CLOSED MINIMAL SURFACES IN CUSPED HYPERBOLIC THREE-MANIFOLDS

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Abstract. We present a new proof for the following theorem, originally proved in [**CHMR14**]: if any oriented complete (non compact) hyperbolic three-manifold of finite volume M^3 admits a closed, oriented, embedded and incompressible surface S with genus greater than one, then M^3 admits a closed embedded incompressible minimal surface Σ which is of least area in the isotopy class of S. Our argument highlights how special structures of these three-manifolds prevent such a minimal surface going too deep into the cusped region.

1. Introduction

1.1. **Minimal surfaces in hyperbolic three-manifolds.** Minimal surfaces are fundamental objects in geometry. In three-manifold theory, the existence and multiplicity of minimal surfaces often offer important geometrical insight into the structure of the ambient three-manifold (see for instance [**Rub05, Mee06**]), they also have important applications in Teichmüller theory, Lorentzian geometry and many other mathematical fields (see for example [**Rub07, KS07**]). By Thurston's geometrization theory, the most common geometry in a three-manifold is hyperbolic ([**Thu80**]), and this paper is a part of a larger goal of studying closed incompressible minimal surfaces in hyperbolic three-manifolds.

Before we state our main result, we briefly motivate our effort by making some historic notes on minimal surface theory in three different types of hyperbolic three-manifolds, namely, compact hyperbolic three-manifolds, quasi-Fuchsian manifolds, and cusped hyperbolic three-manifolds (complete, noncompact, and of finite volume).

Let M^3 be a Riemannian three-manifold, and let Σ be a closed surface which is immersed or embedded in M^3, then Σ is called a *minimal surface* if its mean curvature vanishes identically, further we call it *least area* if the area of Σ with respect to the induced metric from M^3 is no greater than that of any other surface which is homotopic or isotopic to Σ in M^3.

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A closed surface is called *incompressible* in M^3 if the induced map between the fundamental groups is injective. Throughout this paper, we always assume a closed incompressible surface is of genus at least two and is oriented.

In the case when M^3 is compact, Schoen and Yau ([**SY79**]) and Sacks and Uhlenbeck ([**SU82**]) showed that if $S \subset M^3$ is a closed incompressible surface, then S is homotopic to an immersed least area minimal surface Σ in M^3. The techniques of [**SY79, SU82**] extend to the case M^3 is a compact (negatively curved) three-manifold with mean convex boundary (i.e. ∂M^3 has non-negative mean curvature with respect to the inward normal vector), then there still exists an immersed least area minimal surface Σ in any isotopy class of incompressible surfaces (see [**Uhl83, MSY82, HS88**]).

Recall that a quasi-Fuchsian manifold is a complete (of infinite volume) hyperbolic three-manifold diffeomorphic to the product of a closed surface and $\mathbb{R}$. Since the convex core of any geometrically finite quasi-Fuchsian manifold is compact with mean convex boundary, one finds the existence of closed incompressible surface of least area in this class of hyperbolic three-manifolds. In [**Uhl83**], Uhlenbeck initiated a systematic study of the moduli theory of minimal surfaces in hyperbolic three-manifolds, where she also studied a subclass of quasi-Fuchsian manifolds which we call *almost Fuchsian*. M^3 is called almost Fuchsian if it admits a closed minimal surface of principal curvatures less than one in magnitude. Such a minimal surface is unique and embedded in the almost Fuchsian manifold (see also [**FHS83**]), and therefore one can study the parameterization of the moduli of almost Fuchsian manifolds via data on the minimal surface (see for instance [**GHW10, HW13, San13**]). For the uniqueness and multiplicity questions of minimal surfaces in quasi-Fuchsian manifolds, or in general hyperbolic three-manifolds, one can refer to [**And83, Wan12, HL12, HW15**] and references within.

This paper will address the existence question for embedded closed incompressible minimal surfaces in another important class of hyperbolic three-manifolds: cusped hyperbolic three-manifolds. M^3 is called a *cusped hyperbolic three-manifold* if it is a complete non-compact hyperbolic three-manifold of finite volume. There are many examples of this type, frequently the complements of knots and links in the 3-sphere $\mathbb{S}^3$. Mostow rigidity theorem ([**Mos73**]) extends to this class of hyperbolic three-manifolds by Prasad ([**Pra73**]), however the techniques used in [**SY79, SU82**] to find incompressible minimal surfaces do not. It is well-known that any cusped hyperbolic three-manifold admits infinitely many immersed closed minimal surfaces ([**Rub05**]), however, they may not be embedded, nor incompressible. Using min-max theory, very recently, Collin, Hauswirth, Mazet and

Rosenberg in [**CHMR14**, Theorem A] proved the existence of an embedded (not necessarily incompressible) compact minimal surface in M^3. It has been a challenge to show the existence of closed embedded incompressible minimal surface in hyperbolic three-manifolds.

For the rest of the paper, we always assume M^3 is an oriented cusped hyperbolic three-manifold.

1.2. **Main result.** In three-manifold theory, it is a question of basic interest to ask if one can deform an embedded surface in its isotopy class to some minimal surface. Among several remarkable results on minimal surface in cusped hyperbolic three-manifold, Collin, Hauswirth, Mazet and Rosenberg proved the following existence theorem:

Theorem 1.1 ([**CHMR14**, Theorem B])**.** *Let S be a closed orientable embedded surface in a cusped hyperbolic three-manifold which is not a 2-sphere or a torus. If S is incompressible and non-separating, then S is isotopic to an embedded least area minimal surface.*

We present a new proof of this theorem by using relatively elementary tools. In the proof, we obtain quantitative estimates on how deep this least area minimal surface can reach into the cusped region of M^3 (see Remark 2.2 and Corollary 5.5). The geometric structures both in the upper-half space $\mathbb{H}^3$ and the cusped hyperbolic three-manifold M^3 play crucial role in our arguments to keep the least area minimal surface in the region not arbitrarily far into the cusp.

While there exist some cusped hyperbolic three-manifolds which do not admit any closed embedded essential surfaces ([**MR92**]), it has been shown recently that any cusped hyperbolic three-manifold must admit an *immersed* closed essential quasi-Fuchsian surface ([**MZ08, BC15**]).

1.3. **Outline of the proof.** There are essentially two parts for our proof of Theorem 1.1. First we modify the hyperbolic metric in $\mathbb{H}^3$ to obtain a submanifold of M^3 in the quotient with sufficiently long cusped regions, and the modified metric around all boundaries so that the submanifold is a compact negatively curved manifold with totally geodesic boundaries. By results of [**MSY82, HS88, Uhl83**], there is a least area minimal surface Σ (with respect to the new metric, not the hyperbolic metric) in the isotopy class of a closed incompressible surface S in this compact submanifold. The heart of the argument is then to guarantee it does not drift into infinity of M^3. We deploy a co-area formula (see Lemma 5.6) as our main tool for this. We can then show that Σ is actually contained in the subregion of the submanifold which is still equipped with the hyperbolic metric. Hence Σ

is a least area minimal surface with respect to the hyperbolic metric. It is oriented as well since the surface S is non-separating.

The organization of the paper is as follows: in §2, we cover necessary background material and fix some notations; in §3, we modify the upper-half space model of $\mathbb{H}^3$ to set up semi-spheres as barriers for minimal surfaces in $\mathbb{H}^3$; in §4, we move down to the cusped hyperbolic three-manifold M^3 and its maximal cusped regions. Using the modification in previous section we obtain a truncated Riemannian three-manifold of negative curvature. Finally in §5, we prove our main result.

