Twister ([**BHS14**]) and SnapPy ([**CDGW**]) programs to produce some explicit examples of fibered hyperbolic three-manifolds with our conditions satisfied. Taking finite covers of these lower genera examples one finds more examples for higher genera surfaces of which mapping tori are made.

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2. Tool Box

2.1. **Kleinian surface groups and hyperbolic mapping tori.** We will mostly work with the upper-half space model of hyperbolic three-space: $\mathbb{H}^3 = \{z + tj : z \in \mathbb{C},\, t > 0\}$, equipped with the standard hyperbolic metric: $ds^2 = \frac{|dz|^2+dt^2}{t^2}$. The orientation preserving isometry group of $\mathbb{H}^3$, denoted by $\mathrm{PSL}(2,\mathbb{C})$, is the set of Möbius transformations on $\mathbb{H}^3$, namely, for each element $\tau \in \mathrm{PSL}(2,\mathbb{C})$, we have

$$\tau(z) = \frac{az+b}{cz+d}\ , \quad \forall z \in \mathbb{C}\ ,$$

with $ad - bc = 1$. Its Poincaré extension is given by:

$$\tau(z+tj) = \frac{(az+b)(\overline{cz+d}) + a\bar{c}t^2 + tj}{|cz+d|^2 + |c|^2t^2}\ , \quad \forall (z,t) \in \mathbb{H}^3\ .$$

Suppose that S is an oriented closed surface of genus ≥ 2. Let $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ be a discrete and faithful representation, then the image $G = \rho(\pi_1(S))$, a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$, is called a **Kleinian surface group**. The quotient manifold $M_\rho = \mathbb{H}^3/\rho(\pi_1(S))$ is a complete hyperbolic three-manifold. By the work of Thurston and Bonahon ([**Bon86**]), we know that M_ρ is diffeomorphic to $S \times \mathbb{R}$.

Two Kleinian surface groups are **equivalent** if the corresponding representations are conjugate in $\mathrm{PSL}(2,\mathbb{C})$. The **algebraic deformation space** of S, denoted by $\mathrm{AH}(S)$, is the space of equivalence classes. A Kleinian surface group is called **quasi-Fuchsian** if its limit set is a topological circle. The resulting quotient of $\mathbb{H}^3$ by a quasi-Fuchsian group is called a **quasi-Fuchsian manifold**. We abuse our notation to denote both the space of quasi-Fuchsian manifolds and the space of quasi-Fuchsian groups by $\mathcal{QF}(S)$. This space plays a fundamental role in hyperbolic three-manifold theory.

Let $\{\rho_n : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})\}$ be a sequence of representations, then the sequence of Kleinian surface groups $\{G_n = \rho_n(\pi_1(S))\}$ **converges algebraically** if $\lim\limits_{n\to\infty} \rho_n(\gamma)$ exists as a Möbius transformation for all $\gamma \in \pi_1(S)$. Since the Kleinian surface group is finitely generated, the algebraic limit of Kleinian surface groups is also Kleinian (see [**JK82**]). Equipping the deformation space $\mathrm{AH}(S)$ with the topology of algebraic convergence, the space $\mathrm{AH}(S)$ is closed as a subspace of the space of equivalence classes of all homomorphisms into the isometry group of $\mathbb{H}^3$ (see [**Chu68, Wie77**] or [**Ota01**, Proposition 1.1.3]). One of the fundamental theorems in Kleinian surface group theory is that $\mathcal{QF}(S)$ is in fact the interior of $\mathrm{AH}(S)$ (see [**Mar74, Sul85, Min03**]). Moreover, if we denote $\overline{\mathcal{QF}(S)}$ the closure of $\mathcal{QF}(S)$ in $\mathrm{AH}(S)$ with respect to the algebraic topology, then we have (see [**BB04, Bro07**]):

$$\overline{\mathcal{QF}(S)} = \mathrm{AH}(S)\ . \tag{2.1}$$

A mapping torus with monodromy $\psi : S \to S$, denoted by $\mathcal{M}_\psi$, can be constructed by taking the quotient $S \times [0,1]/\sim$, where we identify $(x,0)$ and $(\psi(x),1)$. The automorphism ψ of S defines an element of the mapping class group Mod(S), it is **pseudo-Anosov** if no power of ψ preserves the isotopy class of any essential simple closed geodesic on S. Thurston's hyperbolization theorem (see [**Thu98, McM96, Ota96, Ota01**]) shows that the mapping torus $\mathcal{M}_\psi$ carries a hyperbolic structure if and only if ψ is pseudo-Anosov, in this case $\mathcal{M}_\psi$ or simply $\mathcal{M}$, is an oriented closed hyperbolic three-manifold that fibers over the circle with fiber S. Though the hyperbolic mapping tori and quasi-Fuchsian manifolds are very different geometrically, Thurston has shown that a certain covering of the hyperbolic mapping tori arises as the limit of quasi-Fuchsian manifolds: Let $\mathcal{M}_\infty$ be the infinite cyclic cover of $\mathcal{M}$ corresponding to the subgroup $\pi_1(S) \subset \pi_1(\mathcal{M})$, then $\mathcal{M}_\infty$ is a doubly degenerated hyperbolic three-manifold diffeomorphic to $S \times \mathbb{R}$ which arises as an algebraic limit of manifolds in $\mathcal{QF}(S)$, hence it lies on the boundary of quasi-Fuchsian space within $\mathrm{AH}(S)$.

2.2. **Family of Helicoids in hyperbolic three-space.** First let us describe a construction of a helicoid in $\mathbb{H}^3$, which will descend to a minimal annulus in a maximal solid tube in §3.2.

Definition 2.1. *The* ***helicoid*** $\mathscr{H}_a$ *in* $\mathbb{H}^3$*, the upper-half space model of the hyperbolic* 3*-space, is the surface parametrized by the* (u,v)*-plane as follows:*

$$\mathscr{H}_a = \left\{ z + tj \in \mathbb{H}^3 : z = e^{v+\sqrt{-1}\,av}\tanh(u)\ ,\ t = \frac{e^v}{\cosh(u)} \right\}\ , \tag{2.2}$$

where $-\infty < u, v < \infty$*. In this model, the axis of* $\mathscr{H}_a$ *is the* t*-axis.*

The first fundamental form can be written as

$$I = du^2 + (\cosh^2(u) + a^2\sinh^2(u))dv^2\ . \tag{2.3}$$

Mori proved $\mathscr{H}_a$ is indeed a minimal surface in $\mathbb{H}^3$ (see [**Mor82**]). We will soon choose $a = |\theta|/\ell$. We want to remark that this minimal surface is a beautiful analog of the helicoid in Euclidean space, namely, it is a ruled surface (see [**Tuz93**]) that is stratified into straight lines with respect to the hyperbolic metric. This property is used in the proof of our key Lemma 3.10.

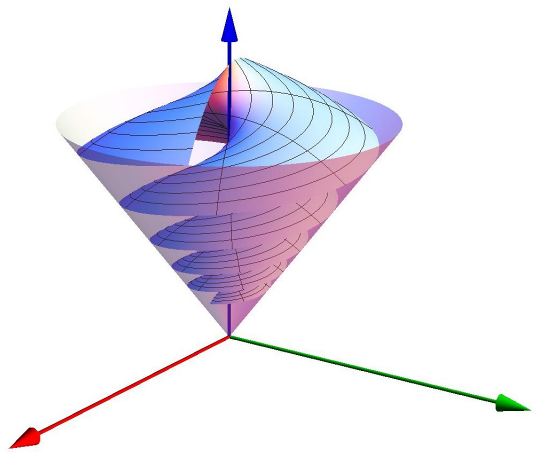

FIGURE 1. The helicoid $\mathscr{H}_{10}$ defined by (2.2) for $-\log(2) \leq u \leq \log(2)$ and $0 \leq v \leq \log(5)$ in the upper-half space model. The cone is the $\log(2)$-neighborhood of the t-axis. The curves perpendicular to the spirals are geodesics in $\mathbb{H}^3$.

3. Tubes of short curves and minimal surfaces

In this section, we start to develop techniques for proving our main theorems. We will work with maximal solid tubes associated with short simple closed geodesics in complete hyperbolic three-manifolds.

3.1. **Short curves and deep tubes.** As mentioned in the introduction, we make use of the maximal solid tubes around short curves, instead of the Margulis tubes. This approach makes our calculations more explicit. In this subsection, we construct such maximal solid tubes (following [**Mey87**]).

