λ-structures and s-structures: Translating the models

Gunter Fuchs
Westfälische Wilhelms-Universität Münster
http://wwwmath.uni-muenster.de/logik/fuchs

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Abstract

I develop a translation procedure between λ-structures, which correspond to premice in the Friedman-Jensen indexing convention on the one hand and s-structures, which are essentially the same as premice in the Mitchell-Steel indexing scheme.

1 Introduction

In the course of the nineteen-nineties, two approaches for constructing extender models became accepted as the most fruitful ones. They are known under the terms "s-indexing" and "λ-indexing". The former was created by William Mitchell and John Steel ([MS94]), the latter by Sy Friedman and Ronald Jensen ([Jen97]).

Both approaches aim at constructing fine structural models that approximate the set theoretic universe V very well, in the sense that as many large cardinals of V as possible remain large cardinals in the model. Since the isolation of the question about the existence of $0^\sharp$ as a crucial dichotomy, large cardinal axioms concerning the existence of elementary embeddings of (segments of) the universe were focal. These can be coded by extenders, which makes possible a formulation of these concepts within ZFC ([Mit79]). There are many different ways of coding such embeddings, and consequently, there are many ways of defining an extender. This is one respect in which the λ and the s approach differ.

These structures are of the form J^E_α where E codes a sequence of extenders. In s-indexing, the index α of an extender on the sequence is the cardinal successor of the support in the extender ultrapower of the structure cut back to α. In λ-indexing, α is the cardinal successor of the image of the critical point under the extender ultrapower embedding, again computed within the extender ultrapower. So the index of an extender in an s-indexed structure will be less than or equal to the index of the "corresponding" extender in a λ-indexed structure. What can happen is that certain extenders appearing on the sequence of a λ structure have no corresponding extender on the sequence of the corresponding s-structure. But those extenders will be coded by the extenders on the sequence of the corresponding s-indexed structure and appear after applying the right extenders of the s-structure.

Up to now I was mainly talking about the extender sequences, and it was indicated that there is a canonical way in which one can produce from a λ indexed sequence a corresponding one in s indexing and vice versa. If one doesn't demand of an extender sequence more than the right indexing, some form of an initial segment condition (which is needed to show that the comparison process terminates) and some level of coherence of the sequence, then this is not too hard to see and also not new (albeit unpublished, but see [Ste00, remark before def. 2.6]).

But I want a correspondence between the entire structures $\langle \mathrm{J}^E_\alpha, F\rangle$ that are referred to as premice both in the s- and the λ-approach. Taken literally, such a correspondence doesn't exist, but mainly for reasons of different tastes of the creators of the two theories - e.g., Mitchell-Steel demand that proper initial segments of premice be solid, while Jensen omits this requirement. Iterable premice are solid in both approaches, so that this difference vanishes if one looks at the structures of real interest. So in order for a significant correspondence to exist, I will define new structures. If possible, the requirements I make are weaker than the original ones, but still sufficient to get the theory going. I call these structures λ- and s-structures.

The main problem when analyzing the *structures* is a property that one could call *pre-soundness*: Proper segments of a λ- or s-structure have to be sound. So this property must be preserved by the translation function. In order to prove this, one has to analyze definability in the corresponding structures very closely; for instance, it is highly relevant which parameters are necessary for the definition of a $\mathbf{\Sigma}_1$ set (in Section 8.3 I deal solely with the problem that in the case that the heights of the structures considered are successor ordinals, an additional parameter is needed in the translation of a $\mathbf{\Sigma}_1$ formula).

The aim of being less restrictive when defining λ- and s-structures cannot be realized in three respects: Firstly, I demand that proper segments of λ-structures which are not of type III be not only sound but also 1-solid (i.e., solid above the first projectum) – without this additional assumption, it is not clear that the standard parameters of corresponding structures coincide, which is crucial for our proof that soundness is preserved. I demand the same of s-structures, and in this context, this is less than the usual requirement, namely full soundness and solidity.

Secondly, all active segments of s-structures have to be extendible. This means that applying the top extender to such a structure, truncated at the cardinal successor of its critical point, has to yield a well-founded model. In the original Mitchell-Steel approach, it is only demanded that the height of the structure $+1$ be contained in the well founded part of the ultrapower of the full structure. If the stronger requirement is violated, there obviously cannot be a corresponding λ structure, since these are long coherent well founded structures. So this is a natural requirement.[1] Since iterable premice in the sense of Mitchell-Steel (iterable even in the weakest possible sense) obviously have this property, this is harmless, since, after all, iterable premice are the creatures I care about most.

Thirdly, the λ-structures are equipped with an additional predicate which essentially makes it possible to define the corresponding s structure within the λ structure, which is used for translating $\mathbf{\Sigma}_1$ formulae. Section 3.3 deals with the fine structure of enhanced structures. Note that this change of the definition only refers to the λ side, as does the first point.

The choice of the form of the initial segment condition for the structures at hand is somewhat difficult. Both original approaches suffered initially of an erroneous formulation of this condition; see [SSZ02] for the problems in the s approach, and [Jen99, §I] for a corrected formulation with λ indexing. By now, there is a whole variety of different versions of the ISC, and for the current paper, I came up with yet another one. The reason for this will become clear in the second part of this paper, where it is shown that the translation functions map iterable structures to iterable structures, with respect to appropriate notions of iterability. The philosophy is that I tried to impose as few restrictions on the structures as possible.

I will also investigate classes of weaker structures (on both sides) between which there is a one-to-one correspondence. Roughly, a "p" in the beginning (which stands for "potential") means that instead of "*pre-soundness*" and "*pre-solidity*" it is only required that proper segments

[1] The property of hereditary extendibility has another advantage: It enables us to give a simplified treatment of iterations of s-structures: When forming an ultrapower of a type III structure I don't have to deal with the squash of the structure. Instead, I always pass over to the maximal extension. That these procedures yield the same result is shown in the follow-up article to this paper.

have a very good parameter (this guarantees that the structures are at least acceptable), and a "P" (for "Pseudo-") means that no ISC is required.

In section 8.2 it is shown that the translation functions manifest a correspondence between pPλ- and pPs-structures, as well as between pλ- and ps-structures. The basic ingredients for the proofs are developed in Section 6. In order to prove the corresponding result for Pλ and Ps, or finally for λ- and s-structures, a lot more work has to be done. The problems have to do with the additional parameters appearing in the translations of Σ_1 formulae. The main obstacle is to show that the standard parameters of corresponding structures (or, more precisely, their (pseudo) Σ_0-codes) coincide. This problem is solved in the rest of Section 8, and the desired result is finally proved in Section 8.7. In the second part of this paper, I show that iterable structures are mapped to iterable structures as well.

The reader should be familiar with Σ^* fine structure, which can be applied to arbitrary acceptable J-structures. In [MS94], a fine structure theory is used which is tailor-made for the structures studied there. But the fine structural notions do not coincide, in general, with the ones defined in the Σ^* approach.[2] The form of extender-ultrapowers chosen suits the Σ^*-fine structure best, namely the $*$-extender-ultrapowers ([Jen97],[Zem02],[Zem97]). In [MS94], so-called k-extender-ultrapowers are used, which could be imitated in the Σ^*-context but without any particular gain. Since the Mitchell-Steel-premice, as defined in [Ste00] (as opposed to the exhibition in [MS94]), are amenable, the Σ^* theory is applicable, so there is no reason not to apply it. It allows for a simpler and more uniform description of the process of a fine structural iteration.

A word on notation: The terminology used is quite standard and follows [Jec03] or [Kan94]. Small Greek letters denote ordinals, that is, members of the class On. The least upper bound of a set A of ordinals, $\operatorname{lub} A$, is the least ordinal that's strictly greater than all members of A. M and N are reserved for models, $|M|$ and $|N|$ are their universes, φ and ψ usually are formulae, V is the set theoretical universe and κ is always a cardinal, at least in a context-specific model. Functions are identified with their graphs, where here, the second component of a pair in the graph of f is in the domain, and the first component is the corresponding value from the range of f. If x is a set, then $\dot{x}$ is usually a predicate symbol which is interpreted by x (in a context-specific model, in which x may be a proper class). I use lists quite frequently. Thus, $\vec{x} = x_0, \ldots, x_{n-1}$ is an abbreviation, and doesn't really denote a mathematical object. E.g., $\langle\vec{x}\rangle = \langle x_0, \ldots, x_{n-1}\rangle$ is the ordered n-tuple. Sometimes I will just write $a \cap \vec{x}$ for the list $a \cap x_0, \ldots, a \cap x_{n-1}$, etc. I set: $\operatorname{lh}(\vec{x}) = n$. I should also define: A *J-structure* is an amenable model of the form $M = \langle \mathrm{J}_\alpha^{\vec{A}}, \vec{B}\rangle$ (i.e., $\vec{B} \subseteq |M|$). Note that $\mathrm{J}_\alpha^{\vec{A}} = \langle |\mathrm{J}_\alpha^{\vec{A}}|, \vec{A} \cap |\mathrm{J}_\alpha^{\vec{A}}|\rangle$. Hence, $M = \langle |\mathrm{J}_\alpha^{\vec{A}}|, \vec{A} \cap |\mathrm{J}_\alpha^{\vec{A}}|, \vec{B}\rangle$. I set: $\operatorname{ht}(M) = \alpha$.

I would like to point out that there is an index at the end of the paper, for the reader's convenience. Instead of describing how the paper is organized, I also added a table of contents.

This paper is based on a part of my PhD Thesis which I wrote under the supervision of Prof. Dr. Ronald Jensen. He stated the problem, advised me very well whenever I had questions, and made doubts about the project that I had from time to time disappear. I am very grateful for this.

[2]The structures studied in [MS94] are not amenable, hence the general fine structure theory is not applicable to them. Instead of analyzing $\boldsymbol{\Sigma}_1$ definability in these structures, the authors introduce the class of $r\Sigma_1$-formulae, and investigate Σ_1 definability. In a remark on p. 13, though, an alternative coding of the top extender is given which yields amenable structures. Σ_1 definability over these structures corresponds to rΣ_1-definability over the original structures; in [Ste00] the amenable coding is used. So the first projectum corresponds to the classical one. In the appendix to §2 (p.24ff) of [MS94] it is shown that the $n+1$st projectum in the sense of Mitchell-Steel coincides with the $n+1$st classical projectum if the structure is n-sound. Whenever in the Mitchell-Steel approach fine structural extender-ultrapowers are formed, then the amount of soundness demanded is sufficient to ensure that the projecta which are relevant for the ultrapower construction coincide with the classical ones. So the differences in the fine structure are not of high relevance.

Contents

2 Basics on extenders

In this section, I will fix notions and notations concerning extenders. I assume familiarity with the concept of an extender, though. As references, one can consult [Zem02, P. 47-56] (here, the focus is on extenders in the functional representation), [Kan94, 352-358] (for the hypermeasure representation), [MS94, §1] (hypermeasure representation in a fine structural context), [Mit79] (from here the concept originates).

2.1 Extenders

The next definition captures the characteristic quantities of extenders used in the Jensen approach to inner model theory.

Definition 2.1. Let F be an extender in the functional representation on M at $\langle\kappa,\gamma\rangle$. Then

(a) $\mathrm{crit}(F)=\kappa$.

(b) $\mathrm{lh}(F)=\gamma$.

(c) $\lambda(F)=\pi(\kappa)$, where $\pi : M \longrightarrow_F N$ is the Σ_0 extender embedding.[3]

(d) $\tau(F)=(\kappa^+)^M$.

I need notations for the switch from an extender in *hypermeasure* representation to the corresponding extender in the functional representation, and vice versa.

Definition 2.2. Let F be an extender in functional representation at $\langle\kappa,\gamma\rangle$, where γ is primitive recursively closed (p.r. closed, for short). Then the extender in hypermeasure representation derived from F is

$$F^{\mathfrak{h}} := \{\langle a,x\rangle \mid \exists n<\omega \quad a\in[\gamma]^n \wedge x\subseteq[\kappa]^n \wedge a\in F(x)\}.$$

I regard F as a function whose domain is contained in $\mathcal{P}(\kappa)$. Since γ was required to be p.r. closed, F can be canonically extended to a function whose domain is a subset of $\bigcup_{n<\omega}\mathcal{P}(\kappa^n)$, so that the above definition makes sense – see [Zem02, P. 48].