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2. Preliminary

2.1. **Kleinian groups and cusped hyperbolic three-manifolds.** We will work with the upper-half space model of the hyperbolic space $\mathbb{H}^3$, i.e.

$$\mathbb{H}^3 = \{(x, y, t) \in \mathbb{R}^3 \mid t > 0\} \ ,$$

equipped with metric

$$ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2} \ . \tag{2.1}$$

The hyperbolic space $\mathbb{H}^3$ has a natural compactification: $\overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \widehat{\mathbb{C}}$, where $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. The orientation preserving isometry group of the upper-half space $\mathbb{H}^3$ is given by $\mathsf{PSL}_2(\mathbb{C})$, which consists of linear fractional transformations that preserve the upper-half space.

A (torsion free) discrete subgroup Γ of $\mathsf{PSL}_2(\mathbb{C})$ is called a *Kleinian group*, and the quotient space $M^3 = \mathbb{H}^3/\Gamma$ is a complete hyperbolic three-manifold whose fundamental group $\pi_1(M^3)$ is isomorphic to Γ. Conversely, if M^3 is a complete hyperbolic three-manifold, then there exists a holonomy $\rho : \pi_1(M^3) \to \mathsf{PSL}_2(\mathbb{C})$ such that $\Gamma = \rho(\pi_1(M^3))$ is a (torsion free) Kleinian group and $M^3 = \mathbb{H}^3/\rho(\pi_1(M^3))$.

Mostow-Prasad's Rigidity Theorems imply that hyperbolic volume is a topological invariant for hyperbolic three-manifolds of finite volume, that is to say, these hyperbolic three-manifolds are completely determined by their fundamental groups. Jørgensen and Thurston (see [**Thu80**, Chapter

5–6]) proved that the set of volumes of orientable hyperbolic three-manifolds is well ordered and of order type ω^ω. Since any non-orientable hyperbolic three-manifold is double-covered by an orientable hyperbolic three-manifold, then the set of volumes of all hyperbolic three-manifolds is also well ordered.

Many examples of the cusped hyperbolic three-manifold come from the complements of hyperbolic knots [**Thu82**, Corollary 2.5] on $\mathbb{S}^3$. In general cusped hyperbolic three-manifolds can be described as follows (see [**Thu80**, Theorem 5.11.1]):

Theorem 2.1. *A cusped hyperbolic three-manifold is the union of a compact submanifold which is bounded by tori and a finite collection of horoballs modulo $\mathbb{Z} \oplus \mathbb{Z}$ actions.*

2.2. **Maximal cusps and maximal cusped regions.** In this subsection, we briefly describe the *maximal cusps and maximal cusped regions* of the cusped hyperbolic three-manifold M^3, and they will play important roles in our construction. For more details, one can go to for instance [**Ada05, Mar07**].

Suppose that M^3 has been decomposed into a compact component (which is called the *compact core* of M^3) and a finite set of cusps (or ends), each homeomorphic to $T^2 \times [0, \infty)$, where T^2 represents a torus. Each cusp can be realized geometrically as the image of some horoball $\mathcal{H}$ in $\mathbb{H}^3$ under the covering map from $\mathbb{H}^3$ to M^3. If we lift any such cusp to the upper-half space model $\mathbb{H}^3$ of the hyperbolic space, we obtain a parameter family of disjoint horoballs.

Assume first that M^3 has exactly one cusp, and we lift it to the corresponding set of disjoint horoballs, each of which is the image of any other by some group element. Expand the horoballs equivariantly until two first become tangent. The projection of these expanded horoballs back to M^3 is called the *maximal cusped region* of M^3, denoted by $\mathcal{C}$.

Assume that one such horoball $\mathcal{H}$ is centered about ∞. We may normalize the horoball $\mathcal{H}$ so that $\partial\mathcal{H}$ is a horizontal plane with Euclidean height *one* above the xy-plane. Thus $\mathcal{H} = \{(x, y, t) \mid t \geq 1\}$.

Let $\rho : \pi_1(M^3) \to \mathsf{PSL}_2(\mathbb{C})$ be the holonomy of M^3. Then $\Gamma = \rho(\pi_1(M^3))$ is a (torsin free) Kleinian group with parabolic elements. Let Γ_∞ be the parabolic subgroup of Γ which fixes ∞, it's then well-known that Γ_∞ is generated by two elements $z \to z + \mu$ and $z \to z + \nu$, where μ and ν are non-trivial complex numbers which are not real multiples of each other. Obviously $\mathcal{H}$ is invariant under Γ_∞, and the quotient $\mathcal{H}/\Gamma_\infty$ is just the maximal cusped region $\mathcal{C}$ of M^3 described above. Also $T^2 = \partial\mathcal{H}/\Gamma_\infty$ is a torus.

The fundamental domain of the parabolic group Γ_∞ in the horoball $\mathcal{H}$ is denoted by $A \times [1, \infty)$, where $A \subset \partial\mathcal{H}$ is a parallelogram spanned by the complex numbers μ and ν. It is not hard to see that the Euclidean area of A, which is given by $Re(\mu\bar{\nu})$, is the same as that of the torus T^2.

Recall that we may equip the horoball $\mathcal{H}$ with the *warped product metric* $ds^2 = e^{-2\tau}(dx^2 + dy^2) + d\tau^2$, by letting $\tau = \log t$ for $t \geq 1$. Then the metric on the maximal cusped region $\mathcal{C}$ can be written in the form

$$ds^2 = e^{-2\tau}\, ds^2_{\text{eucl}} + d\tau^2 \ , \tag{2.2}$$

where ds^2_{eucl} is the standard flat metric on the torus T^2 induced from that of $\partial\mathcal{H}$.

If M^3 has more than one cusp, we define the maximal cusped region for each cusp exactly as above. It's possible that the maximal cusped regions in a cusped hyperbolic three-manifold can intersect.

Now suppose that the cusped hyperbolic three-manifold M^3 has k cusps, whose maximal cusped regions are denoted by $\mathcal{C}_i = T_i^2 \times [0, \infty)$, $i = 1, \dots, k$. Let $\tau_0 > 0$ be the smallest number such that each maximal cusped region $T_i^2 \times (\tau_0, \infty)$, $i = 1, 2, \dots, k$, is disjoint from any other maximal cusped regions of M^3.

For any constant $\tau \geq \tau_0$, let $M^3(\tau)$ be the compact subdomain of M^3 which is given by

$$M^3(\tau) = M^3 - \bigcup_{i=1}^{k} \left(T_i^2 \times (\tau, \infty)\right) \ . \tag{2.3}$$

By this construction, $M^3(\tau)$ is a compact submanifold of M^3 with concave boundary components with respect to the inward normal vectors.

For each i with $1 \leq i \leq k$, we lift M^3 to the upper-half space model of the hyperbolic space $\mathbb{H}^3$ such that one horoball $\mathcal{H}_i$ corresponding to the maximal cusped region $\mathcal{C}_i$ is centered at ∞ and $\partial\mathcal{H}_i$ passes through the point $(0, 0, 1)$. Suppose that Γ^i_∞ is the subgroup of Γ, which is generated by two elements $z \to z + \mu_i$ and $z \to z + \nu_i$, where μ_i and ν_i are non-trivial complex numbers that are not real multiples of each other.