We consider loxodromic elements in the Kleinian surface group, namely, $\tau(z) = \alpha z$, up to conjugacy, where $\alpha = \exp(\ell + \sqrt{-1}\,\theta)$ with $\ell > 0$ and $\theta \in [-\pi, \pi)$. Such a loxodromic element translates points on the t-axis by the (hyperbolic) distance ℓ and twists a normal plane by the angle θ. For a simple closed geodesic γ in any complete hyperbolic three-manifold, a lift $\tilde{\gamma}$ of γ in $\mathbb{H}^3$ is the axis of a loxodromic element $\tau \in \mathrm{PSL}(2, \mathbb{C})$ representing γ. We note that a different lift just gives rise to another element that is conjugate to τ in $\mathrm{PSL}(2, \mathbb{C})$.

Definition 3.1. *We denote*

$$\mathscr{N}_r(\tilde{\gamma}) = \{x \in \mathbb{H}^3 : \mathrm{dist}(x, \tilde{\gamma}) < r\} \ ,$$

*as the r-neighborhood of the geodesic $\tilde{\gamma}$ in $\mathbb{H}^3$. We call $r_0(\gamma)$ the **tube radius** of γ if it is the supremum of numbers $r > 0$ such that $\mathscr{N}_r(\tilde{\gamma}) \cap \mathscr{N}_r(\tilde{\gamma}') = \emptyset$, for all lifts $\tilde{\gamma}'$ of γ different from $\tilde{\gamma}$. The **maximal solid tube** of γ is then defined by, for τ loxodromic in $\mathrm{PSL}(2,\mathbb{C})$ representing γ, and $\tilde{\gamma}$ the axis of the element τ,*

$$\mathbb{T}(\gamma) = \mathscr{N}_{r_0}(\tilde{\gamma})/\langle\tau\rangle \ . \tag{3.1}$$

We have the following basic result of Meyerhoff:

Theorem 3.2 ([**Mey87**])**.** *If γ is a simple closed geodesic in a complete hyperbolic three-manifold with real length ℓ less than the constant*

$$\varepsilon_2 = \mathbb{W}(1) = \frac{\sqrt{3}}{4\pi}\left(\log(\sqrt{2}+1)\right)^2 \approx 0.107071 \ , \tag{3.2}$$

then there exists an embedded maximal solid tube around γ whose tube radius is given by

$$\cosh^2(r_0(\gamma)) = \frac{1}{2}\left(\frac{\sqrt{1-2\kappa(\ell)}}{\kappa(\ell)} + 1\right) \ , \tag{3.3}$$

where the function

$$\kappa(\ell) = \cosh\left(\sqrt{\frac{4\pi\ell}{\sqrt{3}}}\right) - 1 \ . \tag{3.4}$$

Moreover, maximal solid tubes around different simple closed geodesics do not intersect if their real lengths are both less than ε_2.

Note that Meyerhoff's constant (3.2) is the maximum of the $\mathbb{W}$-function, therefore the real length condition in Theorem 3.2 is satisfied by short curves in our main results.

We now justify the geometry behind the introduction of "Otal's constant":

Proposition 3.3. *Let γ be a simple closed geodesic in a complete hyperbolic three-manifold which is diffeomorphic to $S \times \mathbb{R}$, such that its real length ℓ is less than "Otal's constant" (1.2), namely, $\ell < \varepsilon_{\mathrm{Otal}}(g) = \mathbb{W}(g)$, where $g \geq 2$ is the genus of S, then the area of the meridian disk of the maximal solid tube $\mathbb{T}(\gamma)$, defined in (3.1), is greater than the hyperbolic area of S.*

Proof. Recall that

$$\varepsilon_{\mathrm{Otal}}(g) = \mathbb{W}(g) = \frac{\sqrt{3}}{4\pi}\left[\cosh^{-1}\left(\frac{1}{1+\sqrt{1+(8g^2-8g+1)^2}} + 1\right)\right]^2 \ .$$

From (3.4), we have

$$\kappa(\ell) < \frac{1}{1+\sqrt{1+(8g^2-8g+1)^2}} \ .$$

Then the tube radius $r_0(\gamma)$ satisfies

$$\cosh^2(r_0(\gamma)) = \frac{1}{2}\left(\frac{\sqrt{1-2\kappa(\ell)}}{\kappa(\ell)}+1\right) > \frac{8g^2-8g+2}{2} \ .$$

The area of the meridian disk of the tube $\mathbb{T}(\gamma)$ is $2\pi(\cosh(r_0)-1)$. Therefore we have

$$2\pi(\cosh(r_0)-1) > 2\pi(2g-2) \ , \tag{3.5}$$

which is the hyperbolic area of the surface S. □

With this estimate, if $\ell < \varepsilon_{\mathrm{Otal}}(g) = \mathbb{W}(g) < \mathbb{W}(1)$, the arguments in [**Ota95, Ota03**] imply that γ is unknotted (i.e., γ lies on an embedded surface isotopic to S), and any collection of simple closed geodesics with real lengths shorter than $\mathbb{W}(g)$ are not linked (i.e., they can be isotoped such that each lies on a surface $S \times n$, where n is an integer).

Using the computation in Proposition 3.3, we also have the following proposition, which will be used for proving Theorem 1.5 and Theorem 1.6.

Proposition 3.4. *Let $\psi : S \to S$ be a pseudo-Anosov map of a closed surface S with genus $g \geq 2$, and $\mathcal{M}_\psi$ be the hyperbolic mapping torus with respect to ψ. If $\mathcal{M}_\psi$ contains a simple closed geodesic γ whose real length satisfies $\ell < \varepsilon_{\mathrm{Otal}}(g)$, then γ lies on an embedded surface isotopic to S.*

Proof. Recall that the fundamental group of $\mathcal{M}_\psi$ is given by

$$\pi_1(\mathcal{M}_\psi) = \langle \pi_1(S), t \ : \ \forall \alpha \in \pi_1(S), \ t\alpha t^{-1} = \psi_*(\alpha)\rangle \ , \tag{3.6}$$

where $\psi_* : \pi_1(S) \to \pi_1(S)$ is the induced isomorphism.

According to [**SY79, SU82**], $\mathcal{M}_\psi$ contains an embedded least area incompressible minimal surface Σ, which is isotopic to some fiber of $\mathcal{M}_\psi$. By the Gauss-Codazzi equation, we have $K_\Sigma = -1-\lambda^2$, where K_Σ and $\pm\lambda$ are the Gauss curvature and principal curvatures of Σ with respect to the induced metric respectively. By Gauss-Bonnet Theorem, the area of Σ is bounded from above by the area of a hyperbolic closed surface of genus g, i.e., $\mathrm{Area}(\Sigma) \leq 2\pi(2g-2) = 4\pi(g-1)$.

Obviously the minimal surface Σ is non-separating in $\mathcal{M}_\psi$, so there exists an essential loop $\gamma' \subset \mathcal{M}_\psi$ intersects Σ transversely at a single point. Let $\gamma^* \subset \mathcal{M}_\psi$ be the simple closed geodesic homotopic to γ', then the mod 2 intersection number of γ^* and Σ, denoted by $I_2(\gamma^*, \Sigma)$, is equal to one.

We claim that the length of γ^* is $\geq \varepsilon_{\mathrm{Otal}}(g)$. Our argument below actually can show that any such kind of geodesic has length greater than $\varepsilon_{\mathrm{Otal}}(g)$.

In particular, the geodesic representing the element t in (3.6) has length greater than $\varepsilon_{\mathrm{Otal}}(g)$.