E is an extender in hypermeasure representation, if there is an extender F in functional representation s.t. $E=F^{\mathfrak{h}}$. For such an extender E, set:

$$\begin{aligned}\mathrm{crit}(E)&=\mathrm{crit}(F), & \mathrm{lh}(E)&=\mathrm{lh}(F),\\ \lambda(E)&=\lambda(F), & \tau(E)&=\tau(F),\\ (E)_a&:=\{x\subseteq[\mathrm{crit}(E)]^n \mid a\in F(x)\} & &\text{for } a\in[\mathrm{lh}(E)]^n.\end{aligned}$$

Finally, the extender in functional representation derived from E, $E^{\mathfrak{f}}$, is

$$E^{\mathfrak{f}}(x)=\{\alpha<\mathrm{lh}(E)\mid x\in(E)_\alpha\}.$$

Here, $x\in\mathrm{dom}(E):=\bigcup_{\alpha<\mathrm{lh}(E)}((E)_\alpha\cup\{\mathrm{crit}(E)\setminus y\mid y\in(E)_\alpha\})$. I identified $[\kappa]^1$ with κ here.

Definition 2.3. Let F be an extender in functional representation and α an ordinal. Then $F|\alpha$, the *truncation of* F *to* α, is the function with domain $\mathrm{dom}(F)$, defined by

$$(F|\alpha)(x)=F(x)\cap\alpha.$$

If F is given in hypermeasure representation, then set:

$$F{\restriction}\alpha := \{\langle a,x\rangle \mid \langle a,x\rangle\in F\wedge a\subseteq\alpha\}.$$

Definition 2.4. Let F be an extender on M at $\langle\kappa,\gamma\rangle$. Let $\pi : M \longrightarrow_F N$ be the extender-embedding. An ordinal $\delta\in N$ is a *generator of* F iff there is no function $f\in M$ with $f:\kappa^n\longrightarrow\kappa$ and there are no ordinals $\alpha_1,\ldots,\alpha_n<\delta$ such that $\pi(f)(\vec{\alpha})=\delta$. I denote the set of generators of F by gen_F, and define the *support* of F to be

$$s(F):=\mathrm{lub}(\tau(F)\cup\mathrm{gen}_F);$$

this quantity is also called the *natural length of* F. Further, let $s^+(F):=(s(F)^+)^N$; so in the terminology of Mitchell-Steel, $s^+(F)$ is the length of the trivial completion of F.

[3]For extenders appearing in weak j-pre-premice (see Definition 3.6), this is equivalent to the definition $\lambda(F)=F(\kappa)$, since these extenders are *whole* (in the sense of [Jen97, Chapter 1, p. 14]).

2.2 Extender Ultrapowers

Again, the following definition's main purpose is the introduction of some notation. For an introduction to the formation of extender ultrapowers, the reader may consult the references given above.

Definition 2.5. Let F be an extender at $\langle\kappa,\lambda\rangle$ on M. The the Σ_0-extender ultrapower of M by F is defined as follows. Set:

$$\begin{aligned}\Gamma(M,\kappa) &:= |M|\cap(\bigcup_{n\in\omega}{}^{\kappa^n}|M|),\\ D(M,\kappa,\lambda) &:= \{\langle\vec{\alpha},f\rangle \mid f\in\Gamma(M,\kappa)\wedge\exists n<\omega \quad \mathrm{dom}(f)=\kappa^n\wedge\vec{\alpha}\in\lambda^n\}.\end{aligned}$$

If F is given in functional representation, an equivalence relation $\simeq_0$ on $D(M,\kappa,\lambda)$ is defined by:

$$\langle\vec{\alpha},f\rangle\simeq_0\langle\vec{\beta},g\rangle \iff \prec\vec{\alpha},\vec{\beta}\succ\in F(\{\prec\vec{\gamma},\vec{\delta}\succ<\kappa\mid f(\vec{\gamma})=g(\vec{\delta})\});$$

it should be clear how one proceeds if one uses extenders in hypermeasure representation. Denote the equivalence class of $\langle\vec{\alpha},f\rangle$ as $[\vec{\alpha},f]$. Then let

$$I\!\!D(M,F):=\{[\vec{\alpha},f]\mid\langle\vec{\alpha},f\rangle\in D(M,\kappa,\lambda)\}.$$

Further, define binary relations I and E on $I\!\!D(M,F)$ by setting:

$$\begin{aligned}[\vec{\alpha},f]I[\vec{\beta},g] &\iff \langle\vec{\alpha},f\rangle\simeq_0\langle\vec{\beta},g\rangle,\\ [\vec{\alpha},f]E[\vec{\beta},g] &\iff \prec\vec{\alpha},\vec{\beta}\succ\in F(\{\prec\vec{\gamma},\vec{\delta}\succ<\kappa\mid f(\vec{\gamma})\in g(\vec{\delta})\}).\end{aligned}$$

Then $\mathrm{Ult}(M,F)$ is isomorphic to $\langle I\!\!D(M,F),I,E\rangle$, and its well founded part is transitive. Hence, $\mathrm{Ult}(M,F)$ is uniquely determined if $\langle I\!\!D(M,F),I,E\rangle$ is well founded. I write $\pi:M\longrightarrow_F N$ to express that $N=\mathrm{Ult}(M,F)$ and π is the extender embedding.

The construction of the $*$-extender ultrapower is completely analogous. For details, the reader is referred to [Zem02, Chapter 3] and [Jen97, §2].

Definition 2.6. Let F be an extender at $\langle\kappa,\lambda\rangle$ on the J-structure M. Then the $*$-extender ultrapower of M by F (or the *fine structural* extender ultrapower) is defined as follows. Let $\Gamma^*(M,\kappa)$ to be the set of functions f from κ^m to $|M|$ (for some $m<\omega$), so that either $f\in|M|$, or f is a good $\mathbf{\Sigma}_1^{(n)}(M)$ function, for an $n<\omega$ with $\omega p_M^{n+1}>\kappa$. Now $D^*(M,\kappa,\lambda)$, $\simeq^*$, $I\!\!D^*(M,F)$ are defined like $D(M,\kappa,\lambda)$, $\simeq$, $I\!\!D(M,F)$, respectively, where $\Gamma(M,\kappa)$ has to be replaced by $\Gamma^*(M,\kappa)$ always. I write $\mathrm{Ult}^*(M,F)$ for the $*$-extender ultrapower. The notation $\pi:M\longrightarrow_F^* N$ then says: $N=\mathrm{Ult}^*(M,F)$, and π is the corresponding embedding.

3 The structures

3.1 Extender structures

Definition 3.1. A model $M=\langle \mathrm{J}_\alpha^A,\vec{B},F\rangle$ is an *extender structure* iff $\langle \mathrm{J}_\alpha^A,\vec{B}\rangle$ is acceptable and amenable, and either F is a pre-extender [4] in functional or hypermeasure representation on M so that $(\mathrm{crit}(F)^+)^M$ exists, or if $F=\emptyset$. If F is a pre-extender, M is *active*, otherwise it is *passive*. In the active case, F is called the *top extender* of M (in short: E_{top}^M). Let $\kappa=\mathrm{crit}(F)$ and $\tau=(\kappa^+)^M$. Further, for $\xi\in[\tau,s(F)]$, define $\pi_\xi=\pi_\xi^M$ and $[M]_\xi$, as follows:

[4] A pre-extender F satisfies all requirements an extender has to fulfill, except that the extender product of M by F doesn't have to be well founded. But $\mathrm{lh}(F)$ has to be contained in the well founded part of the extender product. See [MS94, §1].

- $\pi_\xi : \langle \mathrm{J}^A_\tau, \overrightarrow{B \cap |\mathrm{J}^A_\tau|}\rangle \longrightarrow_{F|\xi} M'$,
- $[M]_\xi := \langle M', \pi_\xi \restriction \mathcal{P}(\kappa)\rangle$,

if this structure is well founded. Otherwise, $[M]_\xi$ is undefined. For $\xi, \zeta \in [\tau, s(F)]$ with $\xi < \zeta$, so that $[M]_\xi$ and $[M]_\zeta$ are defined, there is a canonical embedding

$$\sigma_{\xi,\zeta} = \sigma^M_{\xi,\zeta} : [M]_\xi \longrightarrow [M]_\zeta, \text{ defined by } \sigma_{\xi,\zeta}(\pi_\xi(f)(\vec{\alpha})) = \pi_\zeta(f)(\vec{\alpha}),$$

for $\langle \vec{\alpha}, f\rangle \in D(\mathrm{J}^{E^M}_\tau, \kappa, \xi)$. Now the *maximal continuation* $\widehat{M}$ of M is:

$$\widehat{M} := \begin{cases} M & \text{if } M \text{ is passive,} \\ [M]_{s(F)} & \text{if this structure is defined,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

M is called *continuable*, if $\widehat{M}$ is defined. If M is active and continuable, and $\widehat{M} = \langle M', F'\rangle$, then F' is the *maximal continuation* of E^M_{top}. I write $\widehat{E^M_{\mathrm{top}}}$ for that extender. I also write M^{passive} for the structure $\langle \mathrm{J}^A_\alpha, \vec{B}, \emptyset\rangle$. Finally, given an active extender structure M, set:

$$\begin{array}{ll} \lambda(M) = \lambda(E^M_{\mathrm{top}}), & s(M) = s(E^M_{\mathrm{top}}), \\ \tau(M) = \tau(E^M_{\mathrm{top}}), & \kappa(M) = \mathrm{crit}(E^M_{\mathrm{top}}), \end{array}$$

and if M is continuable, then let $s^+(M) = (s(M)^+)^{\widehat{M}}$, where, as usual, this is the height of $\widehat{M}$ if there is no cardinal above $s(M)$ in $\widehat{M}$. Call $s(M)$ the *natural length* of M.

Remark 3.2. It follows from an observation of Sy Friedman that $[M]_\xi$ is always amenable; see the proof of [Jen97, §1, Lemma 4].

Definition 3.3. Let $M = \langle \mathrm{J}^A_\alpha, \vec{B}, F\rangle$ be an active extender structure, where F is given in hypermeasure representation. Then the amenable coding $F^c = F^c_{M^{\mathrm{passive}}}$ of F is defined to be the set of quadruples $\langle \gamma, \xi, a, x\rangle \in |M|$ with the following properties:

1. $\gamma > s(F)$.
2. $\mathrm{crit}(F) < \xi < \mathrm{crit}(F)^{+M}$.
3. $F \cap ([s]^{<\omega} \times \mathrm{J}^{E^M}_\xi) \in \mathrm{J}^{E^M}_\gamma$.
4. $\langle a, x\rangle \in F \cap ([\gamma]^{<\omega} \times \mathrm{J}^{E^M}_\xi)$.

I also define $(\emptyset)^c := \emptyset$.

3.2 pPs-structures

Definition 3.4. Let $E = \langle E_\beta \mid \beta \leq \omega\alpha\rangle$ be a sequence s.t. for $\beta \leq \omega\alpha$ either $E_\gamma = \emptyset$, or E_γ is a pre-extender in hypermeasure representation. Set:

$$A = A_E := \{\langle \beta, z\rangle \mid z \in E_\beta\}.$$

If M is a structure of the form $\langle \mathrm{J}^{A_E}_\delta, \vec{B}\rangle$, then let $E^M := E \restriction \delta$.

N is a *potential Pseudo-s-structure (pPs-structure)*, if the following conditions are satisfied, for a suitable sequence as above:

1. For $\gamma \leq \alpha$ let $N||\gamma := \langle \mathrm{J}_\gamma^{A_E\restriction\gamma}, (E_{\omega\gamma})^c_{\mathrm{J}_\gamma^{A_E\restriction\gamma}} \rangle$ be the truncation of N to γ. Then $N = N||\alpha$. Moreover, the structure $\widetilde{N||\gamma} := \langle \mathrm{J}_\gamma^{A_E\restriction\gamma}, E_{\omega\gamma} \rangle$ is a continuable extender structure. I will write $[N||\gamma]_\delta$, $\pi_\delta^{N||\gamma}$, $\sigma_{\delta,\mu}^{N||\gamma}$, $\widehat{N||\gamma}$ to denote the corresponding objects defined with respect to $\widetilde{N||\gamma}$. Using this convention, I can define: $\lambda(\gamma)^N = \lambda(N||\gamma)$, $\tau(\gamma)^N = \tau(N||\gamma)$, etc. If $E_\gamma \neq \emptyset$, then $\gamma = \omega\gamma$, and $E\restriction\gamma = E^{\mathrm{Ult}(N||\gamma, E_\gamma)}\restriction(\gamma+1)$.