Now we may define a constant as follows:

$$L_0 = \max\left\{e^{\tau_0},\, |\mu_1| + |\nu_1|,\, \dots,\, |\mu_k| + |\nu_k|\right\} \ > 0. \tag{2.4}$$

Remark 2.2. *Note that this constant only depends on M^3. We will prove that the closed incompressible least area minimal surface Σ in Theorem 1.1 is contained in $M^3(\tau_3)$, where $\tau_3 = \log(3L_0)$, if the prescribed embedded incompressible surface S is contained in $M^3(\tau_1)$.*

3. Constructing Barriers in Hyperbolic Three-space

In this section we work entirely in the hyperbolic space $\mathbb{H}^3$ instead of the quotient cusped hyperbolic three-manifold M^3. Our goal will be to construct semi-spheres in $\mathbb{H}^3$ which can be used as barriers for minimal surfaces. To do this, we will first modify the standard hyperbolic metric on $\mathbb{H}^3$ to get a new metric which is non-positively curved. This procedure gives us the flexibility we need to obtain barriers.

3.1. **Modifying the hyperbolic space.** For fixed constants $L_2 > L_1 > 0$, we define a smooth cut-off function $\varphi : (0, \infty) \to [0, \infty)$ as follows:

(i) $\varphi(t) = \frac{1}{t}$, if $0 < t \leq L_1$;
(ii) $\varphi(t)$ is is strictly decreasing on $[L_1, L_2)$, with $\varphi(L_1) = \frac{1}{L_1}$ and $\varphi(L_2) = 0$;
(iii) $\varphi(t) \equiv 0$ if $t \geq L_2$;
(iv) We also require φ to satisfy the following inequality:

$$0 \leq \varphi(t) \leq \frac{1}{t}\ , \quad \text{for all } t > 0\ . \tag{3.1}$$

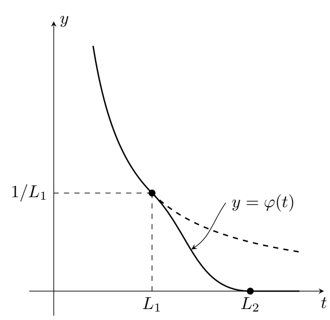

Figure 1. A graph of $\varphi(t)$

We now define another smooth function $f(t) : (0, \infty) \to (0, \infty)$ by solving the following equation:

$$\frac{f'(t)}{f(t)} = \varphi(t)\ , \quad \text{for all } t > 0\ . \tag{3.2}$$

And it is then easy to see that we may require $f(t)$ to satisfy the following:

(i) $f(t) = t$, if $0 < t \leq L_1$;
(ii) $f(t)$ is strictly increasing on the interval (L_1, L_2);
(iii) $f(t)$ is a constant, if $t \geq L_2$.

Now we consider an upper-half space model of the *modified* hyperbolic space $(\mathbb{U}^3, \bar{g})$, constructed as follows:

(i) $\mathbb{U}^3 = \mathbb{R}^3_+ = \{(x, y, t) \in \mathbb{R}^3 \mid t > 0\}$,

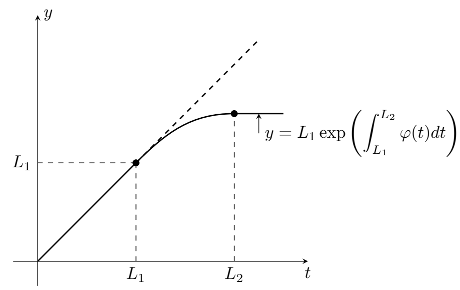

FIGURE 2. A graph of $f(t)$

(ii) with the new metric given by

$$\bar{g}(x,y,t) = \frac{dx^2 + dy^2 + dt^2}{(f(t))^2}. \tag{3.3}$$

Comparing with the standard hyperbolic metric (2.1) on $\mathbb{H}^3$, one sees that $\bar{g}$ is just the hyperbolic metric for $t \in (0, L_1]$, and flat beyond $t = L_2$. In fact, we have the following result, which was not explicitly listed but can be derived from the proof of [**Zho99**, Theorem 4.1]. We include a proof here for the sake of completeness.

Proposition 3.1. [**Zho99**] *The upper-half space* $(\mathbb{U}^3, \bar{g})$ *is non-positively curved.*

Proof. Recalling from (3.3), we may choose a local coordinate system such that $\bar{g}_{ij} = \frac{\delta_{ij}}{f(t)^2}$, for $\{i,j\} = \{1,2,3\}$. We can then workout the Christoffel symbols $\{\bar{\Gamma}_{ij}^k\}$ with respect to this metric $\bar{g}$ according to the formula:

$$\bar{\Gamma}_{ij}^k = \frac{1}{2}\bar{g}^{km}(\bar{g}_{mi,j} + \bar{g}_{mj,i} - \bar{g}_{ij,m}).$$

We find these Christoffel symbols are:

(i) $\overline{\Gamma}_{13}^1 = \overline{\Gamma}_{31}^1 = \overline{\Gamma}_{23}^2 = \overline{\Gamma}_{32}^2 = \overline{\Gamma}_{33}^3 = -\frac{f'(t)}{f(t)}$,
(ii) $\overline{\Gamma}_{11}^3 = \overline{\Gamma}_{22}^3 = \frac{f'(t)}{f(t)}$, and
(iii) all others are equal to 0.

One can then verify the sectional curvatures of the space $(\mathbb{U}^3, \bar{g})$ at a point (x, y, t) are given by

$$K_{12} = -(f'(t))^2 \ , \quad \text{and} \quad K_{13} = K_{23} = f''(t)f(t) - (f'(t))^2 \ . \tag{3.4}$$

Note that, by (3.2), we have

$$\frac{f''(t)f(t) - (f'(t))^2}{f^2(t)} = \left(\frac{f'(t)}{f(t)}\right)' = \varphi'(t) \le 0 \ , \quad \text{for all } t > 0 \ .$$

Therefore the space $(\mathbb{U}^3, \bar{g})$ is non-positively curved. □

We need to calculate the principal curvatures of some surfaces immersed in $(\mathbb{U}^3, \bar{g})$, if these surfaces are special with respect to to a metric that is conformal to $\bar{g}$ in $\mathbb{U}^3$. The tool can be found in the following more general lemma:

Lemma 3.2 ([**Lóp13**]). *For $m \geq 3$, let $(\mathcal{M}, g)$ be an m-dimensional Riemannian manifold and let $\sigma : \mathcal{M} \to \mathbb{R}^+$ be a smooth positive function on $\mathcal{M}$. Define the metric $\bar{g} = \sigma^2 g$. Let $\iota : S \to \mathcal{M}$ be an immersion of an orientable hypersurface. If κ is a principal curvature of $(S, \iota^* g)$ with respect to the unit normal vector field N, and then*

$$\bar{\kappa} = \frac{\kappa}{\sigma} - \frac{1}{\sigma^2} \nabla_N \sigma \tag{3.5}$$

is a principal curvature of $(S, \iota^* \bar{g})$ with respect to the unit normal vector field $\overline{N} = N/\sigma$, and $\nabla_N \sigma$ is the covariant derivative of σ along N.

By Proposition 3.1, we know that the space $(\mathbb{U}^3, \bar{g})$ is non-positively curved. We now want to understand the structure of some special figures in $(\mathbb{U}^3, \bar{g})$. This will become important in Theorems 4.1 and 4.2: we need to construct a submanifold in M^3 is of negative curvature and it is a quotient from a subregion in $\mathbb{H}^3$ by the same Kleinian group.