Let $\mathbb{T}(\gamma^*) \subset \mathcal{M}_\psi$ be the maximal solid tube of γ^* with tube radius r_0. Similar to Lemma 3.9, any component of $\Sigma \cap \mathscr{N}_s(\gamma^*)$ is either a disk or an annulus for $0 \le s \le r_0$. Recall $I_2(\gamma^*, \Sigma) = 1$, so for any $0 \le s \le r_0$, $\Sigma \cap \mathscr{N}_s(\gamma^*)$ must contain at least one disk component, whose boundary is a simple closed curve which is essential on $\partial\mathscr{N}_s(\gamma^*)$ and is isotopic to the meridian of $\partial\mathscr{N}_s(\gamma^*)$. The meridian of $\partial\mathscr{N}_s(\gamma^*)$ is a geodesic with respect to the induced metric on the torus $\partial\mathscr{N}_s(\gamma^*)$, whose length equals $2\pi\sinh(s)$, so we have the following inequality

$$\mathrm{Length}(\Sigma \cap \partial\mathscr{N}_s(\gamma^*)) \ge 2\pi\sinh(s)\ ,\ \ 0 \le s \le r_0\ .$$

By the coarea formula [**CG06**, p.399] or [**Wan16**, Lemma 5.4], we have

$$\begin{aligned}\mathrm{Area}(\Sigma \cap \mathbb{T}(\gamma^*)) &= \int_0^{r_0}\int_{\Sigma\cap\partial\mathscr{N}_s(\gamma^*)} \frac{1}{\cos\theta} dl ds \\ &\ge \int_0^{r_0} \mathrm{Length}(\Sigma \cap \partial\mathscr{N}_s(\gamma^*)) ds \\ &\ge \int_0^{r_0} 2\pi\sinh(s) ds = 2\pi(\cosh(r_0) - 1)\ ,\end{aligned}$$

where $\theta = \theta(q)$ is the angle between the tangent space to Σ at q, and the radial geodesic that is through q (emanating from γ^*) and is perpendicular to γ^*. If the real length of γ^* satisfies $\ell < \varepsilon_{\mathrm{Otal}}(g)$, then by the computation in Proposition 3.3, we have the inequality (3.5), which implies the inequality $\mathrm{Area}(\Sigma \cap \mathbb{T}(\gamma^*)) > 4\pi(g-1)$. But this is a contradiction, since we have $\mathrm{Area}(\Sigma \cap \mathbb{T}(\gamma^*)) \le \mathrm{Area}(\Sigma) \le 4\pi(g-1)$.

Therefore if $\gamma \subset \mathcal{M}_\psi$ is a geodesic with $\ell < \varepsilon_{\mathrm{Otal}}(g)$, then the mod 2 intersection number of γ and Σ (or any fiber of $\mathcal{M}_\psi$) is zero, so we may lift γ to a closed geodesic, denoted by $\widetilde{\gamma}$, in the cyclic covering space of $\mathcal{M}_\psi$ which is diffeomorphic to $S \times \mathbb{R}$. Now we can apply the results in [**Ota95, Ota03**] to show that $\widetilde{\gamma}$ lies on an embedded surface isotopic to $S \times \{0\}$. In the quotient space $\mathcal{M}_\psi$, therefore γ is contained in some fiber of $\mathcal{M}_\psi$. □

3.2. **Minimal annuli in maximal solid tubes.** We now construct a minimal annulus inside a maximal solid tube of a short simple closed geodesic in a metrically complete hyperbolic three-manifold, this is done by using the helicoid in $\mathbb{H}^3$ defined in (2.2).

Let γ be a simple closed geodesic in a complete hyperbolic 3-manifold M, and $\mathbb{T}(\gamma)$ be its maximal solid tube with tube radius r_0. Let $\tau \in \mathrm{PSL}(2,\mathbb{C})$ be a loxodromic element of complex length $\mathcal{L} = \ell + \sqrt{-1}\,\theta$ representing γ, with $\ell > 0$ and $\theta \in [-\pi, \pi)$. Suppose $\tilde{\gamma}$ is a lift of γ in $\mathbb{H}^3$ which is the axis of τ, letting $a = |\theta|/\ell$, we define a surface in $\mathbb{T}(\gamma)$ as follows:

$$\mathscr{A}_a = \frac{\mathscr{H}_a \cap \mathscr{N}_{r_0}(\tilde{\gamma})}{\langle \tau \rangle} \ . \tag{3.7}$$

It is not hard to see that each component of $\mathscr{A}_a \cap \partial \mathscr{N}_r(\gamma)$ is a closed geodesic with respect to the induced metric on $\partial \mathscr{N}_r(\gamma)$, for each $r \in (0, r_0]$, with $\mathscr{N}_r(\gamma) = \mathscr{N}_r(\tilde{\gamma})/\langle \tau \rangle$. It is proven in [**Wan12**] that $\mathscr{A}_a$ is indeed a minimal annulus in $\mathbb{T}(\gamma)$, moreover, its area is explicitly computed as:

$$\text{Area}(\mathscr{A}_a) = 2\int_0^{r_0} \sqrt{\ell^2 \cosh^2(u) + \theta^2 \sinh^2(u)}\, du \ . \tag{3.8}$$

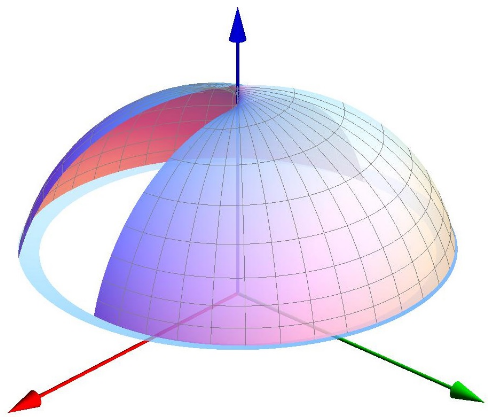

FIGURE 2. The fundamental domains of the maximal tube of the closed geodesic γ with complex length $\mathcal{L} = 0.01 + 0.25i$ (the radius $r_0 \approx 1.98272$) and the lifting of the minimal annulus $\mathscr{A}_{25}$ contained in a piece of the helicoid $\mathscr{H}_{25}$ which is given by (2.3) for $a = 25$, $-r_0 \leq u \leq r_0$ and $0 \leq v \leq 0.1$. In this case, $0.25/\sqrt{0.01} = 2.5$, $\text{Area}(\partial\mathbb{T}(\gamma)) \approx 0.828202$ and $\text{Area}(\mathscr{A}_{25}) \approx 1.35306$.

Now we prove the following technical estimate, where we introduce a constant $\varepsilon_1 = \mathbb{W}\left(\frac{3}{2}\right)$ to guarantee a key inequality (3.9), when the real length of γ is less than this constant and the inequality (1.3) is satisfied.

Theorem 3.5. *If a complete hyperbolic three-manifold M contains a simple closed geodesic γ whose complex length $\mathcal{L} = \ell + \sqrt{-1}\,\theta$ satisfies:*

(i) $\ell < \varepsilon_1 = \mathbb{W}\left(\frac{3}{2}\right) = \frac{\sqrt{3}}{4\pi}\left[\cosh^{-1}\left(\frac{1}{1+5\sqrt{2}} + 1\right)\right]^2 \approx 0.03347$ *;*

(ii) $|\theta|/\sqrt{\ell} > \sqrt[4]{3\pi^2} \approx 2.33268$ *,*

then we have

$$\text{Area}(\partial\mathbb{T}(\gamma)) = \pi\ell\sinh(2r_0) < |\theta|\cosh(r_0) < \text{Area}(\mathscr{A}_a) \ , \tag{3.9}$$

where $a = |\theta|/\ell$.

Proof. The area formula $\mathrm{Area}(\partial\mathbb{T}(\gamma)) = \pi\ell\sinh(2r_0)$ is well-known, see for instance [**GMM01**, Lemma 1.4].

Recall from (3.4), we have, once $\ell < \varepsilon_1$,

$$\kappa(\ell) = \cosh\left(\sqrt{\frac{4\pi\ell}{\sqrt{3}}}\right) - 1 < \frac{1}{1+5\sqrt{2}}\ . \tag{3.10}$$

By the tube radius formula in (3.3), we then have:

$$\cosh(r_0) > \sqrt{\frac{1}{2}\left(1+\left(1+5\sqrt{2}\right)\sqrt{1-\frac{2}{1+5\sqrt{2}}}\right)} = 2\ .$$

Applying the explicit area formula for the minimal annulus (3.8), we find:

$$\begin{aligned}\mathrm{Area}(\mathscr{A}_a) &= 2\int_0^{r_0}\sqrt{\ell^2\cosh^2(u)+\theta^2\sinh^2(u)}\,du\\ &> 2|\theta|\int_0^{r_0}\sinh(u)du = 2|\theta|(\cosh(r_0)-1)\\ &\geq |\theta|\cosh(r_0)\ .\end{aligned}$$

Therefore to establish (3.9), it suffices to show

$$\ell^2\sinh^2(r_0) < \frac{\theta^2}{4\pi^2}\ . \tag{3.11}$$

First we note that $\kappa(\ell) \geq \frac{2\pi\ell}{\sqrt{3}}$. Also $\ell < \varepsilon_1$ implies

$$0 < \sqrt{1-2\kappa(\ell)} - \kappa(\ell) < 1\ .$$

Now we use (3.3) again to find:

$$\ell^2\sinh^2(r_0) = \frac{\ell^2}{2\kappa(\ell)}\left(\sqrt{1-2\kappa(\ell)}-\kappa(\ell)\right) < \frac{\ell^2}{2\kappa(\ell)} \leq \frac{\sqrt{3}\ell}{4\pi}\ .$$

We then complete the proof by using condition (1.3). ☐

3.3. **Separation.** Let us recall some notations we will use in this subsection. Let $M = \mathbb{H}^3/G$ be a (metrically) complete hyperbolic three-manifold, and let γ be a simple closed geodesic in M, whose real length $\ell < \varepsilon_2$, the Meyerhoff constant. Also let r_0 be the tube radius of γ, $\mathbb{T}(\gamma)$ be the maximal solid tube around γ, and $\mathscr{N}_r(\tilde{\gamma})$ (as in Definition 3.1) be the r-neighborhood of the lift $\tilde{\gamma}$ in $\mathbb{H}^3$, with $r \in (0, r_0]$.