2. If $N||\gamma$ is active, then E_γ is a pre-extender of length γ in hypermeasure representation, and $\gamma = (s(E_\gamma)^+)^{\widehat{N||\gamma}}$.

3. For $\gamma < \alpha$, $R^*_{N||\gamma} \neq \emptyset$, if $N||\gamma$ is acceptable.

Remark 3.5. $N||\gamma$ is amenable, for all $\gamma \leq \omega\alpha$, as shown in [MS94, p. 13, Remark] or [Ste00, p. 13-14].

From 3. it follows by induction on $\gamma \leq \alpha$, that $N||\gamma$ is acceptable. The proof that L is acceptable and sound (see [Zem02, Lemma 1.10.1-2]) can be used for this – soundness is more than needed as induction hypothesis for the proof to go through. Since the general fine structure theory presupposes acceptability of the structures, so does the notion of a very good parameter in 3.

3.3 Enhancement Functions

It turned out that the desired correspondence between the Friedman-Jensen and the Mitchell-Steel style premice does not hold literally. The Friedman-Jensen type premice have to be enhanced by an additional predicate. So one has to pass to an expansion of these structures. Such an expansion alters in general the whole definability analysis of the model, in particular, the projecta, the reducts, hence the entire fine structure. In this section, a general criterion for when such an enhancement yields fine structural structures is developed, before defining the concrete enhancement to work with in the next section.

Definition 3.6. Let $\bar{E} = \langle E_\gamma \mid \gamma \leq \omega\alpha \rangle$ be a sequence s.t. for $\gamma \leq \omega\alpha$ either E_γ is an extender in functional representation, or $E_\gamma = \emptyset$. If E_γ is an extender, then let $\gamma = \omega\gamma$. Set:

$$E = \{\langle \nu, \xi, X \rangle \mid \xi \leq \nu \leq \omega\alpha \wedge \xi \in E_\nu(X)\}.$$

Then $M = \langle \mathrm{J}_\alpha^E, E_{\omega\alpha} \rangle$ is a weak Jensen-pre-premouse (weak j-ppm), if the following conditions are satisfied:

(a) For $\gamma \leq \alpha$ let $M||\gamma := \langle \mathrm{J}_\gamma^E, E_{\omega\gamma} \rangle$ be the truncation of M to γ. Then $M||\gamma$ is a continuable extender structure, and $M||\gamma = \widehat{M||\gamma}$.

(b) For $\gamma < \alpha$, $R^*_{M||\gamma} \neq \emptyset$, if $M||\gamma$ is acceptable.[5]

Set $E^M = \bar{E}\restriction\mathrm{On}_M$. I also write $s(\gamma)^M$, $s^+(\gamma)^M$, etc. for $s(M||\gamma)$, $s^+(M||\gamma)$, etc., if $M||\gamma$ is active.

Lemma 3.7. *Let M be an active, weak j-ppm. Then $|M| = h^1_M(s(M))$, in particular $\omega\rho^1_M \leq s(M)$. Moreover, if $\mu < \nu \leq \mathrm{ht}(M)$, then $s^+(\mu)^M \neq s^+(\nu)^M$.*

[5]See the remark concerning item 3 in definition 3.4.

Proof. Using the top extender of M, it is possible to define without parameters a Σ_1-surjection from $D(M, \kappa(M), s(M)) \subseteq |M||s(M)|$ onto $|M|$: $\langle \vec{\alpha}, f \rangle \mapsto \pi_s^M(f)(\vec{\alpha})$. Since this result is well known, I omit the exact analysis of the complexity of this definition – see [Jen97, §1, S. 13]. In order to see the second part of the claim, note that it follows that $(s(\mu)^M, \mu] \cap \mathrm{Card}^{M||\nu} = \emptyset$. But $s^+(\nu)^M \in \mathrm{Card}^{M||\nu}$. So $s^+(\nu)^M$ is either $\leq s(\mu)^M$, or $> \mu$, while $s^+(\mu)^M \in (s(\mu)^M, \mu]$. $\square$

What has to be done in the following is not just to look at an expansion of one particular model. Instead, every weak j-ppm has to be assigned its additional predicate. I will in the following describe a class of functions which I call enhancement functions. It is rich enough for my purposes, and contains only functions that behave nicely. What matters is that the functions in question "commute with the formation of fine structural extender ultrapowers".

Definition 3.8. An enhancement is a function of the form $\mathfrak{A} = \langle A_M \mid M \text{ is a weak j-ppm} \rangle$ with the following properties:

(a) (*Closure*) $A_M \subseteq \{\lambda \in M \mid \lambda \text{ is a limit ordinal } \vee \lambda = 0\}$ and A_M is closed in On_M.

(b) (Π_1-*definability*) Let $M = \langle \mathrm{J}_\alpha^E, E_{\omega\alpha} \rangle$. Then A_M is uniformly $\Pi_1(\mathrm{J}_\alpha^E)$.

(c) (*Coherency*) If $\omega\beta \in A_M$, then $A_M \cap \omega\beta = A_{M||\beta}$.

Let $M = \langle \mathrm{J}_\alpha^E, E_{\omega\alpha} \rangle$ be a weak j-ppm. The enhancement associated to M via $\mathfrak{A}$ is the structure $M^* = M^*_{\mathfrak{A}} := \langle \mathrm{J}_\alpha^E, E_{\omega\alpha}, A_M \rangle$.

In the following, fix an enhancement $\mathfrak{A}$.

Lemma 3.9. *Let M be a weak j-ppm. Then $M^*_{\mathfrak{A}}$ is amenable.*

Proof. Let $u \in M$. It must be shown that $A_M \cap u \in M$. Let $\beta = \sup\{\nu \mid \omega\nu \in A_M \cap u\}$. Then $A_M \cap u = A_{M||\beta} \cap u \in M$, since $\omega\beta \in A_M$ (by (a)), and since A_M is $\Pi_1(M||\beta)$ by (b) – the first identity follows from (c). $\square$

Lemma 3.10. *Let $\pi : \langle M, A_M \rangle \longrightarrow_F \langle N, A \rangle$, N transitive. Then $A = A_N$.*

Proof. (1) $A \subseteq A_N$.
Proof of (1). This is because π, being an extender-ultrapower embedding, is Σ_1-preserving. More precisely, let φ be the uniform $\Pi_1(M)-$definition of A, and let $\dot{A}$ be a unary predicate symbol, which is interpreted in M^* as A_M. Then:

$$M^* \models \forall x \quad (\dot{A}(x) \longrightarrow \varphi(x)).$$

This statement is $\Pi_1(M^*)$. So it is preserved by π, and hence it holds in $N^* := \langle N, A \rangle$. Since φ is a definition of $\mathfrak{A}$ uniform for weak j-ppm, this means that $A \subseteq A_N$. $\square_{(1)}$

(2) *Let $\alpha \in A_M$. Then $A \cap \pi(\alpha) = A_N \cap \pi(\alpha)$.*

Proof of (2). This is an immediate consequence of part (c) of definition 3.8. Letting $\alpha = \omega\bar{\alpha}$,

$$A \cap \pi(\alpha) = \pi(A_M \cap \omega\bar{\alpha}) = \pi(A_{M||\bar{\alpha}}) = A_{N||\pi(\bar{\alpha})} = A_N \cap \pi(\alpha)$$

by (1), as $\pi(\alpha) \in A \subseteq A_N$. $\square_{(2)}$

If A_M is cofinal in On_M, then A is cofinal in On_N, by (2), since π is cofinal. So in this case, the lemma is clear. So let A_M be bounded in On_M. By Definition 3.8, part (a), I can set: $\alpha = \max A_M$. The statement "$\alpha = \max \dot{A}$" is $\Pi_1(M^*)$, so this formula holds in N of $\pi(\alpha)$. Because $\dot{A}^N = A$, this shows that $\pi(\alpha) = \max A$. Using (1) and (2), it follows that

$$\pi(A_M \cap (\alpha + 1)) = A_N \cap (\pi(\alpha) + 1).$$

Moreover, $\pi(A_M \cap (\alpha + 1)) = \pi(A_M) = A$. So $A = A_N \cap (\pi(\alpha) + 1)$. Thus, it suffices to show:

(3) $A_N \subseteq \pi(\alpha) + 1$.

Proof of (3). Assume the contrary. Let β be minimal s.t. $\pi(\alpha) < \omega\beta \in A_N$. Set:

$$Z := \{\xi \mid \pi(\xi) \leq \beta\} \text{ and } \bar{\beta} := \sup Z.$$

(3.1) $\pi(\bar{\beta}) > \beta$.

Proof of (3.1). First, I am going to show that $\pi(\bar{\beta}) \neq \beta$. To see this, assume that $\pi(\bar{\beta}) = \beta$. Then $\pi(\omega\bar{\beta}) = \omega\beta \in A_N$, i.e. $N \models \varphi[\omega\beta]$, and this implies by the preservation properties of π that $M \models \varphi[\omega\bar{\beta}]$, hence $\omega\bar{\beta} \in A_M$. But then $\omega\bar{\beta} \leq \alpha$, i.e. $\omega\beta = \pi(\omega\bar{\beta}) \leq \pi(\alpha)$, contradicting our choice of β.

Now assume that $\pi(\bar{\beta}) \leq \beta$, hence, by the above, $\pi(\bar{\beta}) < \beta$. Then $\pi(\bar{\beta}+1) = \pi(\bar{\beta}) + 1 \leq \beta$, so $\bar{\beta}+1 \in Z$. But $\bar{\beta} = \sup Z$, hence $\bar{\beta}+1 \leq \bar{\beta}$, a contradiction. $\square_{(3.1)}$

(3.2) $A_N \cap \omega\pi(\bar{\beta}) \subseteq A_{N||\pi(\bar{\beta})}$.

Proof of (3.2). This is trivial, as φ is a uniform Π_1-definition of A which doesn't use the top extender predicate. Hence, Π_1−reflection can be applied. $\square_{(3.2)}$

Now, $\omega\beta < \omega\pi(\bar{\beta})$ and $\omega\beta \in A_N$. So, $\omega\beta \in A_N \cap \omega\pi(\bar{\beta}) \subseteq A_{N||\pi(\bar{\beta})}$, by (3.2). Since $\pi(\alpha) < \omega\beta \in A_{N||\pi(\bar{\beta})}$ it follows that

$$N \models (\exists\gamma > \pi(\alpha) \quad \varphi(\gamma))_{\pi(\mathrm{J}^{E^M}_{\bar{\beta}})}.$$

This formula is Σ_0 in $\pi(\alpha)$ and $\pi(\mathrm{J}^{E^M}_{\bar{\beta}})$. Hence, it follows that

$$M \models (\exists\gamma > \alpha \quad \varphi(\gamma))_{\mathrm{J}^{E^M}_{\bar{\beta}}}.$$

So let $\alpha < \gamma \in A_{M||\bar{\beta}}$. It then follows that $\pi(\gamma) \in A_{N||\pi(\bar{\beta})}$, and that $\pi(\alpha) < \pi(\gamma)$. Since $\mathrm{J}^{E^N}_{\beta} \subseteq \mathrm{J}^{E^N}_{\pi(\bar{\beta})}$, I can again apply Π_1 reflection to see that $\omega\beta \cap A_{N||\pi(\bar{\beta})} \subseteq A_{N||\beta}$. But since $\omega\beta \in A_N$, it follows that $A_{N||\beta} = A_N \cap \omega\beta$. Now, $\gamma \in A_{M||\bar{\beta}}$. So γ is a limit ordinal. Let $\gamma = \omega\gamma'$. Then $\gamma' < \bar{\beta}$. So $\pi(\gamma') \leq \beta$, because $\bar{\beta} = \sup Z$. But $\pi(\gamma') \neq \beta$, since $\omega\gamma' = \gamma > \alpha$, so that $\omega\gamma' \notin A_M$, but $\omega\beta \in A_N$. Hence, it follows that $\pi(\gamma') < \beta$, and hence, $\pi(\gamma) = \pi(\omega\gamma') < \omega\beta$.

So we get: $\pi(\alpha) < \pi(\gamma) < \omega\beta \cap A_{N||\pi(\bar{\beta})} \subseteq A_{N||\beta} = \omega\beta \cap A_N$. But this contradicts the minimality of $\omega\beta$ with the property that $\pi(\alpha) < \omega\beta \in A_N$, because $\pi(\gamma)$ has this property too, and $\pi(\gamma) < \omega\beta$. $\square$

Note that the proof used only that π is cofinal and Σ_0-preserving.