Theorem 3.3. *The subspace $\{(x, y, t) \in \mathbb{U}^3 \mid 0 < t \leq L_2\}$ is a negatively curved space (with respect to the metric $\bar{g}$), with a totally geodesic boundary $\{(x, y, t) \in \mathbb{U}^3 \mid t = L_2\}$. Furthermore, any horizontal plane in $(\mathbb{U}^3, \bar{g})$ is either convex with respect to the upward normal vector $N = (0, 0, 1)$, or totally geodesic.*

Proof. To apply Lemma 3.2, on the space $\mathbb{U}^3$, the metric g will be designated as the Euclidean metric, and the conformal factor $\sigma(x, y, t) = \frac{1}{f(t)}$, where $f(t)$ is defined previously, and $\bar{g} = \frac{g}{f^2(t)}$ is the modified metric on $\mathbb{U}^3$ which is nonpositively curved in Proposition 3.1.

For any horizontal plane that passes through $(0, 0, t)$, its unit normal vector at the point (x, y, t) with respect to the Euclidean metric g is given by $N(x, y, t) = (0, 0, 1)$.

Since

$$\nabla_N (1/f(t)) = \operatorname{grad}(1/f(t)) \cdot N = -\frac{f'(t)}{f^2(t)},$$

where grad is the gradient with respect to the Euclidean metric g and $\cdot$ denotes the Euclidean inner product of vectors, then by (3.5), we find the principal curvatures of the plane with respect to the new metric $\bar{g}$

$$\overline{\kappa}_i(x, y, t) = 0 - f^2(t)(-\frac{f'(t)}{f^2(t)}) = f'(t) \ , \quad i = 1, 2 \ .$$

By the construction of the function $f(t)$, we have

- $f'(t) > 0$ if $0 < t < L_2$, and
- $f'(t) \equiv 0$ if $t \geq L_2$.

Therefore any horizontal plane through the $(0,0,t)$ is either convex with respect to the normal vector $N = (0,0,1)$ if $0 < t < L_2$, or totally geodesic if $t \geq L_2$. □

Remark 3.4. *Similarly one can show that any vertical plane is totally geodesic, and any vertical straight line is a geodesic with respect to the new metric $\bar{g}$.*

3.2. **Barriers.** The following result guarantees that semi-spheres in $(\mathbb{U}^3, \bar{g})$ can be used as the barrier surfaces to prevent the least area minimal surface Σ from entering into each cusped region of M^3 too far.

Theorem 3.5. *For any positive constant r, let*

$$S^2_+(r) = \{(x,y,t) \mid x^2 + y^2 + t^2 = r^2,\ t > 0\}$$

be a semi-sphere in $(\mathbb{U}^3, \bar{g})$ with radius r. Then $S^2_+(r)$ is non-concave with respect to the inward normal vector field, i.e. the principal curvatures of $S^2_+(r)$ are nonnegative with respect to the inward normal vector field.

Proof. Let g again denote the standard Euclidean metric on $\mathbb{R}^3_+$. At a point $p = \left(x, y, \sqrt{r^2 - x^2 - y^2}\right)$ on $S^2_+(r)$, the inward normal vector field on the semi-sphere $S^2_+(r)$ with respect to the Euclidean metric g is given by

$$N(p) = \left(-\frac{x}{r}, -\frac{y}{r}, -\frac{\sqrt{r^2 - x^2 - y^2}}{r}\right) .$$

The principal curvatures κ_1 and κ_2 of $S^2_+(r) \subset (\mathbb{R}^3_+, g)$ with respect to the normal vector N are identically equal to $\frac{1}{r}$.

As in the proof of Theorem 3.3, we set $\sigma(x,y,t) = \frac{1}{f(t)}$, where the positive function $f(t)$ is defined by solving the equation (3.2). Let $\overline{\kappa}_i$ $(i = 1,2)$ be the principal curvatures of $S^2_+(r) \subset (\mathbb{U}^3, \bar{g})$ at p with respect to an orientation $\overline{N}(p) = f\left(\sqrt{r^2 - x^2 - y^2}\right) N(p)$.

Now we apply (3.5), the principal curvatures $\overline{\kappa}_i$ $(i = 1,2)$ at p are then given by:

$$\begin{aligned}
\overline{\kappa}_i(p) &= f\left(\sqrt{r^2 - x^2 - y^2}\right) \cdot \frac{1}{r} - f'\left(\sqrt{r^2 - x^2 - y^2}\right) \cdot \frac{\sqrt{r^2 - x^2 - y^2}}{r} \\
&= \frac{f\left(\sqrt{r^2 - x^2 - y^2}\right)}{r} \left\{1 - \varphi\left(\sqrt{r^2 - x^2 - y^2}\right)\sqrt{r^2 - x^2 - y^2}\right\} \\
&\geq 0 ,
\end{aligned}$$

where we use the property (3.1). This completes the proof. □

4. Truncating Cusped Hyperbolic three-manifold

We want to construct a submanifold in a cusped hyperbolic three-manifold M^3 whose boundary components are concave with respect to the inward normal vectors. The idea is to remove some horoballs of certain sizes from $\mathbb{H}^3$ in §4.1, then modify the hyperbolic metric in the remaining regions according to previous section, and we have to of course verify, in §4.2, that the Kleinian group Γ of M^3 preserves the new metric (otherwise we get a different hyperbolic three-manifold in the quotient).

4.1. **Truncated hyperbolic space.** As before we assume that the cusped hyperbolic three-manifold M^3 has k cusps, whose maximal cusped regions are denoted by $\mathcal{C}_i = T_i^2 \times [0, \infty)$, $i = 1, \dots, k$. We also denote $\rho : \pi_1(M^3) \to \mathsf{PSL}_2(\mathbb{C})$ as the holonomy so that $\Gamma = \rho(\pi_1(M^3))$ is a Kleinian group.

For the i-th cusped region $T_i^2 \times [\tau, \infty)$, let $\mathcal{H}_i(\tau)$ be the corresponding horoball centered at ∞, whose boundary is a horizontal plane passing through the point $(0, 0, e^\tau)$, i.e.

$$\mathcal{H}_i(\tau) = \{(x, y, t) \in \mathbb{H}^3 \mid t \geq e^\tau\}. \tag{4.1}$$

In particular, $\mathcal{H}_i(0)$ is the corresponding (maximal) horoball $\mathcal{H}_i$ centered at ∞. We also denote $\mathcal{H}_i^\circ(\tau)$ as the interior of (4.1).

Recall that $\tau_0 > 0$ is the smallest number such that each maximal cusped region $T_i^2 \times (\tau_0, \infty)$, $i = 1, 2, \dots, k$, is disjoint from any other maximal cusped regions of M^3. When $\tau \geq \tau_0$, the subset $\Omega(\tau)$ of $\mathbb{H}^3$ is obtained by removing a disjoint collection of open horoballs, namely,

$$\Omega(\tau) = \mathbb{H}^3 - \bigcup_{i=1}^{k} \bigcup_{\gamma \in \Gamma} \gamma\left(\mathcal{H}_i^\circ(\tau)\right) \tag{4.2}$$

is called a *truncated hyperbolic* 3*-space* (see [**BH99**, p.362]).

It is clear that $\Omega(\tau)$ is invariant under Γ, so

$$\Omega(\tau)/\Gamma = M^3(\tau)\ . \tag{4.3}$$

We define four constants

$$\tau_j = \log(j \cdot L_0)\ , \quad \text{for } j = 1, 2, 3, 4\ , \tag{4.4}$$

where the constant L_0 is defined by (2.4). Note that by this definition (4.4) and by (2.4), we have $\tau_4 > \tau_3 > \tau_2 > \tau_1 \geq \tau_0 > 0$.