We will need the following result, proven in [**Wan12**], using arguments similar to [**MY82a, MY82b**], as well as [**FHS83**]:

Theorem 3.6 ([**Wan12**, Lemma 6])**.** *Using the above notations, and let* $\mathscr{N}_r(\gamma) = \mathscr{N}_r(\tilde{\gamma})/\langle\tau\rangle \subset M$*, where* τ *is the element in* G *representing the geodesic* γ*. If* C *is a smooth simple curve which is null-homotopic on* $\partial\mathscr{N}_r(\gamma)$ *whose length is less than* $2\pi\sinh(r)$*, with* $0 < r < r_0$*, then* C *bounds an embedded least area minimal disk* $\Delta \subset \mathscr{N}_r(\gamma)\backslash\gamma$*.*

For any $r \in (0, r_0]$, we let $D(r)$ be a disk on $\partial\mathscr{N}_r(\gamma)$ of radius r in the induced metric on $\partial\mathscr{N}_r(\gamma)$, and $\mathscr{B}(r)$ be the least area minimal disk in M bounding the closed curve $\partial D(r)$. We define

$$\delta = \delta(\gamma, r_0) = \min\left\{\operatorname{dist}(\gamma, \mathscr{B}(r)) : \frac{r_0}{2} \le r \le r_0\right\}, \tag{3.12}$$

where the distance is measured in hyperbolic metric. We re-write Theorem 3.6 into the following corollary to quantify the separation of the minimal disk Δ and the curve γ:

Corollary 3.7. *Same notations as in above Theorem 3.6. If* $r \in \left[\frac{r_0}{2}, r_0\right]$*, then we have* $\delta \in \left(0, \frac{r_0}{2}\right)$*, and the least area disk* $\Delta \subset \mathscr{N}_r(\gamma)\backslash\gamma$*, obtained in Theorem 3.6, satisfies* $\operatorname{dist}(\gamma, \Delta) \ge \delta$*.*

We prove the following existence result for a closed minimal surface with a specific property: it is separated from a simple closed geodesic if the curve satisfies our conditions in main theorems. More specifically,

Theorem 3.8. *Let* M *be a closed or quasi-Fuchsian hyperbolic three-manifold, and let* γ *be a simple closed geodesic contained in* M *whose complex length* $\mathcal{L} = \ell + \sqrt{-1}\,\theta$*, where* $\ell > 0$ *and* $\theta \in [-\pi, \pi)$*, satisfies:*

(i) $\ell < \varepsilon_0 = \mathbb{W}\left(\frac{2+\sqrt{3}}{2}\right) \approx 0.01822$*;*
(ii) $|\theta|/\sqrt{\ell} > \sqrt[4]{3\pi^2} \approx 2.33268$*,*

if S *is an embedded closed incompressible surface of genus* $g \ge 2$ *in* $M\backslash\gamma$*, then there exists an embedded least area minimal surface* $T \subset M\backslash\gamma$ *isotopic to* S*. Here* T *is of least area means its area is the smallest among all surfaces in* $M\backslash\gamma$ *in the isotopy class of* S*.*

This can be easily applied to the case of quasi-Fuchsian manifolds. Note that any quasi-Fuchsian manifold always contains embedded, closed, incompressible surfaces, a fact not always shared by some other classes of hyperbolic three-manifolds. The proof of this theorem is along the lines of the arguments in [**Wan12**], but we need to take special care at places with improved estimates. We also need the following lemma, whose proof can be found in [**HW17**]. We note that this is a purely topological lemma.

Lemma 3.9 ([**HW17**, Lemma 5.2])**.** *Let* $\mathbf{T}$ *be a solid torus with the core curve removed, and let* $S_{g,n}$ *be a surface of type* (g, n) *such that the Euler characteristic of* $S_{g,n}$ *is negative* [1]*. If* $S_{g,n}$ *is embedded in* $\mathbf{T}$ *such that* $\partial S_{g,n} \subset \partial \mathbf{T}$*, then there exists at least one essential non-peripheral simple closed curve* $\alpha \subset S_{g,n}$ *such that* α *bounds a disk* $D \subset \mathbf{T}$*.*

Proof of Theorem 3.8. Our strategy will be first to invoke a technique modifying the hyperbolic metric called "shrink-wrapping", developed by Calegari-Gabai ([**CG06**]) in their work on the tameness conjecture. We use this to conformally modify the hyperbolic metric of M inside a solid tube so that we can use the resulting totally geodesic boundary tori as barriers. We then construct a minimal surface and prove it is minimal with respect to the hyperbolic metric.

Consider the solid torus $\mathscr{N}_\sigma(\gamma) \subset M$, as before, where $\sigma < r_0$ is a positive constant, and r_0 is the tube radius of γ. For each $t \in [0, 1)$, one can define a family of Riemannian metrics g_t on M such that it coincides with the hyperbolic metric on $M\backslash\mathscr{N}_{(1-t)\sigma}(\gamma)$, while conformally equivalent to the hyperbolic metric on $\mathscr{N}_{(1-t)\sigma}(\gamma)$. Then by [**CG06**, Lemma 1.18], for each $t \in [0, 1)$, there is a function $f(t)$ satisfying $\frac{2}{3}(1-t)\sigma < f(t) < \frac{3}{4}(1-t)\sigma$, such that the torus $\partial\mathscr{N}_{f(t)}(\gamma)$ is totally geodesic with respect to the metric g_t, and the metric g_t dominates the hyperbolic metric on 2-planes. Moreover, by the standard result in [**SY79, FHS83, HS88**], for each $t \in [0, 1)$, there exists an embedded surface S_t in $M\backslash\mathscr{N}_{f(t)}(\gamma)$, isotopic to S, which is globally g_t-least area among all such surfaces. Our first goal is to show, for t sufficiently close to 1, and γ satisfying our conditions on complex length, the S_t produced as a globally least area surface with respect to the metric g_t is also of least area with respect to the hyperbolic metric.

Letting $r \in [\frac{r_0}{2}, r_0]$, by Corollary 3.7, there is $\delta > 0$ defined in (3.12), only depending on r_0 and γ, such that the least area disk $\Delta \subset \mathscr{N}_r(\gamma)\backslash\gamma$, obtained in Theorem 3.6, satisfies $\operatorname{dist}(\gamma, \Delta) \geq \delta$. Here $\delta \in (0, \frac{r_0}{2})$ and from above we have a constant σ with $\sigma < r_0$. Now we choose t sufficiently close to 1 such that $(1-t)\sigma < \delta$. For instance we can define $t_0 \in (0, 1)$, such that we have $(1-t)\sigma < \delta$ for all $t > t_0$.

Now we pause to prove a technical lemma which will be used later for our applications:

Lemma 3.10. *Under the conditions of Theorem 3.8, for all* $t > t_0$*, where* t_0 *is the threshold defined above, the* g_t*-least area surface* S_t *is disjoint from* $\mathscr{N}_\delta(\gamma)$*, i.e.,* $S_t \cap \mathscr{N}_\delta(\gamma) = \emptyset$*, where* δ *is defined in* (3.12)*.*

[1]The Euler characteristic of $S_{g,n}$ is $\chi(S_{g,n}) = 2 - 2g - n$. Thus $\chi(S_{g,n}) < 0$ if one of the conditions is satisfied: $g \geq 2$ and $n \geq 0$, $g = 1$ and $n \geq 1$ or $g = 0$ and $n \geq 3$.