Lemma 3.11. *Let M be a weak j-ppm, and let $\pi : \langle M, A_M\rangle \longrightarrow^*_F \langle N, A\rangle$ or $\pi : \langle M, A_M\rangle \longrightarrow_F \langle N, A\rangle$. Let N be transitive. Then $A = A_N$.*

Proof. If π is a Σ_0-extender embedding, then lemma 3.10 gives the claim. Otherwise, π is even Σ_2-preserving, because $\omega\rho^1_M > \mathrm{crit}(F)$ (now [Zem02, Lemma 3.1.11] can be applied). So the claim is a consequence of the uniform Π_1 definability of A_M. $\square$

Lemma 3.12. *Let N be a weak j-ppm. Let $\pi : \langle M, A\rangle \longrightarrow_{\Sigma_1} \langle N, A_N\rangle$, M transitive. Then M is a weak j-ppm, and $A = A_M$.*

Proof. It suffices to show that $A = A_M$; that M is a weak j-ppm is well known. For this, two directions have to be shown: If $a \in A$, then $\pi(a) \in A_N$, i.e., $N \models \varphi[a]$, where φ is a uniform Π_1 definition of A_N. Hence, it follows that $M \models \varphi[a]$, and this means by uniformity of the definition that $a \in A_M$. Vice versa, if $a \in A_M$, then $M \models \varphi[a]$, hence $N \models \varphi[\pi(a)]$. By uniformity of φ it follows that $\pi(a) \in A_N$, i.e.: $\langle N, A_N\rangle \models \dot{A}(\pi(a))$, where $\dot{A}$ is a symbol for A_N. Since π is an embedding, $\langle M, A\rangle \models \dot{A}(a)$, and this means $a \in A$. $\square$

3.4 pPλ-structures

In this section I will describe the enhancement function which will yield the expanded weak j-ppm corresponding to s-structures. I will also derive some of its basic properties.

Definition 3.13. Let M be a weak j-ppm. Then let D_M be the set defined by:

$$D_M := \{\tau \in M \mid (\mathrm{Lim}(\tau) \vee \tau = 0) \wedge \neg(\exists \nu \in M \quad E^M_{\omega\nu} \neq \emptyset \wedge s^+(\nu)^M < \tau \leq \nu)\}.$$

For $\nu, \gamma \leq \mathrm{ht}(M)$, say that ν *hides* γ *in* M iff $M||\nu$ is active and $s^+(\nu)^M < \gamma \leq \nu$. So D_M consists of 0 and those limit ordinals of M that are not hidden by any $\nu < \mathrm{ht}(M)$.

Lemma 3.14. *The function* $\langle D_M \mid M \text{ is a weak j-ppm} \rangle$ *is an enhancement.*

Proof. Letting M be a weak j-ppm, let's verify properties (a)-(c) of Definition 3.8:
Closure: Let $\tau \in M$ be a limit point of D_M. Assume τ not to be an element of D_M. Then let $\nu \in M$ have the property that $s^+(\nu)^M < \tau \leq \nu$. As τ is a limit point of D_M, we can choose $\bar{\tau}$ in such a way that $s^+(\nu)^M < \bar{\tau} \in D_M \cap \tau$. But then $s^+(\nu)^M < \bar{\tau} < \nu$, hence $\bar{\tau} \notin D_M$, a contradiction.
Π_1-*definability:* By definition, τ belongs to D_M iff the following formula holds in M:

$$(\mathrm{Lim}(\tau) \vee \tau = 0) \wedge \forall \nu \; \forall x (x = \langle \mathrm{J}^{E^M}_\nu, E^M_\nu \rangle \text{ is active } \rightarrow ((\tau \leq s^+(\nu))^x \vee \tau \notin x)).$$

Since „$x = \langle \mathrm{J}^{E^M}_\nu, E^M_\nu \rangle$" is $\Sigma_1(M)$, this definition is clearly Π_1.
Coherency: Let $\omega\beta \in D_M$. It has to be shown that $\omega\beta \cap D_M = D_{M||\beta}$. The direction from left to right is trivially satisfied: One just has to apply Π_1-reflection, since the uniform Π_1-definition of D_M makes no use of the top extender predicate. Suppose the other direction of this inclusion fails. Pick $\tau \in D_{M||\beta} \setminus D_M$. Since $\tau \notin D_M$, ν can now be chosen so that $s^+(\nu)^M < \tau \leq \nu$. It follows that $\nu \geq \omega\beta$, because otherwise we would get that $s^+(\nu)^M = s^+(\nu)^{M||\beta}$, which implies that τ is not an element of $D_{M||\beta}$. But then $s^+(\nu)^M < \omega\beta \leq \nu$, so $\omega\beta \notin D_M$, a contradiction. □

Definition 3.15. A *potential Pseudo-λ-structure* (pPλ-structure) is a structure of the form $\langle M, D_M \rangle$, where M is a weak j-ppm, and for every $\alpha < \mathrm{ht}(M)$, $R^*_{\langle M||\gamma, D_{M||\gamma}\rangle} \neq \emptyset$. For a pP$\lambda$-structure $P = \langle M, D_M \rangle$ let $P^- := M$, and for $\alpha < \mathrm{ht}(M)$ let $P||\alpha = \langle M||\alpha, D_{M||\alpha}\rangle$. Finally, I use the notations $\mathrm{ht}(P)$, E^P_{top}, $\lambda(P)$, etc. for $\mathrm{ht}(P^-)$, $E^{P^-}_{\mathrm{top}}$, $\lambda(P^-)$, etc. In connection with pPλ-structures I will use a language with an additional predicate symbol $\dot{D}$, and interpret it by $\dot{D}^P = D_P$.

Lemma 3.16. *Let M be a pPλ-structure, $\alpha \in \mathrm{On}_M$ a limit ordinal and $\alpha \notin D_M$. Then there is a maximal $\nu \in M$ such that $s^+(\nu)^M < \alpha \leq \nu$. This ν has the additional property that $\nu + \omega \in D_M$, if $\nu + \omega \in M$.*

Proof. By definition of D_M there is a $\nu \in M$ which hides α. To see that there is a maximal ν with this property, note that if $\nu < \nu'$ and both hide α, it follows that

$$s^+(\nu')^M < s^+(\nu)^M,$$

since $s^+(\nu')^M \in \alpha \cap \mathrm{Card}^{M||\nu'}$ and $(s(\nu)^M, \nu] \cap \mathrm{Card}^{M||\nu'} = \emptyset$, because $\omega\rho^1_{M||\nu} \leq s(M||\nu)$; see Lemma 3.7. So an increasing ω-sequence of ordinals hiding α would yield a descending ω-sequence of ordinals. So let $\nu \in M$ be maximal hiding α. Suppose $\nu + \omega \notin D_M$. Then let ν' hide $\nu + \omega$. Then ν' hides α too, for the above reason. This contradicts the maximality of ν. □

Lemma 3.17. *Let M be a pPλ-structure s.t.* $\mathrm{ht}(M)$ *is a limit ordinal. Then D_M is closed and unbounded in* On_M.

Proof. This is an immediate consequence of Lemma 3.16. □

Definition 3.18. For a pPλ-structure M, let $\langle \eta_\xi^M \mid \xi < \mathrm{otp}(D_M)\rangle$ be the monotone enumeration of D_M.

Lemma 3.19. *Let M be a pPλ-structure and $\xi + 1 < \mathrm{otp}(D_M)$. Then:*

(a) If $\eta_\xi^M = s^+(\nu)^M$, for some $\nu \in M$, then $\eta_{\xi+1}^M = \nu + \omega$.

(b) If $\eta_\xi^M \neq s^+(\nu)^M$ for all $\nu \in M$, then $\eta_{\xi+1}^M = \eta_\xi^M + \omega$.

Proof. Set $\alpha := \eta_\xi^M$. First let $\alpha = s^+(\nu)^M$ for a (unique) ordinal $\nu \in M$. Then obviously, $(\alpha, \nu] \cap D_M = \emptyset$, as all elements of this interval are hidden by ν. I show now:

($*$) *ν is the maximal ordinal hiding $\alpha + \omega$.*

Proof of ($*$). Suppose $\nu' > \nu$ and ν' hides $\alpha + \omega$. Then $s^+(\nu')^M \leq s(\nu)^M < \alpha < \nu'$, as $(s(\nu)^M, \nu] \cap \mathrm{Card}^{M||\nu'} = \emptyset$. So α is hidden by ν', i.e., $\alpha \notin D_M$. $\square_{(*)}$

So by Lemma 3.16, $\nu + \omega \in D_M$, and hence $\eta_{\xi+1}^M = \nu + \omega$. Now let $\alpha \neq s^+(\nu)^M$ for all $\nu \in M$. Again, $(\alpha, \alpha + \omega) \cap D_M = \emptyset$. It remains to be shown that $\alpha + \omega \in D_M$. Assume the contrary. Then let ν hide $\alpha + \omega$. Since $s^+(\nu)^M$ is a cardinal in $M||\nu$ and $s^+(\nu)^M < \alpha + \omega$, it follows that $s^+(\nu)^M \leq \alpha$. But by assumption, $s^+(\nu)^M \neq \alpha$, so $s^+(\nu)^M < \alpha$. But then ν hides α, contradicting that $\alpha \in D_M$. □

Lemma 3.20. *Let M be a pPλ-structure and $s^+ = s^+(\gamma)^M \in D_M$ ($\gamma \leq \mathrm{ht}(M)$). Then s^+ is a limit point of D_M.*

Proof. Otherwise it would be the case that $s^+ = \eta_{\xi+1}^M$ for some ξ. Since s^+ is a cardinal in $M||\gamma$, $s^+ \neq \alpha + \omega$ ($s^+ > \omega$). But by Lemma 3.19, every $\eta_{\xi+1}^M$ is of the form $\alpha + \omega$ for some α. Hence, $s^+ \neq \eta_{\xi+1}^M$. □

Lemma 3.21. *Let M be a pPλ-structure. Then the sequence $\langle \eta_\xi^M \mid \xi < \mathrm{otp}(D_M)\rangle$ is a $\Sigma_1(M)$-function.*

Proof. The idea is of course to define η^M by: $\gamma = \eta_\xi^M \iff M \models \exists f \quad (\psi(f) \wedge \gamma = f(\xi))$, where

$$\begin{aligned}\psi \quad = \quad & (\text{“}f \text{ is a function”} \wedge \mathrm{dom}(f) \in \mathrm{On} \wedge \\ & \quad \forall \alpha \in \mathrm{dom}(f) \quad (\dot{D}(f(\alpha)) \wedge \\ & \qquad (\forall \delta < \alpha \quad f(\alpha) > f(\delta)) \wedge \\ & \qquad (\forall \mu < f(\alpha) \exists \nu < \alpha \quad (\dot{D}(\mu) \rightarrow \mu = f(\nu))))).\end{aligned}$$

Obviously then, ψ is a Σ_0−formula, and hence the formula defining η^M is Σ_1, as wished. I show by induction on μ that there are arbitrarily long proper initial segments of $\eta^{M||\mu}$ in $M||\mu$, finishing the proof. If this holds for μ, this means in particular that $\eta^{M||\mu}$ is $\Sigma_1(M||\mu)$ (even without using the top extender predicate). In the successor step this is easy to see, since

$$\eta^{M||\mu+1} = \begin{cases} \eta^{M||\mu} \cup \{\langle \omega\mu, \mathrm{dom}(\eta^{M||\mu})\rangle\} & \text{If } M||\mu \text{ is passive,} \\ \eta^{M||\mu} & \text{If } M||\mu \text{ is active and} \\ & s^+(\mu)^M = \mu, \\ \eta^{M||\mu}\restriction\mathrm{otp}(D_M) & \text{If } M||\mu \text{ is active and} \\ & s^+(\mu)^M < \mu. \end{cases}$$

We know that $\eta^{M||\mu} \in M||\mu+1$, because it is definable in $M||\mu$.