We are particularly interested in the subregion $\Omega(\tau_4)$, and we define a new metric on it as follows:

(i) We equip the subregion $\Omega(\tau_3)$ with the standard hyperbolic metric.

(ii) The subregion $\Omega(\tau_4)\backslash\Omega^\circ(\tau_3)$ (where $\Omega^\circ(\tau_3)$ is the interior of $\Omega(\tau_3)$) consists of countably infinitely many disjoint subregions which can be divided into k families $\mathscr{H}_1, \ldots, \mathscr{H}_k$, such that each family $\mathscr{H}_i$ is the lift of the cusped subregion $T_i^2 \times [\tau_3, \tau_4]$.

For an element $U_i \in \mathscr{H}_i$, we may assume that it can be described as

$$U_i = \{(x, y, t) \in \mathbb{H}^3 \mid 3L_0 \le t \le 4L_0\} \ . \tag{4.5}$$

We equip the region U_i with the new metric

$$d\bar{s}^2 = \frac{dx^2 + dy^2 + dt^2}{(f(t))^2} \ , \tag{4.6}$$

where the function f is defined on $[3L_0, 4L_0]$ just as in §2 (i.e. $L_1 = 3L_0$ and $L_2 = 4L_0$). Similarly we may define the same new metric on the other elements in $\mathscr{H}_i$, and so on the elements from the other families.

We denote $\bar{g}$ the new metric on the space $\Omega(\tau_4)$. Now we apply Theorem 3.3 to arrive at the following:

Theorem 4.1. *The compact space $(\Omega(\tau_4), \bar{g})$ is a negatively curved space with (countably infinitely many) totally geodesic boundary components.*

4.2. **The Kleinian group.** We now show the Kleinian group Γ preserves the new metric $\bar{g}$ on $\Omega(\tau_4)$. More precisely,

Theorem 4.2. *The group Γ is a subgroup of $\mathsf{Isom}(\Omega(\tau_4), \bar{g})$, the isometry group of $\Omega(\tau_4)$ with respect to the negatively curved metric $\bar{g}$.*

Proof. Let p and q be two points in $\Omega(\tau_4)$, and we need to show that $d(p, q) = d(\gamma(p), \gamma(q))$ for any element $\gamma \in \Gamma$, where $d(\cdot, \cdot)$ denotes the distance function with respect to the new metric $\bar{g}$.

By Theorem 4.1, the manifold $(\Omega(\tau_4), \bar{g})$ is negatively curved. Then there is a unique geodesic $c : [0, L] \to (\Omega(\tau_4), \bar{g})$ parameterized by arc length, such that $c(0) = p$ and $c(L) = q$. If the geodesic $c([0, L])$ is totally contained in $\Omega(\tau_3)$, we are done by the definition of the function $f(t)$ (note that $f(t) = t$ for $t \in (0, 3L_0)$). If $c([0, L])$ is entirely contained in any component of $\Omega(\tau_4) - \Omega^\circ(\tau_3)$, then $f(t)$ is a strictly increasing function and γ preserves the distance.

In general the geodesic c is expressed as a chain of non-trivial paths $c_1, \ldots, c_n$, each parameterized by arc length, such that

(i) each of the paths c_i is either a hyperbolic geodesic or else its image is contained in one component of $\Omega(\tau_4) - \Omega^\circ(\tau_3)$;
(ii) if c_i is a hyperbolic geodesic then the image of c_{i+1} is contained in one component of $\Omega(\tau_4) - \Omega^\circ(\tau_3)$, and vice versa.

Suppose that each geodesic segment c_i is parameterized by $c_i(s) = c(s)$ for $s \in [s_{i-1}, s_i]$, where $0 = s_0 < s_1 < \cdots < s_n = L$ is a partition of the interval $[0, L]$. Then we write $c = c_1 * c_2 * \cdots * c_n$ in the sense that $c(s) = c_i(s)$ if $s \in [s_{i-1}, s_i]$. By the above argument, we have that each curve $\gamma \circ c_i : [s_{i-1}, s_i] \to (\Omega(\tau_4), \bar{g})$ is a geodesic for $i = 1, \ldots, n$.

We need to show that the curve $\gamma \circ c = (\gamma \circ c_1) * \cdots * (\gamma \circ c_n)$ is a geodesic from $\gamma(p)$ to $\gamma(q)$. We will proceed by induction. To start, $(\gamma \circ c_1)$ is a geodesic segment. Now suppose that $(\gamma \circ c_1) * \cdots * (\gamma \circ c_{j-1})$ is a geodesic segment, and $(\gamma \circ c_1) * \cdots * (\gamma \circ c_{j-1}) * (\gamma \circ c_j)$ is not a geodesic segment, then there exists a (unique) geodesic $c' : [0, s_j] \to (\Omega(\tau_4), \bar{g})$ such that $c'(0) = \gamma(p)$ and $c'(s_j) = \gamma(c(s_j))$, and furthermore the $\bar{g}$-length of $c'([0, s_j]) < s_j$. However, Γ is a subgroup of $\mathsf{PSL}(2, \mathbb{C})$, whose elements are conformal, therefore they preserve the angle. Now three geodesic segments $(\gamma \circ c_1) * \cdots * (\gamma \circ c_{j-1})([s_0, s_{j-1}])$, $\gamma \circ c_j([s_{j-1}, s_j])$ and $c'([0, s_j])$ would form a geodesic triangle whose sum of its inner angles is $\geq \pi$. This is a contradiction.

Therefore $\gamma \circ c = (\gamma \circ c_1) * \cdots * (\gamma \circ c_n)$ is a geodesic segment from $\gamma(p)$ to $\gamma(q)$, and then $d(\gamma(p), \gamma(q)) = L = d(p, q)$. □

As a corollary, we consider the resulting quotient manifold:

Corollary 4.3. *The manifold $M^3(\tau_4) = \Omega(\tau_4)/\Gamma$ can be equipped with a new metric induced from the covering space, still denoted by $\bar{g}$, such that $(M^3(\tau_4), \bar{g})$ is a compact negatively curved three-manifold with totally geodesic boundary components.*

By [**MSY82**, **HS88**, **Uhl83**], we have a closed incompressible least area minimal surface Σ contained in $(M^3(\tau_4), \bar{g})$, which is isotopic to S in $(M^3(\tau_4), \bar{g})$. It is oriented since we assumed S is nonseparating. We will prove that actually $\Sigma \subset (M^3(\tau_3), \bar{g})$, which means that Σ is a least area minimal surface with respect to the hyperbolic metric. We now make a special remark here on the submanifold $(M^3(\tau_3), \bar{g})$.

Remark 4.4. *By the definition of $f(t)$ and the definition of four constants (4.4), $\bar{g}$ in $M^3(\tau_3)$ is the hyperbolic metric. The submanifold $(M^3(\tau_3), \bar{g})$ is a compact hyperbolic three-manifold whose boundary components are concave with respect to the inward normal vectors.*

5. Proof of Theorem 1.1

Last section we constructed a submanifold $M^3(\tau_4) = \Omega(\tau_4)/\Gamma$ in any cusped hyperbolic three-manifold $M^3 = \mathbb{H}^3/\Gamma$, and a modified metric $\bar{g}$, such that $(M^3(\tau_4), \bar{g})$ is a compact negatively curved three-manifold with totally geodesic boundary components. We now have a closed incompressible least area minimal surface Σ in $(M^3(\tau_4), \bar{g})$. In this section, we prove Theorem 1.1 by showing that the minimal surface Σ is contained in $(M^3(\tau_3), g)$, a hyperbolic subregion of $(M^3(\tau_4), \bar{g})$.