Proof of Lemma 3.10. We may assume $S_t \cap \mathscr{N}_\delta(\gamma) \neq \emptyset$, for all $t > t_0$, and we will get a contradiction.

By the shrink-wrapping argument, it's easy to see that S_t is disjoint from γ for $t \in [0,1)$, hence

$$S_t \cap \mathbb{T}(\gamma) = S_t \cap \mathbf{T} \ , \quad \text{for } t \in [0,1) \ ,$$

where $\mathbb{T}(\gamma)$ is the maximal solid tube of γ with radius r_0 and $\mathbf{T} = \mathbb{T}(\gamma) \setminus \gamma$. Let Σ be a component of $S_t \cap \mathbb{T}(\gamma)$ which intersects $\mathscr{N}_\delta(\gamma)$. We claim that we always have

$$\text{Length}(\Sigma \cap \partial\mathscr{N}_s(\gamma)) \geq 2|\theta|\sinh(s) \ , \quad \text{where } \frac{r_0}{2} \leq s \leq r_0 \ . \tag{3.13}$$

In fact, since S_t is incompressible in M, and as we are dealing with a solid torus with core curve removed, we apply Lemma 3.9, then Σ is either a disk whose boundary is null-homotopic on $\partial\mathbb{T}(\gamma)$ or an annulus whose boundary is essential on $\partial\mathbb{T}(\gamma)$. There are two cases we need to consider:

(i) Case One: Σ is a disk. Then by Corollary 3.7, $\Sigma \cap \mathscr{N}_s(\gamma)$ consists of disjoint disks for all $s \in [\frac{r_0}{2}, r_0]$. If there exists some $s' \in [\frac{r_0}{2}, r_0]$ such that $\text{Length}(\Sigma \cap \partial\mathscr{N}_{s'}(\gamma)) < 2|\theta|\sinh(s')$, then $\text{Length}(\Sigma \cap \partial\mathscr{N}_{s'}(\gamma)) < 2\pi\sinh(s')$, which implies that $\Sigma \cap \mathscr{N}_{s'}(\gamma)$ is disjoint from $\mathscr{N}_\delta(\gamma)$, therefore so is Σ. A contradiction. Therefore (3.13) is true when Σ is a disk.

(ii) Case Two: Σ is an annulus. For any $s \in [\frac{r_0}{2}, r_0]$, $\Sigma \cap \mathscr{N}_s(\gamma)$ either only consists of disjoint disks or contains an annulus, say Σ'. In the former subcase, (3.13) is true according to the argument in Case One. In the latter subcase, $\partial\Sigma'$ consists of two isotopic slopes on $\partial\mathscr{N}_s(\gamma)$, so $\text{Length}(\partial\Sigma')$ is greater than either $2(2\pi\sinh(s)) = 4\pi\sinh(s)$ or $\text{Length}(\mathscr{A}_a \cap \partial\mathscr{N}_s(\gamma))$, where $2\pi\sinh(s)$ is the length of the meridian of the torus $\partial\mathscr{N}_s(\gamma)$ and each component of $\mathscr{A}_a \cap \partial\mathscr{N}_s(\gamma)$ is a geodesic isotopic to the longitude of the torus $\partial\mathscr{N}_s(\gamma)$. It's easy to see the inequalities $4\pi\sinh(s) > 2|\theta|\sinh(s)$ and

$$\begin{aligned}\text{Length}(\mathscr{A}_a \cap \partial\mathscr{N}_s(\gamma)) &= 2\sqrt{\ell^2\cosh^2(s) + \theta^2\sinh^2(s)} \\ &> 2|\theta|\sinh(s) \ .\end{aligned}$$

So (3.13) is also true in this case.

Recalling that the new metric g_t dominates the hyperbolic metric on 2-planes, we apply the co-area formula (see [**Wan12**, Lemma 3]) to obtain the following estimate:

$$\begin{aligned}\text{Area}(\Sigma, g_t) &\geq \text{Area}(\Sigma) \\ &\geq \text{Area}\left(\Sigma \cap \left(\overline{\mathbb{T}(\gamma) \setminus \mathscr{N}_{\frac{r_0}{2}}(\gamma)}\right)\right)\end{aligned}$$

$$\geq \int_{\frac{r_0}{2}}^{r_0} \text{Length}(\Sigma \cap \partial \mathscr{N}_s(\gamma)) ds \tag{3.14}$$

$$\geq 2|\theta| \int_{\frac{r_0}{2}}^{r_0} \sinh(s) ds$$

$$= 2|\theta| \left(\cosh(r_0) - \cosh\left(\frac{r_0}{2}\right)\right) \ . \tag{3.15}$$

where we denote $\text{Area}(\cdot, g_t)$ the g_t-area, and $\text{Area}(\cdot)$ the hyperbolic area.

We now interpret constant ε_0 in (1.5). When $\ell < \varepsilon_0$, we have from (3.4) that:

$$\kappa(\ell) = \cosh\left(\sqrt{\frac{4\pi\ell}{\sqrt{3}}}\right) - 1 < \frac{1}{1 + \sqrt{1 + (7 + 4\sqrt{3})^2}} \ ,$$

therefore we have from Meyerhoff's formula (3.3) for the tube radius:

$$\cosh(r_0) = \sqrt{\frac{1}{2}\left(\frac{\sqrt{1 - 2\kappa(\ell)}}{\kappa(\ell)} + 1\right)} > \sqrt{3} + 1 \ . \tag{3.16}$$

As a result, we find:

$$\cosh(r_0) > 2\cosh\left(\frac{r_0}{2}\right) \ . \tag{3.17}$$

Putting this into the inequality (3.15), we have:

$$\text{Area}(\Sigma) > |\theta| \cosh(r_0) \ . \tag{3.18}$$

Since $\varepsilon_0 = \mathbb{W}\left(\frac{2+\sqrt{3}}{2}\right) < \varepsilon_1 = \mathbb{W}\left(\frac{3}{2}\right)$, conditions (i) and (ii) in the statement allow us to apply the inequality (3.11) in the proof of Theorem 3.5, namely, we have

$$\text{Area}(\partial \mathbb{T}(\gamma)) = \pi\ell \sinh(2r_0) < |\theta| \cosh(r_0) \ .$$

By our choice of $t > t_0$, we have $(1-t)\sigma < \delta$, the metric g_t coincides with the hyperbolic metric outside $\mathscr{N}_{(1-t)\sigma}(\gamma)$, and combining these inequalities, we have established:

$$\text{Area}(\Sigma, g_t) > \text{Area}(\partial \mathbb{T}(\gamma)) = \text{Area}(\partial \mathbb{T}(\gamma), g_t) \ . \tag{3.19}$$

This estimate then allows us to proceed with cut-and-paste, namely, we can replace each component of $S_t \cap \mathbb{T}(\gamma)$ which intersects $\mathscr{N}_\delta(\gamma)$ by either an annulus or a disk on $\partial \mathbb{T}(\gamma)$, to obtain a new surface $S'_t \subset M \backslash \mathscr{N}_\sigma(\gamma)$ such that it has less g_t-area than S_t, disjoint from $\mathscr{N}_\sigma(\gamma)$ and isotopic to S in $M \backslash \gamma$. This is impossible since S_t is the least area surface with these properties. This completes the proof for the lemma. □

Now we continue our proof for Theorem 3.8. By above lemma, we have the g_t-least area surface S_t is separated from $\mathscr{N}_\delta(\gamma)$. But by shrinkwrapping, the metric g_t coincides with the hyperbolic metric outside of $\mathscr{N}_\delta(\gamma)$, therefore S_t

is of least area with respect to the hyperbolic metric for t sufficiently close to 1. Since δ is independent of t, we let $t \to 1$, and complete the proof. □

4. Applications

Previously we have examined how closed incompressible least area minimal surface interacts with maximal solid tubes of short curves (Theorem 3.8 and Separation Lemma 3.10). We now proceed to apply these techniques in the settings of quasi-Fuchsian manifolds and oriented closed hyperbolic three-manifolds that fiber over the circle, respectively.