At limit stages λ, by Lemma 3.17, D_M is unbounded in On_M, and hence it follows from coherency of D_M that

$$D_{M||\lambda} = \bigcup_{\omega\mu\in D_M} D_{M||\mu}, \text{ and hence } \eta^{M||\lambda} = \bigcup_{\omega\mu\in D_M} \eta^{M||\mu}.$$

Again, the inductive hypothesis gives: $\eta^{M||\mu} \in M||(\mu+1) \subseteq M||\lambda$ for $\mu < \lambda$ by definability. $\square$

Definition 3.22. Let $M = \langle \mathrm{J}^E_\alpha, F\rangle$ be a weak j-ppm. Set:

$$D^*_M := \begin{cases} D_M & \text{if } F = \emptyset, \\ D_M \setminus (s^+(\alpha)^M, \alpha) & \text{otherwise.} \end{cases}$$

So if $N = \langle \mathrm{J}^E_{\alpha+1}, \emptyset\rangle$ and $M = N||\alpha$, then $D^*_M = D_N \cap \omega\alpha$.

Definition 3.23. For two pPλ-structures (or j-ppm) M and N let the relation $<_0$ be defined by:

$$M <_0 N \iff M = N^{\text{passive}} \neq N \vee \exists\beta < \mathrm{ht}(N) \quad M = N||\beta.$$

Obviously, $<_0$ is well founded, since $M <_0 N <_0 N' \implies M \in N'$.

Lemma 3.24. *For every weak pPλ-structure (or j-ppm) M, $M = h^1_M(D^*_M)$.*

Proof. I prove the lemma by $<_0$-induction on M. Let $D^*_M \subseteq X$ and $M|X \prec_{\Sigma_1} M$. I must show that $X = M$. The base case $M = \langle\emptyset,\emptyset,\emptyset\rangle$ (this triple stands for $\langle |M|, E^M, E^M_{\text{top}}\rangle$) is clear.

Case 1: M is a $<_0$-successor.

Case 1.1: $M = \langle \mathrm{J}^E_{\alpha+1}, \emptyset\rangle$.

Then $D^*_M = D_M$ and $D^*_{M||\alpha} = D_M \cap \omega\alpha$.

Case 1.1.1: $M||\alpha$ is active.

Then $\omega\alpha = \alpha$. We have: $D^*_{M||\alpha} \subseteq D_M \subseteq X$.

(1) $\alpha \in X$.

Proof of (1).

Case 1: $s^+(\alpha)^M = \alpha$.

Then $D_M = D^*_M = D^*_{M||\alpha} \cup \{\omega\alpha\}$, hence obviously $\alpha \in D^*_M \subseteq X$.

Case 2: $s^+(\alpha)^M < \alpha$.

Then $s^+(\alpha)^M = \max(D^*_{M||\alpha})$, and $D^*_{M||\alpha} = D_M = D^*_M$. Hence $s := s^+(\alpha)^M \in D^*_M \subseteq X$. We have:

$$M \models \exists x\, \exists\mu \quad x = M||\mu \wedge (s = s^+(\mu))^x.$$

This statement is Σ_1 in s, hence valid in $M|X$ as well. Let x and μ be elements of X witnessing this. Then $x = M||\mu$, hence $s = s^+(\mu)^M = s^+(\alpha)^M$, and hence $\mu = \alpha \in X$. I used Lemma 3.7 here. $\square_{(1)}$

(2) $(M||\alpha)|X \prec M||\alpha$.

This is an immediate consequence of the fact that $\alpha \in X$. Firstly, it follows that $M||\alpha$, being Σ_1-definable from α, is an element of X. In order to show that $(M||\alpha)|X \prec M||\alpha$, I verify Tarski's criterion. So let $M||\alpha \models (\exists y \quad \varphi)[a_1,\ldots,a_n]$, where φ is some formula and $\vec{a} \in X \cap |M||\alpha|$. Then

$$M \models (\exists y \quad y \in |M||\alpha| \wedge \varphi^{M||\alpha})[\vec{a}].$$

Since "$(y \in M||\alpha \wedge \varphi^{M||\alpha})(y)[\vec{a}]$" is a Σ_0-formula in the parameters $M||\alpha, a_1, \ldots, a_n \in X$ and $M|X \prec_{\Sigma_1} M$, it follows that there is a $b \in X$ so that

$$M \models (b \in |M||\alpha| \wedge \varphi^{M||\alpha}[b, a_1, \ldots, a_n]).$$

Hence $b \in X \cap |M||\alpha|$, and since the above formula is $\mathbf{\Sigma}_0$ and M is transitive, it follows that $M||\alpha \models \varphi[b, a_1, \ldots, a_n]$, so that b verifies the Tarski criterion. $\square_{(2)}$

Since $D^*_{M||\alpha} \subseteq D_M = D^*_M \subseteq X$, it follows inductively that $|M||\alpha| \subseteq X$. But since $M||\alpha \in X$ as well, it follows that $M \subseteq X \subseteq M$, because $|M| =$ rud $_{E\restriction\omega\alpha, E_{\omega\alpha}}(|M|)$, and because every function rudimentary in $E\restriction\omega\alpha$, $E_{\omega\alpha}$ is Σ_0 in $E\restriction\omega\alpha$, $E_{\omega\alpha}$.

Case 1.1.2: $M||\alpha$ is passive.

Then $\omega\alpha \in D_M$, since there can be no $\nu \in M$ with $s^+(\nu)^M < \omega\alpha \leq \nu$, because if there were, then it would have to be the case that $\nu = \alpha$, but α indexes no extender in M. Hence $\omega\alpha \in D^*_M \subseteq X$, and one can argue as in case 1.1.1.

Case 1.2: $M = \langle \mathrm{J}^E_\alpha, F\rangle$, $F \neq \emptyset$.

Then $D^*_M = D_M \setminus (s^+(\alpha)^M, \alpha)$. So if $s^+(\alpha)^M = \alpha$, then $D^*_M = D_M = D^*_{M^{\mathrm{passive}}} \subseteq X$. Obviously, $M^{\mathrm{passive}}|X \prec_{\Sigma_1} M^{\mathrm{passive}}$, and it follows in this case inductively that $X = |M^{\mathrm{passive}}| = |M|$.

Now let $s^+(\alpha)^M < \alpha$. Then $s^+(\alpha)^M \in D^*_M \subseteq X$. So the proof of (2) shows:

$$(M||s^+(\alpha)^M)|X \prec M||s^+(\alpha)^M.$$

By coherency, $D_{M||s^+(\alpha)^M} = D_M \cap s^+(\alpha)^M = D^*_M \cap s^+(\alpha)^M$. Hence it follows that $D_{M||s^+(\alpha)^M} \subseteq X$, and the induction hypothesis can be applied in order to deduce that $|M||s^+(\alpha)^M| \subseteq X$. In particular, $s^+(\alpha)^M \subseteq X$. But $M = h^1_M(s^+(\alpha)^M)$ – in fact, we even know that $M = h^1_M(s(\alpha)^M)$; see Lemma 3.7. Hence $|M| \subseteq X$.

Case 2: M is a limit point of $<_0$.

Then $M = \langle \mathrm{J}^E_\lambda, \emptyset\rangle$, where λ is a limit. For $\omega\alpha \in D_M = D^*_M$,

$$D_{M||\alpha} = D_{\langle \mathrm{J}^E_\alpha, \emptyset\rangle} = D_M \cap \omega\alpha$$

and $\omega\alpha \in X$, hence $(M||\alpha)|X \prec M||\alpha$ (see (2)), where $D^*_{M||\alpha} \subseteq D_{M||\alpha} \subseteq X$, hence by induction hypothesis, $|M||\alpha| \subseteq X$. This holds whenever $\omega\alpha \in D_M$. But by Lemma 3.17, D_M is unbounded in On_M, and hence $|M| \subseteq X$, which is what was to be shown. $\square$

Corollary 3.25. *Let M be a pPλ-structure. Then $\rho^1_M \leq \mathrm{otp}(D^*_M)$.*

Proof. By Lemma 3.21, the monotone enumeration of D_M is a $\Sigma_1(M)$-function, so this is true in particular for the monotone enumeration of D^*_M, since this is an initial segment of D_M. So $D^*_M \subseteq h^1_M(\mathrm{otp} D^*_M)$. But by Lemma 3.24, $h^1_M(D^*_M) = |M|$ (this is even true for M^-), hence the claim follows because $h^1_M(D^*_M) \subseteq h^1_M(\mathrm{otp}(D^*_M))$, since then we have a $\Sigma_1(M)$-surjection from $\omega \cdot \mathrm{otp}(D^*_M)$ onto $|M|$. $\square$

Corollary 3.26. *Let M be a pPλ-structure. Then $h^1_M(\mathrm{otp}(D^*_M)) = |M|$. If M is active, then $h^1_M(\cup\mathrm{otp}(D^*_M)) = |M|$.*

Proof. That $h^1_M(\mathrm{otp}(D^*_M)) = |M|$ follows from the identity $h^1_M(D^*_M) = |M|$, using the fact that η^M is $\Sigma_1(M)$ – see the proof of Lemma 3.25. Now let M be active. If $s^+(M) = \mathrm{ht}(M)$, then $D^*_M = D_M$ is unbounded in M. So since M^{passive} is a ZF^--model, it follows that $\mathrm{otp}(D^*_M) = \mathrm{ht}(M)$, hence a limit ordinal. So in this case, $\mathrm{otp}(D^*_M) = \cup\mathrm{otp}(D^*_M)$, and we're done.

So let's suppose that $s^+(M) < \mathrm{ht}(M)$. Then $s^+(M) = \max(D^*_M)$, because $s^+(M) \in D_M$ (as $s^+(M)$ is a cardinal in M). Hence by Lemma 3.20, $s^+(M)$ is a limit point of D_M, and hence of

D^*_M. Now let $\xi \in D^*_M$ be chosen in such a way that $s(M) \leq \xi < s^+(M)$. Then $s^+(M)$ is the least cardinal in M greater than ξ. As $s^+(M) < \mathrm{ht}(M)$, it follows that $s^+(M) < \lambda(M)$, and λ is a cardinal of M that surely is an element of $h^1_M(D^*_M \cap s^+(M))$ (because $\lambda = E^M_{\mathrm{top}}(\mathrm{crit}(E^M_{\mathrm{top}}))$). Using λ and ξ, we can now define s^+ in a Σ_1 way: It is the least cardinal of $\mathrm{J}^{E^M}_\lambda$ greater than ξ. Hence $s^+(M) \in h^1_M(D^*_M \setminus \{s^+(M)\})$. This means that $h^1_M(D^*_M) = h^1_M(D^*_M \setminus \{s^+(M)\}) = |M|$, and this yields the second part of the claim immediately, again using the fact that the monotone enumeration of D_M is $\Sigma_1(M)$. □

3.5 The s'-initial segment condition

Because the extenders appearing in pPλ- and pPs-structures are indexed differently, and because the index is essential for the choice of the extenders applied in coiterations, it follows that if one translates coiterations of pPs-structures into coiterations of pPλ-structures, the outcome will be iterations which are not necessarily normal in the sense of λ indexing. So a modified notion of normality will be used on the λ-side, which imitates the way normal iterations on the s-side are formed. Such a notion has been developed by Jensen already, and these iterations are called *s-iterations*. The idea is that every extender appearing in the sequence of a Jensen-premouse is assigned an additional iteration index which determines to which model in an iteration tree the extender must be applied. These s-iterations call for an appropriate initial segment condition which is preserved by them, which guarantees that coiterations terminate, and which is not unduly restrictive.

This condition has to satisfy two requirements that are tightly connected to the notion of a normal iteration. Firstly, it must guarantee that the coiteration (which is normal) of two coiterable structures terminates. The second requirement is really contained in the first one: As the argument showing that coiterations terminate is applied to normal iterates of the premice involved, the initial segment condition must be preserved under normal iterations.

I am first aiming at finding a tailor-made *minimal initial segment condition*. It is a slight modification of the variant given in [Jen01].[6] First, a definition is needed, though.

Definition 3.27. Let F be a pre-extender in functional representation. Then ξ is a cutpoint of F iff $\xi = s(F|\xi)$.[7]

Definition 3.28. Let M be an active extender structure. M satisfies the minimal *s'-initial segment condition* (s'-MISC), iff, letting $F := E^M_{\mathrm{top}}$, for every cutpoint $\xi \in [\tau(F), s(F))$ of F, $(\xi^+)^M \neq (\xi^+)^{[M]_\xi}$.