5.1. **Minimal surface intersecting toric region.** As before, we assume that the oriented cusped hyperbolic three-manifold M^3 has k cusps, such that each maximal cusped region is parametrized by $\mathcal{C}_i = T_i^2 \times [0, \infty)$ for $i = 1, \dots, k$. Suppose $\rho : \pi_1(M^3) \to \mathsf{PSL}_2(\mathbb{C})$ is the holonomy so that $\Gamma = \rho(\pi_1(M^3))$.

Recall that there are four constants only depending on M^3: $\tau_j = \log(jL_0)$, for $j = 1, 2, 3, 4$. Assume that the embedded closed incompressible surface S is contained in $M^3(\tau_1)$. We consider a compact submanifold $M^3(\tau_4)$ of M^3, equipped with the new metric $\bar{g}$ (see Corollary 4.3), so that $(M^3(\tau_4), \bar{g})$ is a compact negatively curved three-manifold whose boundary components are all totally geodesic. By above arguments, we have a closed incompressible least area minimal surface Σ in $(M^3(\tau_4), \bar{g})$. We need to show the least area minimal surface Σ is contained in $M^3(\tau_3)$. If Σ does not intersect with $T_i^2 \times \{\tau_2\}$, then we are done (since it can not be contained entirely in the cusped region). Therefore we can just assume that $\Sigma \cap \left(T_i^2 \times [0, \tau_2]\right)$ is non-empty. We are interested in how it intersects with the region $T_i^2 \times [0, \tau_4]$:

Proposition 5.1. *Each component of the intersection $\Sigma \cap \left(T_i^2 \times [0, \tau_4]\right)$ is either a minimal disk whose boundary is a null-homotopic Jordan curve in $T_i^2 \times \{0\}$, or a minimal annulus whose boundary consists of essential Jordan curves in $T_i^2 \times \{0\}$.*

Proof. Let Σ' be a component of $\Sigma \cap \left(T_i^2 \times [0, \tau_4]\right)$. Since $(M^3(\tau_4), \bar{g})$ is a compact negatively curved three-manifold with totally geodesic boundary components, so the least area minimal surface Σ is disjoint from its boundary. Therefore the boundary of Σ' is contained in $T_i^2 \times \{0\}$. Since Σ is incompressible while T_i^2 is a torus, we have very few cases to consider:

(i) **Case I**: Σ' is a surface of negative Euler characteristic, i.e., it a surface (with or without boundary) of genus ≥ 2, or a torus with more than one boundary component, or a planar surface with more than 3 boundary components (topologically a sphere with more than 3 points removed, i.e., genus zero). Either case contradicts with the assumption that Σ is compressible.

(ii) **Case II**: Σ' is an annulus whose boundary consists of two Jordan curves which are null-homotopic in $T_i^2 \times \{0\}$. In this case, we may pick up a Jordan curve $\alpha \subset \Sigma'$ homotopic to either component of $\partial\Sigma'$, then α is null-homotopic in $T_i^2 \times [0, \tau_4] \subset M^3(\tau_4)$. Then there exists a minimal disk $D \subset \Sigma$ such that $\partial D = \alpha$, since Σ is incompressible and minimal in $M^3(\tau_4)$. By the argument in [**MY82**, pp.155–156], this minimal disk D itself must be contained in the compact three-manifold $T_i^2 \times [0, \tau_4]$. A contradiction.

Thus each component of $\Sigma \cap \left(T_i^2 \times [0, \tau_4]\right)$ is either a minimal disk whose boundary is a null-homotopic Jordan curve in $T_i^2 \times \{0\}$, or a minimal annulus whose boundary consists of two essential Jordan curves in $T_i^2 \times \{0\}$. □

5.2. **Good positioned Jordan curves on tori.** We start by making a definition of Jordan curves being in good position on a torus. This will be important for what follows.

Definition 5.2. *Let M^3 be a cusped hyperbolic three-manifold and $\mathcal{C} = T^2 \times [0, \infty)$ be a maximal cusped region of M^3. A Jordan curve (i.e., simple closed) $\alpha \subset T^2 \times \{\tau\}$ is said to be in "*good position*" if one of the lifts of α to $\mathbb{H}^3$ is contained in $A \times \{e^\tau\}$, where A is the fundamental domain of the parabolic group $\Gamma_\infty = \langle z \mapsto z + \mu,\ z \mapsto z + \nu \rangle$ in the horosphere $\{(x, y, 1) \mid (x, y) \in \mathbb{R}^2\}$.*

It's easy to see from the definition, we have the following:

Proposition 5.3. *A Jordan curve $\alpha \subset T^2 \times \{\tau\}$ is in good position if the Euclidean length of α is less than $\min\{2|\mu|, 2|\nu|, 2|\mu \pm \nu|\}$, while it is not in good position if the Euclidean length of α is at least $\min\{2|\mu|, 2|\nu|, 2|\mu \pm \nu|\}$. In particular, if $\alpha \subset T^2 \times \{\tau\}$ is an essential Jordan curve, then α is* not *in good position.*

Recall from (4.4) that we have 4 constants: $\tau_j = \log(j \cdot L_0)$ for $j = 1, 2, 3, 4$, where the constant L_0 is defined in (2.4). And these constants are ordered: $\tau_4 > \tau_3 > \tau_2 > \tau_1 > 0$. As in the previous subsection, we assume $\Sigma \cap \left(T_i^2 \times [0, \tau_2]\right)$ is non-empty. We first observe the following fact:

Proposition 5.4. *Let Σ' be a component of $\Sigma \cap \left(T_i^2 \times [0, \tau_4)\right)$. If there exists some $\tau \in [0, \tau_2]$, such that $\Sigma' \cap \left(T_i^2 \times \{\tau\}\right)$ consists of Jordan curves in good position, then each component of $\Sigma' \cap \left(T_i^2 \times \{\tau'\}\right)$ is also in good position for all $\tau' \in [\tau, \tau_2]$.*

Proof. By Theorem 4.2, we can lift $(M^3(\tau_4), \bar{g})$ to the truncated negatively curved space $(\Omega(\tau_4), \bar{g})$ such that $T_i^2 \times \{0\}$ is lifted to the horizontal plane passing through the point $(0, 0, 1)$. Suppose that the barycenter of the fundamental domain A_i of the parabolic group generated by $z \mapsto z + \mu_i$ and $z \mapsto z + \nu_i$ is the point $(0, 0, 1)$.

Suppose D is a component of $\Sigma' \cap \left(T_i^2 \times [\tau, \tau_4]\right)$ such that $\partial D \subset T_i^2 \times \{\tau\}$ is in good position, then by the arguments in Proposition 5.1, and

Proposition 5.3, D must be a disk and ∂D must be a null-homotopic Jordan curve in $T_i^2 \times \{\tau\}$. Let $\widetilde{D}$ be a lift of D such that $\partial \widetilde{D} \subset A_i \times \{e^\tau\}$.

We define the following:

$$\mathcal{B}_i = A_i \times [e^\tau, 4L_0] \ . \tag{5.1}$$

We want to show that $\widetilde{D}$ must be contained in $\mathcal{B}_i$. In fact, it is a minimal disk such that $\partial \widetilde{D} \subset A_i \times \{e^\tau\}$ is null-homotopic. Then we are left with very few cases:

(i) The minimal disk $\widetilde{D}$ doesn't have any subdisk below the horizontal plane through the point $(0, 0, e^\tau)$, since such a plane is convex with respect to the upward normal vectors (see the argument in [**MY82**, pp. 155–156]).