4.1. **Multiplicity of minimal surfaces in quasi-Fuchsian manifolds.**

When the complex length of the curve $\gamma \subset \mathscr{M}$ satisfies the conditions in Theorem 1.5, where $\mathscr{M}$ is a quasi-Fuchsian manifold, one would expect multiple minimal surfaces around γ. This indeed the case, for instance, in Figure 3 below. This is because there exist two closed incompressible surfaces S_1 and S_2 in $\mathscr{M}\backslash\gamma$ which are not isotopic to each other. Applying Theorem 3.8, we produce two least area surfaces T_1 and T_2 that are not isotopic.

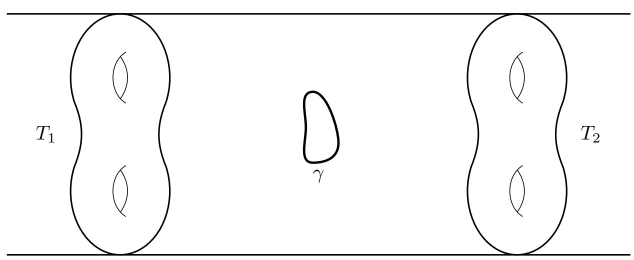

Figure 3. Two minimal surfaces around a short curve.

We now make more precise this observation to the case of multiple short (but unlinked) curves in a quasi-Fuchsian manifold:

Corollary 4.1. *Let $\Gamma = \{\gamma_i\}_{i=1}^n$ be a collection of mutually disjoint simple closed geodesics in a quasi-Fuchsian manifold $\mathscr{M} \cong S \times \mathbb{R}$, each of complex length $\mathcal{L}_i = \ell_i + \sqrt{-1}\,\theta_i$, where $\ell_i > 0$ and $\theta_i \in [-\pi, \pi)$, which satisfy:*

(i) $\ell_i < \varepsilon_{\mathrm{Otal}}(g)$;
(ii) $|\theta_i|/\sqrt{\ell_i} > \sqrt[4]{3\pi^2} \approx 2.33268$.

If Σ is an embedded closed incompressible surface of genus $g \geq 2$ in $\mathscr{M}\backslash\Gamma$ (which is homeomorphic to S), then there exists an embedded least area minimal surface $T \subset \mathscr{M}\backslash\Gamma$ isotopic to Σ. Moreover, the quasi-Fuchsian manifold $\mathscr{M}$ contains at least $n+1$ distinct closed incompressible least area surfaces.

Proof. By Theorem 3.2, and $\varepsilon_{\mathrm{Otal}}(g) < \varepsilon_2$, the tubes $\mathbb{T}(\gamma_i)$ are mutually disjoint. Then the first part of the corollary follows from the proof of Theorem 3.8.

For the second part, with the real length condition $\ell_i < \varepsilon_{\mathrm{Otal}}(g)$, for all $i = 1, 2, \cdots, n$, the collection Γ is unlinked in the following sense ([**Ota03**]): there exists a homeomorphism between $\mathscr{M}$ and $S \times \mathbb{R}$ such that each component of Γ is contained in one of the surfaces $S \times \{i\}$, $1 \leq i \leq n$. Now we count isotopy classes: there are $n+1$ ways one can find closed incompressible surfaces $\Sigma_1, \ldots, \Sigma_{n+1}$ embedded in $\mathscr{M} \backslash \Gamma$ can separate Γ such that they are not isotopic to each other in $\mathscr{M} \backslash \Gamma$ (see Figure 4 for instance). For each arrangement, we apply Theorem 3.8, and then we find $n+1$ embedded closed incompressible least area surfaces $T_1, \ldots, T_{n+1}$ such that T_α is isotopic to Σ_α in $\mathscr{M} \backslash \Gamma$ for $\alpha = 1, 2, \ldots, n+1$. They are distinct since they are not isotopic pair-wisely. □

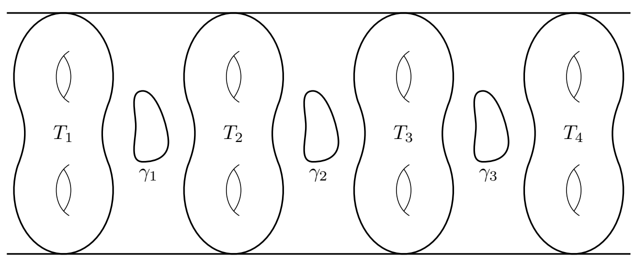

FIGURE 4. Minimal surfaces around multiple short curves.

4.2. **Proof of Theorem 1.5.** We now move to our main interest: oriented closed hyperbolic three-manifolds that fiber over the circle. Theorem 1.5 is proved by taking algebraic limits in the quasi-Fuchsian setting. Now we will use Corollary 4.1 to complete the proof of Theorem 1.5, which we re-state here:

Theorem 1.5. *If an oriented closed hyperbolic three-manifold $\mathcal{M}$ that fibers over the circle with fiber S contains a simple closed geodesic whose complex length $\mathcal{L} = \ell + \sqrt{-1}\,\theta$ satisfies:*

(i) $\ell < \varepsilon_{\mathrm{Otal}}(g)$;
(ii) $|\theta|/\sqrt{\ell} > \sqrt[4]{3\pi^2} \approx 2.33268$,

then for any positive integer N, there exists a quasi-Fuchsian manifold $\mathscr{M} \cong S \times \mathbb{R}$ which contains at least N embedded closed incompressible least area minimal surface.

Proof. Recall that $\mathcal{M}$ is a closed hyperbolic three-manifold fibering over the circle, with fiber S closed surfaces of genus greater than one. We consider a cyclic cover of $\mathcal{M}$, "unwrapping" the circle direction. We denote this cover $\mathcal{M}_\infty \cong S \times \mathbb{R}$. Identifying S with some lift of the fiber, we obtain a discrete and faithful representation $\rho : \pi_1(\mathcal{M}_\infty) = \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$, which is a Kleinian surface group.

Let γ be a simple closed geodesic on $\mathcal{M}$ whose complex length $\mathcal{L} = \ell + \sqrt{-1}\,\theta$ satisfies $0 < \ell < \varepsilon_{\mathrm{Otal}}(g)$ and $|\theta|/\sqrt{\ell} > \sqrt[4]{3\pi^2}$. By Proposition 3.4, the geodesic γ can be lifted to a closed geodesic in $\mathcal{M}_\infty$, denoted by γ too. Let Φ be a deck transformation of $\mathcal{M}_\infty$, then $\mathcal{M}_\infty$ contains a sequence $\{\Phi^k(\gamma)\}_{k\in\mathbb{Z}}$ leaving every compact subset in $\mathcal{M}_\infty$ (see for instance [**Min03**]). This doubly degenerate hyperbolic three-manifold $\mathcal{M}_\infty$ belongs to the Thurston boundary of the deformation space, namely, $\partial\mathcal{QF}(S) = \mathrm{AH}(S)\backslash\mathcal{QF}(S) = \overline{\mathcal{QF}_g(S)}\backslash\mathcal{QF}(S)$, using (2.1). There is a sequence of quasi-Fuchsian groups, each representing a quasi-Fuchsian manifold $\{\mathscr{M}_i\}$, which converges to the manifold $\mathcal{M}_\infty$ algebraically as $i \to \infty$.

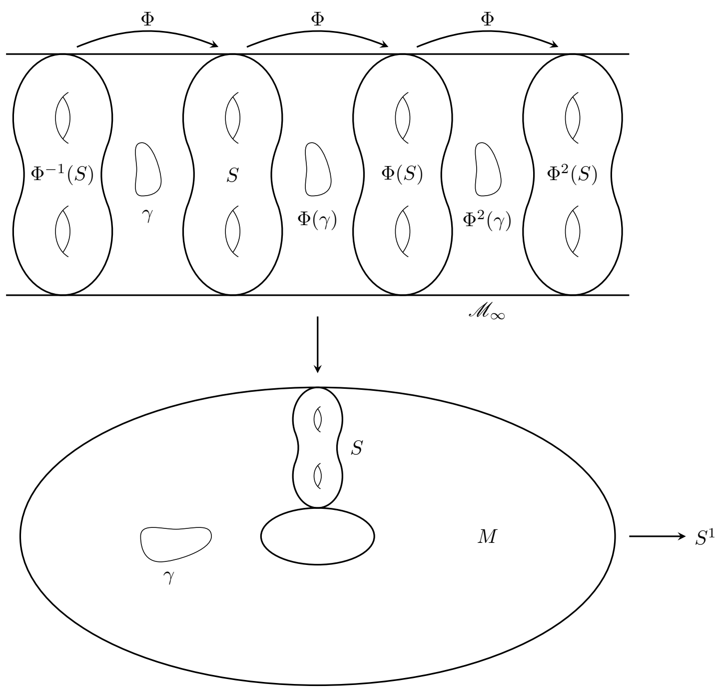

FIGURE 5. Cyclic cover for surface bundle fibering over the circle.