If M satisfies the s'-MISC, then obviously, $(\xi^+)^M > (\xi^+)^{[M]_\xi}$. Modulo the modification of the minimal s-ISC, the s-ISC itself remains practically unchanged, compared to [Jen01, Chapter 1, p. 4] – only the broader context of pPs-structures leads to a more general formulation:

Definition 3.29. Let M be a potential Pseudo-λ- or s-structure. The s'-initial segment condition (s'-ISC) for M says that for every $\alpha \leq \mathrm{ht}(M)$ with $F = E^M_\alpha \neq \emptyset$ and each cutpoint $\xi \in [\tau(F), s(F))$ of F,

(a) If $[M||\alpha]_\xi$ satisfies the s'-MISC, then $[M||\alpha]_\xi \in \widehat{M||\alpha}$.

(b) If $[M||\alpha]_\xi$ satisfies the s'-MISC and $\xi' \in [\tau(F), \xi)$ is such that $[M||\alpha]_{\xi'}$ satisfies the s'-MISC, then $[M||\alpha]_{\xi'} \in [M||\alpha]_\xi$.

[6] The modification was necessary since I was looking for an ISC which is a consequence of the present version of Steel's ISC from [Ste00, Def.2.4., item 3].

[7] In order to avoid possible confusions, I maybe should have referred to these ordinals as s-cutpoints. But since only this kind of cutpoints will play a role, I opted for a somewhat slicker terminology.

Definition 3.30. Let M be an active extender structure. Set:

$$C_M = \{\xi \mid \tau \leq \xi < s(M),\ \xi \text{ is a cutpoint of } E^M_{\text{top}} \\ \text{and } [M]_\xi \text{ satisfies the } s'\text{-MISC } \}.$$

3.6 Potential λ- and s-structures

Definition 3.31. A *potential λ-structure (pλ-structure)* is a potential Pseudo-λ-structure, that satisfies the s'-ISC. Analogously, a *potential s-structure (ps-structure)* is a potential Pseudo-s-structure that satisfies the s'-ISC.

Now I will introduce the different types of structures, which are most important in connection with pPs-structures.

Definition 3.32. Let M be an active pPs- or pPλ-structure. Then M is of...

$$\begin{array}{rll} \text{...type I} & \text{iff} & s(M) = \tau(M), \\ \text{...type II} & \text{iff} & s(M) = \xi + 1 \text{ for some } \xi, \\ \text{...type III} & \text{iff} & \tau(M) < s(M) \text{ is a limit ordinal.} \end{array}$$

3.7 Σ_0-codes

In the following, the Σ_0-codes of the structures involved will be introduced. I follow [Ste00, Def. 2.11], but will need several variants of the codes defined there. First, I am going to define the Pseudo-Σ_0-codes of pPs- and pPλ-structures.

Definition 3.33. Let $\mathcal{L}$ be the language of set theory with additional symbols $\dot{E}$, $\dot{F}$, $\dot{\kappa}$ and $\dot{s}$. Let $N = \langle \mathrm{J}^E_\alpha, F\rangle$ be a pPs-structure. Then its *Pseudo-Σ_0-code,* $\tilde{\mathcal{C}}_0(N)$, is an $\mathcal{L}$-structure, which is defined as follows:

1. If N is passive, then $\tilde{\mathcal{C}}_0(N)$ has the universe $|\mathrm{J}^E_\alpha|$, $\dot{\kappa}^{\tilde{\mathcal{C}}_0(N)} = \dot{s}^{\tilde{\mathcal{C}}_0(N)} = 0$, $\dot{E}^{\tilde{\mathcal{C}}_0(N)} = E{\restriction}\alpha$ and $\dot{F}^{\tilde{\mathcal{C}}_0(N)} = \emptyset$.
2. If N is active of type I or II, then $\tilde{\mathcal{C}}_0(N)$ has the universe $|\mathrm{J}^E_\alpha|$ again, but in that case, $\dot{\kappa}^{\tilde{\mathcal{C}}_0(N)} = \mathrm{crit}(F)$, $\dot{s}^{\tilde{\mathcal{C}}_0(N)} = s(F)$, $\dot{E}^{\tilde{\mathcal{C}}_0(N)} = E{\restriction}\omega\alpha$ and $\dot{F}^{\tilde{\mathcal{C}}_0(N)} = F$.
3. If N is active of type III, then the universe of $\tilde{\mathcal{C}}_0(N)$ is $|\widehat{N}|$, $\dot{\kappa}^{\tilde{\mathcal{C}}_0(N)} = \mathrm{crit}(F)$, $\dot{s}^{\tilde{\mathcal{C}}_0(N)} = 0$, $\dot{E}^{\tilde{\mathcal{C}}_0(N)} = E^{\widehat{N}}{\restriction}\mathrm{ht}(\widehat{N})$ and $\dot{F}^{\tilde{\mathcal{C}}_0(N)} = E^{\widehat{N}}_{\text{top}}$.

In addition, I define $\tilde{\mathcal{C}}_0(N)^{\text{sq}}$, the *squashed-Pseudo-$\Sigma_0$-code* of N, as follows: If N is passive or active of type I or II, then $\tilde{\mathcal{C}}_0(N)^{\text{sq}} = \tilde{\mathcal{C}}_0(N)$. If, on the other hand, N is active of type III, then let $s = s(F)$. The universe of $\tilde{\mathcal{C}}_0(N)^{\text{sq}}$ is then $|\mathrm{J}^E_s|$, $\dot{\kappa}^{\tilde{\mathcal{C}}_0(N)^{\text{sq}}} = \mathrm{crit}(F)$, $\dot{s}^{\tilde{\mathcal{C}}_0(N)^{\text{sq}}} = 0$, $\dot{E}^{\tilde{\mathcal{C}}_0(N)^{\text{sq}}} = E{\restriction}s$ and $\dot{F}^{\tilde{\mathcal{C}}_0(N)^{\text{sq}}} = F^{\mathfrak{h}}{\restriction}s = \{\langle\alpha, X\rangle \mid \alpha \in (F^{\mathfrak{f}}(X)) \cap s\}$.[8]

Analogously, I define $\tilde{\mathcal{C}}_0(\widehat{N})$ as follows.

1. If N is passive, then $\tilde{\mathcal{C}}_0(\widehat{N}) = \tilde{\mathcal{C}}_0(N)$.

[8] In accordance with [Ste00] one really would have to define:

$$\dot{F}^{\tilde{\mathcal{C}}_0(N)^{\text{sq}}} = \{\langle a, X\rangle \mid \exists n < \omega \quad a \in s^n \wedge X \subseteq [\dot{\kappa}^{\tilde{\mathcal{C}}_0(N)^{\text{sq}}}]^n \wedge a \in F^{\mathfrak{f}}(X)\}.$$

But the above coding doesn't contain less information and is easier to work with.

2. If N is active of type I or II, then $\tilde{\mathcal{C}}_0(\widehat{N})$ has universe $|\widehat{N}|$, and I set: $\dot{\kappa}^{\tilde{\mathcal{C}}_0(\widehat{N})} = \mathrm{crit}(F)$, $\dot{s}^{\tilde{\mathcal{C}}_0(\widehat{N})} = s(F)$, $\dot{E}^{\tilde{\mathcal{C}}_0(\widehat{N})} = E^{\widehat{N}}\restriction\mathrm{ht}(\widehat{N})$ and $\dot{F}^{\tilde{\mathcal{C}}_0(\widehat{N})} = E^{\widehat{N}_{\mathrm{top}}}$.

3. If N is active of type III, then $\tilde{\mathcal{C}}_0(\widehat{N}) = \tilde{\mathcal{C}}_0(N)$.

Here are the corresponding codes for (Pseudo)-λ-structures:

Definition 3.34. Let $\tilde{\mathcal{L}}$ be the language of set theory with additional symbols $\dot{D}$, $\dot{E}$, $\dot{F}$, $\dot{\kappa}$ and $\dot{s}$. Let $M = \langle \mathrm{J}^E_\alpha, F, D_M\rangle$ be a pPλ-structure. Then its *Pseudo-*Σ_0*-code,* $\tilde{\mathcal{C}}_0(M)$ is the $\tilde{\mathcal{L}}$-structure defined as follows. The universe of $\tilde{\mathcal{C}}_0(M)$ is $|\tilde{\mathcal{C}}_0(M)| = |\mathrm{J}^E_\alpha|$ and $\dot{D}^{\tilde{\mathcal{C}}_0(M)} = D_M$, and

1. If M is passive, then $\dot{\kappa}^{\tilde{\mathcal{C}}_0(M)} = \dot{s}^{\tilde{\mathcal{C}}_0(M)} = 0$, $\dot{E}^{\tilde{\mathcal{C}}_0(M)} = E\restriction\omega\alpha$ and $\dot{F}^{\tilde{\mathcal{C}}_0(M)} = \emptyset$.

2. If M is active of type I or II, then $\dot{\kappa}^{\tilde{\mathcal{C}}_0(M)} = \mathrm{crit}(F)$, $\dot{s}^{\tilde{\mathcal{C}}_0(M)} = s(F)$, $\dot{E}^{\tilde{\mathcal{C}}_0(M)} = E\restriction\omega\alpha$ and $\dot{F}^{\tilde{\mathcal{C}}_0(M)} = F$.

3. If M is active of type III, then $\dot{\kappa}^{\tilde{\mathcal{C}}_0(M)} = \mathrm{crit}(F)$, $\dot{s}^{\tilde{\mathcal{C}}_0(M)} = 0$, $\dot{E}^{\tilde{\mathcal{C}}_0(M)} = E\restriction\omega\alpha$ and $\dot{F}^{\tilde{\mathcal{C}}_0(M)} = F$.

It is shown in the second part of this paper that for every pPλ-structure M of type II, the set C_M has a maximum. With the proofs given there it follows that the same is true for pPs-structures. Alternatively, one may apply Lemma 4.16 in order to prove this for every ps-structure which has a Λ-image. Finally one shows that every pPs-structure has an image. So I can define:

Definition 3.35. For a pPλ- or pPs-structure M of type II, let $q_M := F|\max C_M$.

Remark 3.36. For a pλ-structure M of type II, $q_M \in M$, as M satisfies the s'-ISC, which implies that even $[M]_{\max C_M} \in M$. Correspondingly, for a ps-structure N of type II, $q_N \in \widehat{N}$.

In order to make sure that transitivized Σ_1-elementary submodels of pλ-structures satisfy the s'-ISC, in the case of an active type II pλ-structure M, I have to demand that q_M be an element of the submodel. For this reason, it is frequently convenient to work with structures equipped in this case with the constant $\dot{q}^M = q_M$. This way, one arrives at the Σ_0-code of potential λ-structures. In the case of active s-structures N of type II, the additional complication arises that q_N may be an element of $|\widehat{N}| \setminus |N|$. In order to deal with this problem, I follow [Ste00, p. 14-15] in a form suitable for the present context.

Definition 3.37. Let M be an acceptable, amenable J-structure and F an extender on M. Let $\kappa = \mathrm{crit}(F)$ and $\tau = (\kappa^+)^M$. Let

$$\Gamma'(M,F) = \bigcup_{n<\omega} \{\langle a, f\rangle \mid f : [\kappa]^n \longrightarrow |M| \wedge f \in |\mathrm{J}^{E^M}_\tau| \wedge a \in [\mathrm{lh}(F)]^n\}.$$

Define a well order $\prec_M$ on $\Gamma'(M,F)$ as follows:

$$\langle a, f\rangle \prec_M \langle b, g\rangle \longleftrightarrow f <_M g \vee (f = g \wedge a <_{\mathrm{lex}} b).$$

Here, $<_M$ is the canonical Σ_1-well order of M.

Definition 3.38. Let $\mathcal{L}^*$ be the language of set theory with additional symbols $\dot{E}$, $\dot{F}$, $\dot{\kappa}$, $\dot{s}$ and $\dot{q}$. Let $N = \langle \mathrm{J}^{E^N}_\alpha, F\rangle$ be a potential s-structure. Then its Σ_0*-code,* $\mathcal{C}_0(N)$, is the $\mathcal{L}^*$-structure defined as follows:

1. If N is passive or active of type I or III, then $\mathcal{C}_0(N)$ is defined like $\tilde{\mathcal{C}}_0(N)$, where, in addition, $\dot{q}^{\mathcal{C}_0(N)} = \emptyset$.

2. If N is active of type II, then, again, $\mathcal{C}_0(N)$ is defined like $\tilde{\mathcal{C}}_0(N)$, with the addition that $\dot{q}^{\mathcal{C}_0(N)}$ is defined as follows.

   $\dot{q}^{\mathcal{C}_0(N)} :=$ the $\prec_N$-minimal pair $\langle a, f\rangle \in \Gamma'(N)$ with the property that
   $\pi^N_{s(N)}(f)(a) = E^{\widehat{N}}_{\mathrm{top}}|\eta$, where $\eta = \max(C_N)$.