(ii) Since all vertical planes are totally geodesic (see Theorem 3.3 and Remark 3.4), the minimal disk $\widetilde{D}$ does not have any subdisk outside $\mathcal{B}_i$ by Hopf's maximum principle.

Thus $\widetilde{D}$ must be contained in the domain $\mathcal{B}_i$. This is certainly true for the other lifts of D which are given by $\gamma(\widetilde{D})$ for $\gamma \in \Gamma$. By definition, for $\tau' \geq \tau$, each component of $\Sigma' \cap \left(T_i^2 \times \{\tau'\}\right)$ is in good position. □

As a corollary, and taking advantage of Theorem 3.5 that we can use semi-spheres as barriers, we find

Corollary 5.5. *If there exists some $\tau \in [0, \tau_2]$, such that $\Sigma' \cap \left(T_i^2 \times \{\tau\}\right)$ consists of Jordan curves in good position, then Σ' is contained in $T_i^2 \times [0, \tau_3]$, i.e. Σ' is a least area disk or annulus with respect to the hyperbolic metric.*

Proof. Recall from (4.5) and (4.6), the modified metric is flat for $t > 4L_0$, and hyperbolic when $t < 3L_0$. For convenience, we denote two new constants: $L_3 = \sqrt{e^{2\tau} + (\frac{L_0}{2})^2}$ and $L_4 = \frac{\sqrt{65}}{2} L_0$. Since $\tau \leq \tau_2 = \log(2L_0)$, so we have

$$L_3 \leq \frac{\sqrt{17}}{2} L_0 < 3L_0 < 4L_0 < L_4. \tag{5.2}$$

Therefore $A_i \times \{L_4\}$ is totally geodesic with respect to the metric $\bar{g}$.

We consider the subregion $\mathcal{B}_i'$ of $\mathcal{B}_i$, which is defined by

$$\mathcal{B}_i' = \mathcal{B}_i \cap \left\{ \bigcup_{L_3 \leq r \leq L_4} S_+^2(r) \right\} \ .$$

by Theorem 3.5, the subregion $\mathcal{B}_i'$ is foliated by the non-concave spherical caps with respect to the inward normal vectors. By the definition of L_0 in (2.4), the spherical cap $\mathcal{B}_i \cap S_+^2(L_3)$ lies above $A_i \times \{e^\tau\}$.

Recall from the proof of Proposition 5.4 that D is a component of $\Sigma' \cap \left(T_i^2 \times [\tau, \tau_4]\right)$ such that $\partial D \subset T_i^2 \times \{\tau\}$ is in good position, and $\widetilde{D}$ be a lift of D such that $\partial\widetilde{D} \subset A_i \times \{e^\tau\}$. Therefore by the maximum principle, $\widetilde{D}$ is contained in $\mathcal{B}_i$ and below the spherical cap $\mathcal{B} \cap S^2_+(L_3)$. In other words, the Euclidean height of $\widetilde{D}$ is at most L_3.

By (5.2), we have $\widetilde{D} \subset A_i \times [e^\tau, 3L_0]$. This is true for other lifts of D which are given by $\gamma(\widetilde{D})$, for all $\gamma \in \Gamma$. Since the Kleinian group preserves the metric $\bar{g}$ (Theorem 4.2), we have $D \subset T^2_i \times [\tau, \tau_3]$, and therefore

$$\Sigma' \subset \big(T^2_i \times [0,\tau]\big) \cup \big(T^2_i \times [\tau,\tau_3]\big) = T^2_i \times [0,\tau_3]\ .$$

□

5.3. **Completing the proof.** First we need a version of the co-area formula modified from that in [**CG06**, p.399]. The proof of (5.3) in the following Lemma 5.6 can be found in [**Wan12**].

Lemma 5.6. *If M^3 is a Riemannian three-manifold with nonempty boundary ∂M^3, and F is a component of ∂M^3 such that its s-neighborhood $\mathscr{N}_s(F) \subset M^3$ is a trivial normal bundle over itself. If $\Sigma_1 \subset M^3$ is a surface such that $\Sigma_1 \cap \mathscr{N}_s(F) \neq \emptyset$, then*

$$\text{Area}(\Sigma_1 \cap \mathscr{N}_s(F)) = \int_0^s \int_{\Sigma_1 \cap \partial\mathscr{N}_\tau(F)} \frac{1}{\cos\theta}\, dl d\tau\ , \tag{5.3}$$

where the angle θ is defined as follows: For any point $q \in \Sigma_1$, set $\theta(q)$ to be the angle between the tangent space to Σ_1 at q, and the radial geodesic which is through q (emanating from q) and is perpendicular to F.

To complete the proof of the main theorem 1.1, we just need to find one $\tau \in [0,\tau_2]$ to satisfy the assumption in Proposition 5.4. And we show this τ may be chosen as just τ_2:

Theorem 5.7. *Let Σ' be a component of $\Sigma \cap \big(T^2_i \times [0,\tau_4)\big)$, then any component of $\Sigma' \cap \big(T^2_i \times \{\tau_2\}\big)$ is a Jordan curve in good position.*

Proof. Assume that Σ' is a component of $\Sigma \cap \big(T^2_i \times [0,\tau_4)\big)$ such that at least one component of $\Sigma' \cap \big(T^2_i \times \{\tau_2\}\big)$ is not in good position, then by Proposition 5.4, for each $\tau \in [0,\tau_2]$, $\Sigma' \cap \big(T^2_i \times \{\tau\}\big)$ has at least one component that is not in good position.

By Proposition 5.3, for all $\tau \in [0,\tau_2]$, we have:

$$\text{Length}\left(\Sigma' \cap \big(T^2_i \times \{\tau\}\big)\right) \geq \min\{2|\mu_i|, 2|\nu_i|, 2|\mu_i \pm \nu_i|\} e^{-\tau}\ . \tag{5.4}$$

To apply the co-area formula (5.3), we choose $F = T^2_i \times \{0\}$, and for $\tau \in [0,\tau_2]$, we set

$$\mathscr{N}_\tau(F) = \left\{p \in T^2_i \times [0,\tau_2]\ \middle|\ \text{dist}(p,F) \leq \tau\right\}\ , \tag{5.5}$$

where dist$(\cdot,\cdot)$ is the hyperbolic distance function. Now we apply the co-area formula (5.3) to find:

$$\begin{aligned}\text{Area}\left(\Sigma' \cap \left(T_i^2 \times [\tau_1,\tau_2]\right)\right) &= \int_{\tau_1}^{\tau_2}\int_{\Sigma'\cap\partial\mathscr{N}_\tau(F)} \frac{1}{\cos\theta}\, dl d\tau \\ &\geq \int_{\tau_1}^{\tau_2} \text{Length}(\Sigma'\cap\partial\mathscr{N}_\tau(F))\, d\tau \\ &\geq \int_{\tau_1}^{\tau_2} \min\{2|\mu_i|, 2|\nu_i, 2|\mu_i \pm \nu_i||\}\, e^{-\tau}\, d\tau \\ &= \frac{\min\{|\mu_i|, |\nu_i, |\mu_i \pm \nu_i|\}}{L_0} \\ &> \frac{\min\{|\mu_i|\cdot|\nu_i|, |\mu_i-\nu_i|\cdot|\mu_i+\nu_i|\}}{L_0^2} \\ &= \min\{|\mu_i|\cdot|\nu_i|, |\mu_i-\nu_i|\cdot|\mu_i+\nu_i|\}e^{-2\tau_1} \\ &\geq \text{Area}\left(T_i^2\times\{\tau_1\}\right) \ .\end{aligned}$$

Here we used the fact that $L_0 \geq |\mu_i| + |\nu_i|$ ((2.4)) and $\tau_j = \log(jL_0)$ for $j = 1, 2$.