Since each element in a Kleinian surface group determines a geodesic, and $\mathcal{M}_\infty$, as a cyclic cover of $\mathcal{M}$, contains infinitely many hyperbolic geodesics $\{\Phi^k(\gamma)\}_{k\in\mathbb{Z}}$, all having the same complex length $\mathcal{L}$. For any $N > 0$, when i is sufficiently large, there is a quasi-Fuchsian manifold $\mathscr{M}_i$ in the sequence such that it contains at least $N-1$ simple closed geodesics each complex length satisfying the conditions in the statement. We then apply Corollary 4.1 to find that $\mathscr{M}_i$ contains at least N embedded closed incompressible least area surfaces. □

4.3. **Proof of Theorem 1.6.** In this subsection, we apply our estimates and cut-and-paste techniques developed in §3 to prove the nonexistence of a C^2-minimal fibration for an oriented closed hyperbolic three-manifold that fibers over the circle, which we re-state here:

Theorem 1.6. *If an oriented closed hyperbolic three-manifold $\mathcal{M}$ that fibers over the circle with fiber S contains a simple closed geodesic whose complex length $\mathcal{L} = \ell + \sqrt{-1}\,\theta$ satisfies:*

(i) $\ell < \varepsilon_{\text{Otal}}(g)$;
(ii) $|\theta|/\sqrt{\ell} > \sqrt[4]{3\pi^2} \approx 2.33268$,

then $\mathcal{M}$ does not admit a minimal fibration.

Proof. We proceed by contradiction. Suppose that the hyperbolic mapping torus $\mathcal{M}$ is foliated by minimal surfaces all isotopic to a closed surface S. We denote this C^2-foliation by $\mathcal{F}$. By theorems of Sullivan ([**Sul79**]), Hass ([**Has86**]) and Harvey-Lawson ([**HL82**]), all leaves of the foliation $\mathcal{F}$ are of least area homologically, and of the same area.

Since $\mathcal{M}$ contains a simple closed geodesic γ whose complex length $\mathcal{L}$ satisfies conditions $\ell < \varepsilon_{\text{Otal}}(g)$ and the inequality (1.3), by Proposition 3.4 the geodesic γ lies on an embedded surface isotopic to some fiber of $\mathcal{M}$. Our strategy is to prove: (1) there exist leaves of $\mathcal{F}$ which are disjoint from γ, and (2) any such leaf of $\mathcal{F}$ can't be very close to γ.

Firstly let's prove (1). By Proposition 3.4, there exists an embedded surface isotopic to the fiber which is disjoint from γ, therefore we may apply Theorem 3.8 to obtain a minimal surface $F' \subset \mathcal{M} \setminus \gamma$ (with respect to the hyperbolic metric of $\mathcal{M}$) which is isotopic to the leaves of $\mathcal{F}$. Since each leaf of $\mathcal{F}$ is a minimal surface, F' must be a leaf of $\mathcal{F}$ by the maximum principle. This means there exist some leaves of $\mathcal{F}$ which are disjoint from γ.

Secondly let's prove (2). There exist some other leaves of $\mathcal{F}$ which must intersect γ since $\mathcal{F}$ is a foliation of the manifold $\mathcal{M}$, therefore there exists a leaf $F \in \mathcal{F}$ such that F is disjoint from γ and the (hyperbolic) distance between F and γ is less than δ, where δ is the constant defined by (3.12). According to the argument in the proof of Theorem 3.8 and Lemma 3.10, this is impossible. Actually let Σ be a component of $F \cap \mathbb{T}(\gamma)$ which intersects the δ-neighborhood of γ, since $F \cap \mathbb{T}(\gamma) = F \cap \mathbf{T}$, where $\mathbb{T}(\gamma)$ is the maximal solid tube (whose radius is still denoted by r_0) for this short geodesic $\gamma \subset \mathcal{M}$

and $\mathbf{T} = \mathbb{T}(\gamma) \setminus \gamma$. Then by Lemma 3.9, Σ is either a disk such that $\partial\Sigma$ is null-homotopic on $\partial\mathbb{T}(\gamma)$ or Σ is an annulus such that $\partial\Sigma$ consists of two isotopic essential slopes on $\partial\mathbb{T}(\gamma)$.

We claim that in both cases

$$\text{Length}(\Sigma \cap \mathscr{N}_s(\gamma)) \geq 2|\theta| \sinh(s) \ , \quad \frac{r_0}{2} \leq s \leq r_0 \ . \tag{4.1}$$

We consider two cases:

(i) Σ is a disk. In this case, if there exists some $s' \in \left[\frac{r_0}{2}, r_0\right]$ such that (4.1) fails, then we have $\text{Length}(\Sigma \cap \mathscr{N}_{s'}(\gamma)) < 2|\theta|\sinh(s') \leq 2\pi \sinh(s')$. We then apply the argument in the proof of Lemma 3.10, and find $\Sigma \cap \mathscr{N}_{s'}(\gamma)$ is disjoint from the δ-neighborhood of γ, so is Σ. A contradiction.

(ii) Σ is an annulus. For any $s \in \left[\frac{r_0}{2}, r_0\right]$, if $\Sigma \cap \mathscr{N}_s(\gamma)$ only consists of disks, then (4.1) is true according to the argument in the previous case. If $\Sigma \cap \mathscr{N}_s(\gamma)$ contains at least one annulus, similar to the argument in the proof of Lemma 3.10, (4.1) is also true.

Therefore each component of $F \cap \mathbb{T}(\gamma)$ which intersects the δ-neighborhood of γ must satisfies (4.1).

Then we have the area comparison as in (3.19):

$$\text{Area}(\Sigma) > \text{Area}(\partial\mathbb{T}(\gamma)).$$

This area estimate allows us to use cut-and-paste technique again, namely, we can replace each component of $F \cap \mathbb{T}(\gamma)$ that intersects the δ-neighborhood of γ by either an annulus or a disk on $\partial\mathbb{T}(\gamma)$, to obtain a new surface F'' such that it has less area than F, and isotopic to F. This is impossible since F is of the least area.

Now we have proved that if a leaf F of the foliation $\mathcal{F}$ is disjoint from γ (recall this kind of leaves always exist by the shrink-wrapping argument), then it must be disjoint from the δ-neighborhood of γ, that is, we have proved (2).

But on the other hand, since $\mathcal{F}$ is a foliation of $\mathcal{M}$, some leaves of $\mathcal{F}$ must intersect γ. By the continuity of $\mathcal{F}$, some leaves of $\mathcal{F}$ which are disjoint from γ must be sufficiently close to γ. This contradiction implies the non-existence of the minimal foliation on $\mathcal{M}$.

□

4.4. **Final remarks.** We make several remarks.

(i) There are many other interesting work related the complex length with the geometry of hyperbolic three-manifolds, see for instance ([**Min99**, **Bre11**, **Mil14**]).

(ii) One may ask Question 1.3 for the existence of C^0 foliations. Our techniques rely on a Theorem of Sullivan ([**Sul79**]) which requires the taut foliation to be at least C^2.

(iii) It is still unknown whether there exists any fibered hyperbolic three-manifold which does admit a minimal foliation (C^0 or C^2). Theorem 1.6 provides sufficient conditions for a negative answer to the existence of minimal foliations, and these conditions are verified in §5.2 for many fibered hyperbolic three-manifolds. One can produce many more examples by varying the number of twists and the number of loops being twisted by Twister and SnapPy programs.

(iv) One may further ask whether a hyperbolic three-manifold always admits a foliation of closed incompressible surfaces of constant mean curvature. We ([**HW13**]) constructed a quasi-Fuchsian manifold which does not contains such a constant mean curvature foliation, but the question remains open for many other cases of hyperbolic three-manifolds.

5. Appendix

In this appendix, we want to explore the computational aspect of the ratio $\frac{|\theta|}{\sqrt{\ell}}$. Note that we will need the short curve γ lying on an incompressible closed surface to work. Nevertheless we first give an upper bound for the ratio when ℓ is small, in terms of the genus of the surface. In the second part, we use computer program Twister ([**BHS14**]) to produce some explicit examples. The first such examples were produced for us by Saul Schleimer, to whom we are most grateful.