The *squashed* Σ_0-*Code* $\mathcal{C}_0(N)^{\mathrm{sq}}$ is defined correspondingly, as follows: If N is passive or active of type I or II, then $\mathcal{C}_0(N)^{\mathrm{sq}} = \mathcal{C}_0(N)$. If, on the other hand, N is active of type III, then $\mathcal{C}_0(N)^{\mathrm{sq}}$ is defined like $\tilde{\mathcal{C}}_0(N)^{\mathrm{sq}}$, with the addition that $\dot{q}^{\mathcal{C}_0(N)^{\mathrm{sq}}} = \emptyset$.

Analogously, I define $\mathcal{C}_0(\widehat{N})$ as follows:

1. If N is passive or active of type I or III, then $\mathcal{C}_0(\widehat{N})$ is defined like $\tilde{\mathcal{C}}_0(\widehat{N})$, with the addition that $\dot{q}^{\mathcal{C}_0(\widehat{N})} = \emptyset$.

2. If N is active of type II, then $\mathcal{C}_0(\widehat{N})$ is defined like $\tilde{\mathcal{C}}_0(\widehat{N})$, with the addition that

$$\dot{q}^{\mathcal{C}_0(\widehat{N})} = q_{\widehat{N}} = E^{\widehat{N}}_{\mathrm{top}}|\eta, \text{ where } \eta = \max C_N.$$

Again, the corresponding codes for potential λ-structures are needed.

Definition 3.39. Let $\tilde{\mathcal{L}}^*$ be the language of set theory with additional symbols $\dot{D}$, $\dot{E}$, $\dot{F}$, $\dot{\kappa}$, $\dot{s}$ and $\dot{q}$. Let $M = \langle \mathrm{J}^{E^M}_\alpha, F\rangle$ be a potential λ-structure. Then its Σ_0-*code,* $\mathcal{C}_0(M)$, is the $\tilde{\mathcal{L}}^*$-structure defined as follows:

1. If M is passive or active of type I or III, then $\mathcal{C}_0(M)$ is defined like $\tilde{\mathcal{C}}_0(M)$, with the addition that $\dot{q}^{\mathcal{C}_0(M)} = \emptyset$.

2. If M is active of type II, then $\mathcal{C}_0(M)$ is defined like $\tilde{\mathcal{C}}_0(M)$, with the addition that $\dot{q}^{\mathcal{C}_0(M)} := q_M$.

3.8 (Pseudo-)λ- and (Pseudo-)s-structures

Now I can finally define the structures that will be the protagonists.

Definition 3.40. A *Pseudo-λ-structure (Pλ-structure)* is a potential Pseudo-λ-structure M with the property that for every $\alpha < \mathrm{ht}(M)$, the structure $\tilde{\mathcal{C}}_0(M||\alpha)$ is sound and 1-solid.[9] Analogously, a *Pseudo-s-structure (Ps-structure)* is a potential Pseudo-s-structure N with the property that for every $\alpha < \mathrm{ht}(M)$, the structure $\tilde{\mathcal{C}}_0(N||\alpha)$ is sound and 1-solid. The definition of s- and λ-structures is like that of Ps- and Pλ-structures, with $\tilde{\mathcal{C}}_0$ replaced by $\mathcal{C}_0$, and with the addition that the s'-ISC must be satisfied.[10]

4 The Translation Functions

In this section, I am going to define the functions $\mathtt{S}$ and Λ that map potential Pseudo-λ-structures to potential Pseudo-s-structures and vice versa.

[9] In the literature, this property is often referred to as *solid above* ωp^1_M. For a definition, see 8.15.

[10] If Q is an active pλ- or ps-structure of type III that is sound, then Q is 1-solid, because in this case, $p_{Q,1} = p^0_Q = \emptyset$. This is shown in the second part of this paper. So one wouldn't have to explicitly demand 1-solidity of active type III-segments of λ- or s-structures. The s'-ISC is essential for this argument, though. The corresponding statement need not be true for Pλ- or Ps-structures.

4.1 From pPλ-structures to pPs-structures...

Definition 4.1. $\mathtt{S}(M)$ is defined for potential Pseudo-λ-structures M by $<_0$ −recursion (see Definition 3.23) as follows:

Case 1: $M = \langle\emptyset,\emptyset,\emptyset,\emptyset\rangle$.

Then $\mathtt{S}(M) = \langle\emptyset,\emptyset,\emptyset\rangle$.

Case 2: $M = \langle \mathrm{J}^E_{\alpha+1},\emptyset, D\rangle$.

Assume $\mathtt{S}(M||\alpha) = \langle \mathrm{J}^{E'}_{\alpha'}, E'_{\omega\alpha'}\rangle$. Let F be the pre-extender in hypermeasure-representation with the property that $E'_{\omega\alpha'} = F^c$, if $E_{\omega\alpha} \neq \emptyset$. Otherwise, let $F = \emptyset$. Now let $\tilde{E} = E' \cup \{\langle F, \alpha'\rangle\}$ and set $\mathtt{S}(M) = \langle \mathrm{J}^{\tilde{E}}_{\alpha'+1},\emptyset\rangle$, if this is a pP$s$-structure.[11]In the future, I shall write for this: $\mathtt{S}(M) = \mathtt{S}(M||\alpha)+1$. If $\langle \mathrm{J}^{\tilde{E}}_{\alpha'+1},\emptyset\rangle$ is not a pPs-structure, then $\mathtt{S}(M)$ remains undefined.

Case 3: $M = \langle \mathrm{J}^E_\alpha, F, D_M\rangle$, where $F \neq \emptyset$.

Then $\mathtt{S}(M)$ is defined if

(a) $\mathtt{S}(\langle \mathrm{J}^E_\alpha,\emptyset, D_M\rangle) = \langle \mathrm{J}^{E'}_{\alpha'},\emptyset\rangle$ for some E', α'.

(b) $|\mathrm{J}^{E'}_{\alpha'}| = |\mathrm{J}^E_\alpha|$ (in particular, $\alpha = \alpha'$).

(c) Let $\kappa = \mathrm{crit}(F)$ and $\tau = (\kappa^+)^M$. Then $\mathrm{Ult}(\mathrm{J}^{E'}_\tau, F) = \mathrm{J}^{E'}_{\alpha'}$, and $E'_{s^+(\alpha)^M} = \emptyset$.

If these conditions are satisfied, then, letting $s^+ := s^+(\alpha)^M$, I define

$$\mathtt{S}(M) = \langle \mathrm{J}^{E'}_{s^+(\alpha)^M}, ((F|s^+)^{\mathtt{h}})^c_{\mathrm{J}^{E'}_{s^+}}\rangle,$$

otherwise $\mathtt{S}(M)$ remains undefined.

Case 4: $M = \langle \mathrm{J}^E_\alpha, D,\emptyset\rangle$, where α is a limit ordinal.

Then $\mathtt{S}(M)$ is defined if the following hold:

(a) For $\beta < \mathrm{ht}(M)$, $\mathtt{S}(M||\beta)$ is defined and a potential Pseudo-s-structure.

(b) For $\omega\beta,\omega\beta' \in D_M$ with $\beta < \beta'$, it follows that $\mathtt{S}(M||\beta) <' \mathtt{S}(M||\beta')$, that is, $\mathrm{ht}(\mathtt{S}(M||\beta)) < \mathrm{ht}(\mathtt{S}(M||\beta'))$ and $E^{\mathtt{S}(M||\beta)}\restriction\mathrm{On}_{\mathtt{S}(M||\beta)} = E^{\mathtt{S}(M||\beta')}\restriction\mathrm{On}_{\mathtt{S}(M||\beta)}$.

If these conditions are satisfied, set $\mathtt{S}(M) := \bigcup_{\omega\beta\in D_M}\mathtt{S}(M||\beta^{\mathrm{passive}})$ in the obvious sense, i.e., $E' = \bigcup_{\omega\beta\in D_M} E^{\mathtt{S}(M||\beta)}$, $\alpha' = \bigcup_{\omega\beta\in D_M}\mathrm{ht}(\mathtt{S}(M||\beta))$, and $\mathtt{S}(M) = \langle \mathrm{J}^{A_{E'}}_{\alpha'},\emptyset\rangle$; for the definition of $A_{E'}$, see Definition 3.4. Otherwise, $\mathtt{S}(M)$ remains undefined.

Remark 4.2.

1. Let M be active, $\alpha = \mathrm{ht}(M)$, and $\mathtt{S}(M)$ be defined. In the notation of Definition 4.1, case 3, we then have $|\widehat{\mathtt{S}(M)}| = |\mathrm{J}^{E'}_{\alpha'}| = |\mathrm{J}^E_\alpha| = |M|$ because of (b) and (c).
2. If $N = \mathtt{S}(M)$ is defined, then N is a pPs-structure. In case 2 of the above definition, this was explicitly demanded. It follows that this is true in case 4 as well, and in case 3 the only additional condition that must be satisfied is coherency. But this follows, using (b) and (c), from case 3 – see Def. 3.4.
3. If M is a passive pPλ-structure of limit height, so that for all $\bar{M} <_0 M$, $\mathtt{S}(\bar{M})$ is defined, then the conditions (a) and (b) from case 4 are met, i.e. $\mathtt{S}(M)$ is defined.

[11]In order to make the definition more readable, I don't distinguish between E and A_E here.

Proof of 3. Condition (a) follows from observation 2. It remains to verify weak monotonicity (b). Firstly, it is easy to see that $\mathrm{ht}(\mathtt{S}(M)) \leq \mathrm{ht}(M)$, if $\mathtt{S}(M)$ is defined. Now it suffices to show for an arbitrary pPλ-structure M:

If $\mathtt{S}(M)$ *is defined, then for* $\beta < \beta'$ *with* $\omega\beta, \omega\beta' \in D_M$*:* $\mathtt{S}(M||\beta) <' \mathtt{S}(M||\beta')$.

Assume the contrary. Then let M be a $<_0$-minimal counterexample. Obviously $M \neq \langle \emptyset, \emptyset, \emptyset, \emptyset \rangle$. Let $M = \langle \mathrm{J}^E_\mu, F, D \rangle$.

Case 1: $\mu = \bar{\mu} + 1$.

Let $\beta < \beta'$, $\omega\beta, \omega\beta' \in D$, and set $\bar{M} = M||\bar{\mu}$.

Case 1.1: $\bar{M}$ is active and $s^+ := s^+(\bar{M}) < \bar{\mu}$.

In this case, $D = D_{\bar{M}} \cap (s^+ + 1)$. Hence $\omega\beta, \omega\beta' \in D_{\bar{M}}$. I.e., by minimality of M, $\mathtt{S}(M||\beta) = \mathtt{S}(\bar{M}||\beta) <' \mathtt{S}(\bar{M}||\beta') = \mathtt{S}(M||\beta')$, so M was not a counterexample after all.

Case 1.2: Case 1.1 fails.

Then $D = D_{\bar{M}} \cup \{\omega\bar{\mu}\}$. If $\omega\beta' < \omega\bar{\mu}$, then one can again use minimality of M as in Case 1.1. So let $\omega\beta' = \omega\bar{\mu}$. As $\omega\beta < \omega\beta'$, it follows that $\omega\beta \in D_{\bar{M}}$. Now two subcases are needed in order to capture every possibility of the definition of $\mathtt{S}(\bar{M}) = \mathtt{S}(M||\bar{\mu})$:

Case 1.2.1: $\bar{\mu} = \tilde{\mu} + 1$.