By Proposition 5.1, Σ' is either a least area disk or a least area annulus, but by above inequality, we may use the cut-and-paste technique to find a minimal surface in the same isotopic class of Σ' (with the same boundary as that of Σ') of less area. This is a contradiction. Hence any component of $\Sigma' \cap \left(T_i^2 \times \{\tau_2\}\right)$ is a Jordan curve in good position, and then any component of $\Sigma \cap \left(T_i^2 \times \{\tau_2\}\right)$ is also in good position. □

We may now complete the proof:

Proof of Theorem 1.1. By Theorem 5.7, all components of $\Sigma \cap \left(T_i^2 \times \{\tau_2\}\right)$ are in good position, then by Corollary 5.5, each component of $\Sigma \cap (T_i^2 \times [0, \tau_4])$ is disjoint from $T_i^2 \times (\tau_3, \tau_4]$. Therefore we have

$$\Sigma \cap (T_i^2 \times [0,\tau_4]) \subset T_i^2 \times [0,\tau_3]\ , \quad \text{for } i = 1,\ldots,k\ ,$$

which implies that Σ is a minimal surface with respect to the hyperbolic metric. It is embedded and oriented since it is isotopic to a closed incompressible surface S which is embedded and non-separating. □

References

[Ada05] Colin Adams, *Hyperbolic knots*, Handbook of knot theory, Elsevier B. V., Amsterdam, 2005, pp. 1–18.

[And83] Michael T. Anderson, *Complete minimal hypersurfaces in hyperbolic n-manifolds*, Comment. Math. Helv. **58** (1983), no. 2, 264–290.

[BC15] Mark D. Baker and Daryl Cooper, *Finite-volume hyperbolic 3-manifolds contain immersed quasi-Fuchsian surfaces*, Algebr. Geom. Topol. **15** (2015), no. 2, 1199–1228.

[BH99] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999.

[CG06] Danny Calegari and David Gabai, *Shrinkwrapping and the taming of hyperbolic 3-manifolds*, J. Amer. Math. Soc. **19** (2006), no. 2, 385–446.

[CHMR14] P. Collin, L. Hauswirth, L. Mazet, and H. Rosenberg, *Minimal surfaces in finite volume non compact hyperbolic* 3*-manifolds*, preprint, arXiv:1405.1324 (2014).

[FHS83] Michael Freedman, Joel Hass, and Peter Scott, *Least area incompressible surfaces in* 3*-manifolds*, Invent. Math. **71** (1983), no. 3, 609–642.

[GHW10] Ren Guo, Zheng Huang, and Biao Wang, *Quasi-Fuchsian three-manifolds and metrics on Teichmüller space*, Asian J. Math. **14** (2010), no. 2, 243–256.

[HL12] Zheng Huang and Marcello Lucia, *Minimal immersions of closed surfaces in hyperbolic three-manifolds*, Geom. Dedicata **158** (2012), 397–411.

[HS88] Joel Hass and Peter Scott, *The existence of least area surfaces in* 3*-manifolds*, Trans. Amer. Math. Soc. **310** (1988), no. 1, 87–114.

[HW13] Zheng Huang and Biao Wang, *On almost-Fuchsian manifolds*, Trans. Amer. Math. Soc. **365** (2013), no. 9, 4679–4698.

[HW15] ———, *Counting minimal surfaces in quasi-fuchsian manifolds*, Trans. Amer. Math. Soc. **367** (2015), 6063–6083.

[KS07] Kirill Krasnov and Jean-Marc Schlenker, *Minimal surfaces and particles in 3-manifolds*, Geom. Dedicata **126** (2007), 187–254.

[Lóp13] Rafael López, *Constant mean curvature surfaces with boundary*, Springer Monographs in Mathematics, Springer, Heidelberg, 2013.

[Mar07] Albert Marden, *Outer circles*, Cambridge University Press, Cambridge, 2007, An introduction to hyperbolic 3-manifolds.

[Mee06] William H. Meeks, III, *Applications of minimal surfaces to the topology of three-manifolds*, Surveys in differential geometry. Vol. X, Surv. Differ. Geom., vol. 10, Int. Press, Somerville, MA, 2006, pp. 95–108.

[Mos73] G. D. Mostow, *Strong rigidity of locally symmetric spaces*, Princeton University Press, Princeton, N.J., 1973, Annals of Mathematics Studies, No. 78.

[MR92] William Menasco and Alan W. Reid, *Totally geodesic surfaces in hyperbolic link complements*, Topology '90 (Columbus, OH, 1990), Ohio State Univ. Math. Res. Inst. Publ., vol. 1, de Gruyter, Berlin, 1992, pp. 215–226.

[MSY82] William H. Meeks, III, Leon Simon, and Shing Tung Yau, *Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature*, Ann. of Math. (2) **116** (1982), no. 3, 621–659.

[MY82] William H. Meeks, III and Shing Tung Yau, *The existence of embedded minimal surfaces and the problem of uniqueness*, Math. Z. **179** (1982), no. 2, 151–168.

[MZ08] Joseph D. Masters and Xingru Zhang, *Closed quasi-Fuchsian surfaces in hyperbolic knot complements*, Geom. Topol. **12** (2008), no. 4, 2095–2171.

[Pra73] Gopal Prasad, *Strong rigidity of* $\mathbf{Q}$*-rank* 1 *lattices*, Invent. Math. **21** (1973), 255–286.

[Rub05] J. Hyam Rubinstein, *Minimal surfaces in geometric 3-manifolds*, Global theory of minimal surfaces, Clay Math. Proc., vol. 2, Amer. Math. Soc., Providence, RI, 2005, pp. 725–746.

[Rub07] J. H. Rubinstein, *Problems around 3-manifolds*, Workshop on Heegaard Splittings, Geom. Topol. Monogr., vol. 12, Geom. Topol. Publ., Coventry, 2007, pp. 285–298.

[San13] Andrew Sanders, *Domains of discontinuity for almost Fuchsian groups*, preprint, arXiv:1310.6412 (2013).

[SU82] Jonathan Sacks and Karen K. Uhlenbeck, *Minimal immersions of closed Riemann surfaces*, Trans. Amer. Math. Soc. **271** (1982), no. 2, 639–652.

[SY79] Richard Schoen and Shing Tung Yau, *Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature*, Ann. of Math. (2) **110** (1979), no. 1, 127–142.

[Thu80] William P. Thurston, *The geometry and topology of three-manifolds*, Princeton University, 1980, available at `http://library.msri.org/nonmsri/gt3m`.

[Thu82] ———, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), no. 3, 357–381.

[Uhl83] Karen K. Uhlenbeck, *Closed minimal surfaces in hyperbolic* 3*-manifolds*, Seminar on minimal submanifolds, Ann. of Math. Stud., vol. 103, Princeton Univ. Press, Princeton, NJ, 1983, pp. 147–168.

[Wan12] Biao Wang, *Minimal surfaces in quasi-Fuchsian 3-manifolds*, Math. Ann. **354** (2012), no. 3, 955–966.

[Zho99] Qing Zhou, *The moduli space of hyperbolic cone structures*, J. Differential Geom. **51** (1999), no. 3, 517–550.

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