5.1. **An upper bound.** We present a proof of a statement by Thurston that if the geodesic is *short*, then its rotation is *small* in the following sense (see the proof of Corollary 4.3 in [**Thu98**]).

Proposition 5.1 (Thurston 1986)**.** *Let $M \in \mathrm{AH}(S)$ be a complete hyperbolic three-manifold, here S is an oriented closed incompressible surface with genus $g(S) \geq 2$. Let $\gamma \subset S$ be a simple closed geodesic with complex length $\mathcal{L} = \ell + \sqrt{-1}\,\theta$, where $\ell > 0$ and $-\pi \leq \theta < \pi$. If its real length is less than the Meyerhoff constant, namely, $\ell < \varepsilon_2 = \mathbb{W}(1) \approx 0.107071$, then*

$$|\theta| < \frac{2\pi(g-1)}{\cosh(r_0) - 1}\ , \tag{5.1}$$

where r_0 is the radius of the maximal solid torus of γ. Furthermore, as $\ell \to 0$, we have $\theta \to 0$.

Proof. According to [**MT98**, Lemma 6.12], there exists a pleated surface $f : \Sigma \to M$ whose pleating locus contains γ, where $\Sigma \in \mathcal{T}(S)$ is a hyperbolic surface. This pleated surface is nevertheless incompressible, so at least one component of $\mathbb{T}(\gamma) \cap f(\Sigma)$ is an annulus whose core is the simple closed geodesic γ. The area of this annulus is greater than that of $\mathscr{A}_a$ (see (3.7)) by the co-area formula, where $a = |\theta|/\ell$.

The hyperbolic area of Σ is $2\pi|\chi(\Sigma)| = 4\pi(g-1)$, then we have

$$\begin{aligned} 4\pi(g-1) &> \text{Area}(\mathbb{T}(\gamma) \cap f(\Sigma)) \\ &\geq \text{Area}(\mathscr{A}_a) = 2\int_0^{r_0} \sqrt{\ell^2 \cosh^2(u) + \theta^2 \sinh^2(u)}\, du \\ &\geq 2|\theta| \int_0^{r_0} \sinh(u) du = 2|\theta|(\cosh(r_0) - 1)\ , \end{aligned}$$

then we get (5.1).

Now we apply explicit formulas (3.3) and (3.4) to examine the asymptotics. Since $r_0 \to \infty$ as $\ell \to 0$, we have $\theta \to 0$ as $\ell \to 0$ by (5.1). □

Note that if we assume $\ell < \varepsilon_{Otal}(g)$, then by Proposition 3.4, we may replace the condition $\gamma \subset S$ in Proposition 5.1 to $\gamma \subset M$.

We examine the asymptotics in Proposition 5.1 more closely and find:

Corollary 5.2. *Same assumption in above Proposition, we have, whenever ℓ is small enough,*

$$\frac{|\theta|}{\sqrt{\ell}} < 4\pi\sqrt{\frac{4\pi}{\sqrt{3}}}\,(g-1). \tag{5.2}$$

Proof. From (5.1), when $\ell < \varepsilon_2$, we have

$$\frac{|\theta|}{\sqrt{\ell}} < \frac{2\pi(g-1)}{\sqrt{\ell}(\cosh(r_0)-1)}. \tag{5.3}$$

Given the explicit nature of r_0 in terms of $\sqrt{\ell}$ in (3.3) and (3.4), we expand the function $\sqrt{\ell}(\cosh(r_0) - 1)$ in terms of $\sqrt{\ell}$ as follows:

$$\sqrt{\ell}(\cosh(r_0) - 1) = \frac{1}{b} - \sqrt{\ell} - \frac{b}{24}\ell - \frac{353b^3}{5760}\ell^2 + o(\ell^{\frac{5}{2}}), \tag{5.4}$$

where $b = \sqrt{\frac{4\pi}{\sqrt{3}}} \approx 2.69355$. Certainly from (5.4), we have, when $\ell > 0$ is small enough,

$$\sqrt{\ell}(\cosh(r_0) - 1) > \frac{1}{2b}, \tag{5.5}$$

Therefore we have

$$\frac{|\theta|}{\sqrt{\ell}} < 4\pi b(g-1). \tag{5.6}$$

□

Remark 5.3. *For genus $g = 2$, this upper bound is approximately* 33.84815*, with a limit* 16.92408 *as ℓ goes to zero. Clearly this upper bound gets worse as ℓ goes from zero to ε_2. In comparison, in our main theorems, the lower bound for this ratio we require is approximately* 2.33268*. This propels us to look for fibered hyperbolic three-manifolds with specific length spectra.*

5.2. **Examples via Twister and SnapPy.** Intuitively, in order to produce high rotational angle θ, one may twist a loop in the three-manifold many times. In this subsection, we produce several explicit examples of fibered hyperbolic three-manifold which contains a closed curve with our conditions satisfied by computer programs.

In the first example, we let S_2 be a closed genus two surface, and M be a mapping torus fibering over the circle. We consider γ a short simple closed geodesic on a fiber. Following a suggestion of Saul Schleimer, we run Twister and SnapPy programs under the system Python. Note that for both main theorems, we look for the complex length to satisfy $\ell < \mathbb{W}(g)$, and $\frac{|\theta|}{\sqrt{\ell}} > 2.33268$. We explore the following:

```
>>> import twister
>>> S2 = twister.Surface('S_2')
>>> S2.info()
>>> A Twister surface of genus 2 with 0 boundary component(s)
Loops: a, b, c, d, e
Arcs:
>>> M = S2.bundle('b'*8 + 'cdea').high_precision()
```

This produces a SnapPy manifold M and we may ask the program to calculate its hyperbolic volume:

```
>>> M.volume()
7.991423345
```

We now ask the program to specify the spectrum:

```
>>> M.length_spectrum(0.5)
```

This returns a curve of complex length $0.1055786 + 0.84482566\sqrt{-1}$ in M, namely we have $\ell \approx 0.1055786$ and $\theta \approx 0.84482566$. Though the ratio $\frac{|\theta|}{\sqrt{\ell}} \approx 2.60003$ is desirable, the real length is not short enough. We then choose to do the twists a few more times:

```
>>> M = S2.bundle('b'*25 + 'cdea').high_precision()
>>> M.volume()
8.142725
>>> M.length_spectrum(0.5)
```

Now this returns a mapping torus M of volume 8.142725 and a closed curve of complex length $0.0098 + 0.25794\sqrt{-1}$. Now we have $\ell < \mathbb{W}(2) \approx 0.01515$, and the ratio $\frac{|\theta|}{\sqrt{\ell}} \approx 2.60572$. This is an explicit example for both Theorems 1.5 and 1.6 for $g = 2$.

One similarly can work with other surfaces and their bundles (over the circle) to obtain more examples of fibered hyperbolic three-manifolds with our conditions satisfied. For instance, we find for the case of genus 3:

```
>>> S3=twister.Surface((3,0))
>>> S3.info()
A Twister surface of genus 3 with 0 boundary component(s)
Loops: a0, b1, b2, b3, b4, b5, c
Arcs:
>>> M = S3.bundle('b1'*40 + 'a0b2b3b4b5c').high_precision()
>>> M.volume()
10.4355474
>>> M.length_spectrum(0.5)
```

Now we obtain a short curve with $\ell = 0.00302$ and $\theta = 0.158958$, which yields the ratio $\frac{|\theta|}{\sqrt{\ell}} \approx 2.892537$. Note that when $g = 3$, Otal's constant $\mathbb{W}(3) \approx 0.00549389$. This is an example for $g = 3$ to satisfy both conditions in Theorems 1.5 and 1.6.

We conclude with an example from the program Twister where fibering surfaces have genus 4 and contains a short curve with conditions in both Theorems satisfied. Note that $\mathbb{W}(4) \approx 0.00280798$.

```
>>> S4 = twister.Surface((4,0))
>>> S4.info()
A Twister surface of genus 4 with 0 boundary component(s)
Loops: a0, b1, b2, b3, b4, b5, b6, b7, c
Arcs:
>>> M = S4.bundle('b1'*45 + 'a0b2b3b4b5b6b7c').high_precision()
>>> M.volume()
11.511256
>>> M.length_spectrum(0.5)
```

One of the closed curves returned has complex length $0.002362+0.140781\sqrt{-1}$, which yields the ratio $\frac{|\theta|}{\sqrt{\ell}} \approx 2.8967$.

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