If $\omega\tilde{\mu} \in D_{M||\bar{\mu}}$, then

$$\mathtt{S}(M||\beta) \leq' \mathtt{S}(M||\tilde{\mu}) <' \mathtt{S}(M||\bar{\mu}) = \mathtt{S}(M||\beta'),$$

as is immediate from the definition of $\mathtt{S}$. So let $\omega\tilde{\mu} \notin D_{M||\bar{\mu}}$. Then $s^+(\tilde{\mu})^M < \tilde{\mu}$ and $D_{M||\bar{\mu}} = D_{M||\tilde{\mu}} \cap (s^+(\tilde{\mu})^M + 1)$. As $s^+(\tilde{\mu})^M \in D_{M||\tilde{\mu}}$, and since $\omega\beta \in D_{M||\bar{\mu}}$, it follows that

$$\mathtt{S}(M||\beta) \leq' \mathtt{S}(M||s^+(\tilde{\mu})^M).$$

So it suffices to show that $\mathtt{S}(M||s^+(\tilde{\mu})) <' \mathtt{S}(M||\bar{\mu})$. Using the notation introduced in case 2 of the definition of $\mathtt{S}$, we get:

$$(*)\ \ \begin{aligned}\mathtt{S}(M||\bar{\mu}) &= \mathtt{S}(M||\tilde{\mu}) + 1 \\ &= \langle (\mathtt{S}(M||\tilde{\mu}^{\text{passive}})||s^+(\tilde{\mu})^M)^2_0, ((E^M_{\tilde{\mu}}|s^+(\tilde{\mu})^M)^{\mathtt{h}})^c_{\mathtt{S}(M||\tilde{\mu})^{\text{passive}}} \rangle + 1;\end{aligned}$$

here, let $(\cdot)^2_0$ be the projection onto the first coordinate. But

$$\mathtt{S}(M||\tilde{\mu}^{\text{passive}}) = \bigcup_{\omega\gamma \in D_{M||\tilde{\mu}}} \mathtt{S}(M||\gamma)^{\text{passive}},$$

and $s^+(\tilde{\mu})^M \in D_{M||\tilde{\mu}}$. Hence, $\mathtt{S}(M||s^+(\tilde{\mu})) <' \mathtt{S}((M||\tilde{\mu})^{\text{passive}})$. But, as noted in the beginning, $\mathrm{ht}(\mathtt{S}(M||s^+(\tilde{\mu}))) \leq s^+(\tilde{\mu})^M$. So

$$\mathtt{S}(M||s^+(\tilde{\mu})) \leq' \mathtt{S}((M||\tilde{\mu})^{\text{passive}})||s^+(\tilde{\mu})^M <' \mathtt{S}(M||\bar{\mu}),$$

as wished; I used $(*)$ in the last step.

Case 1.2.2: $\bar{\mu}$ is a limit ordinal and $\bar{M}$ is passive.

As $\omega\beta \in D_{\bar{M}}$, it follows from the limit case of the definition of $\mathtt{S}(\bar{M})$ that $\mathtt{S}(M||\beta) <' \mathtt{S}(\bar{M})$.

Case 1.2.3: $\bar{\mu}$ is a limit ordinal and $\bar{M}$ is active.

Since Case 1.1 was excluded in Case 1.2, it follows that $s^+(\bar{\mu})^M = \bar{\mu}$. Then $(\mathtt{S}(\bar{M}))^{\text{passive}} = \mathtt{S}(\bar{M}^{\text{passive}})$, and by Case 1.2.2 $\mathtt{S}(M||\beta) <' \mathtt{S}(\bar{M}^{\text{passive}})$. Hence $\mathtt{S}(M||\beta) <' \mathtt{S}(\bar{M})$.

Case 2: μ is a limit ordinal.

Case 2.1: M is passive.

Let $\beta < \beta'$ be s.t. $\omega\beta, \omega\beta' \in D$. As D is unbounded in On_M, $\omega\gamma \in D$ can be chosen in such a way that $\beta' < \gamma$. Then $D_{M||\gamma} = D \cap \omega\gamma$, hence $\omega\beta, \omega\beta' \in D_{M||\gamma}$. Since $M||\gamma <_0 M$, it follows by minimality of M that this case cannot occur.

Case 2.2: M is active.

Then let $\bar{M} := M^{\text{passive}}$. Let $\beta < \beta'$ and $\omega\beta, \omega\beta' \in D_M$. We have that $D_M = D_{\bar{M}}$, and hence, $\mathtt{S}(M||\beta) = \mathtt{S}(\bar{M}||\beta) <' \mathtt{S}(\bar{M}||\beta') = \mathtt{S}(M||\beta')$ by minimality of M, and thus this case is excluded as a possibility as well. □

Lemma 4.3. *Let M be a pPλ-structure for which $N = \mathtt{S}(M)$ exists. Let $\alpha < \mathrm{ht}(M)$. Then the following are equivalent:*

1. *There is no $\mu \leq \mathrm{ht}(M)$ such that $M||\mu$ is active and $s^+(M||\mu) \leq \alpha < \mu$.*
2. *$\mathtt{S}(M||\alpha)$ is a segment of N.*

In particular, this is true if $M||\alpha$ is active and $s^+(M||\alpha) \in D_M$.

Proof. For the direction from 1. to 2. one shows by induction on $\beta \in (\alpha, \mathrm{ht}(M)]$ that $\mathtt{S}(M||\alpha)$ is a segment of $\mathtt{S}(M||\beta)$. The other direction is obvious. □

Lemma 4.4. *If M is a pPλ-structure with $\mathtt{S}(M)$ defined, then $\mathrm{ht}(\widehat{\mathtt{S}(M)}) = \mathrm{otp}(D_M)$.*

Proof. The claim is proven by $<_0$ −induction on M.

Case A: $M = \langle \emptyset, \emptyset, \emptyset, \emptyset \rangle$.

This case is trivial.

Case B: $M = \langle \mathrm{J}^{E^M}_{\alpha+1}, \emptyset, D \rangle$.

Case B.1: $M||\alpha$ is passive.

Then $\mathrm{otp}(D) = \mathrm{otp}(D_{M||\alpha}) + 1 = \mathrm{ht}(\widehat{\mathtt{S}(M||\alpha)}) + 1 = \mathrm{ht}(\mathtt{S}(M)) = \mathrm{ht}(\widehat{\mathtt{S}(M)})$.

Case B.2: $M||\alpha$ is active.

Let $s = s^+(\alpha)^M$. By definition of $\mathtt{S}(M)$ it follows that $\mathtt{S}(M) = \widehat{\mathtt{S}(M)}$ and $\mathrm{ht}(\mathtt{S}(M)) = s + 1$. I distinguish two subcases.

Case B.2.1: $s < \alpha$.

Then $D = D_{M||\alpha} \setminus (s, \alpha)$. By Lemma 3.17, $D_{M||s}$ is unbounded in $\mathrm{On}_{M||s}$. As $s < \alpha$ and s is a successor cardinal in $M||\alpha$, it follows easily that $\mathrm{otp}(D_{M||s}) = s$. Moreover, $s \in D_{M||\alpha}$ (for the same reason), hence it follows from coherency that $D_{M||s} = D_{M||\alpha} \cap s$. We get: $D = D_{M||s} \cup \{s\}$, i.e., $\mathrm{otp}(D) = s + 1 = \mathrm{ht}(\widehat{\mathtt{S}(M)})$.

Case B.2.2: $s = \alpha$.

Then $D_{M||\alpha+1} = D_{M||\alpha} \cup \{\omega\alpha\}$, so $\mathrm{otp}(D_M) = \mathrm{otp}(D_{M||\alpha}) + 1 = \mathrm{ht}(\widehat{\mathtt{S}(M||\alpha)}) + 1 = \mathrm{ht}(\widehat{\mathtt{S}(M)})$.

Case C: M is active.

Let $M = \langle \mathrm{J}^E_\alpha, F, D \rangle$. Further, let $\bar{M} = \langle \mathrm{J}^E_\alpha, \emptyset, D \rangle$ (note that $D = D_M = D_{\bar{M}}$). Remark 4.2 and case 3 of Definition 4.1 entail that $|M| = |\widehat{\mathtt{S}(M)}| = |\mathtt{S}(\bar{M})|$. In particular, it follows that $\alpha = \mathrm{ht}(M) = \mathrm{ht}(\widehat{\mathtt{S}(M)}) = \mathrm{ht}(\widehat{\mathtt{S}(\bar{M})})$. Since $\bar{M} <_0 M$, it follows by our inductive hypothesis that

$$\mathrm{ht}(\widehat{\mathtt{S}(M)}) = \mathrm{ht}(\widehat{\mathtt{S}(\bar{M})}) = \mathrm{otp}(D_{\bar{M}}) = \mathrm{otp}(D_M).$$

Case D: M is passive, and $\mathrm{ht}(M)$ is a limit.

Then M is a limit of $<_0$. We have: $\widehat{\mathtt{S}(M)} = \mathtt{S}(M) = \bigcup_{\omega\mu\in D_M} \mathtt{S}(M||\mu^{\text{passive}})$. Hence, due to property (b) of case 4 of the definition of S, the height of $\mathtt{S}(M)$ is a limit ordinal, and

$$\begin{aligned} \mathrm{ht}(\mathtt{S}(M)) &= \bigcup_{\omega\mu\in D_M} \mathrm{ht}(\mathtt{S}(M||\mu^{\text{passive}})) \\ &= \bigcup_{\omega\mu\in D_M} \mathrm{otp}(D_{M||\mu}) \\ &= \bigcup_{\omega\mu\in D_M} \mathrm{otp}(D_M \cap \omega\mu) \\ &= \mathrm{otp}(D_M). \end{aligned}$$

So, inductively, $\mathrm{ht}(\mathtt{S}(M||\mu^{\text{passive}})) = \mathrm{otp}(D_{M||\mu})$. □

Corollary 4.5. *Let M be a pPλ-structure for which $N = \mathtt{S}(M)$ exists. Then*

$$\mathrm{ht}(N) = \begin{cases} \mathrm{otp}(D_M) & \textit{if } M \textit{ is passive,} \\ \cup\,\mathrm{otp}(D^*_M) & \textit{otherwise.} \end{cases}$$

Thus, by Corollary 3.26, $h^1_M(\mathrm{ht}(N)) = |M|$.

Proof. If M is passive, then N is passive as well, and the corollary follows from Lemma 4.4: $\mathrm{otp}(D_M) = \mathrm{ht}(\widehat{N}) = \mathrm{ht}(N)$.

So assume now that M is active. Then $\mathrm{ht}(N) = s^+(M)$, by definition of S. If $s^+(M) < \mathrm{On}_M$, then $D^*_M = D_{M||s^+(M)} \cup \{s^+(M)\}$. Further, $D_M||s^+(M)$ is unbounded in $s^+(M)$, and one can deduce that $\mathrm{otp}(D_{M||s^+(M)}) = s^+(M)$. Hence $s^+(M) = \cup\mathrm{otp}(D^*_M) = \mathrm{ht}(N)$, as wished. On the other hand, if $s^+(M) = \mathrm{On}_M$, then $D^*_M = D_M$, and $s^+(M) = \mathrm{ht}(\mathtt{S}(M)) = \mathrm{ht}(\mathtt{S}(M^{\text{passive}})) = \mathrm{otp}(D_{M^{\text{passive}}}) = \mathrm{otp}(D_M) = \mathrm{otp}(D^*_M) = \cup\mathrm{otp}(D^*_M)$. □

4.2 ...and back to pPλ-structures

In this section, the inverse function Λ of S is introduced. It will be defined by recursion on the following relation.

Definition 4.6. For two potential Pseudo-s-structures M and N, let $M <_1 N$ iff

$$(\exists\xi < \mathrm{ht}(N) \quad M = N||\xi) \vee (N \text{ is active and } M = \widehat{N}^{\text{passive}}).$$

Remark 4.7. Let M_0, M_1 and M_2 be potential Pseudo-s-structures.

(a) If M_0 is active and $M_1 <_1 M_0$, then M_1 is either a segment of M_0, or $M_1 = \widehat{M_0}^{\text{passive}}$.

(b) If M_0 is passive and $M_1 <_1 M_0$, then M_1 is a segment of M_0.

(c) If M_0 and M_1 are passive and $M_2 <_1 M_1 <_1 M_0$, then $M_2 <_1 M_0$ and $M_2 < M_1 \triangleleft M_0$.[12]

Lemma 4.8. *The relation $<_1$ is well founded and set-like.*

[12] $M \triangleleft N$ means that M is a proper initial segment of N.

Proof. Assuming $<_1$ to be ill founded, let $\langle M_i \mid i<\omega\rangle$ be descending in $<_1$. Let $A=\{i \mid M_i \text{ is active }\}$. By Remark 4.7(b), A is unbounded in ω, since otherwise, if j were s.t. $A\subseteq j<\omega$, then $\langle M_{j+i} \mid i<\omega\rangle$ would be descending in $\in$. So let $a:\omega\longrightarrow A$ be the monotone enumeration of A. It is now easy to see that $\widehat{M_{a(i)}}\ni\widehat{M_{a(i+1)}}$, for all $i<\omega$:

If $a(i+1)=a(i)+1$, then by Remark 4.7 (a), $M_{a(i+1)}$ is a segment of $M_{a(i)}$, since obviously, $M_{a(i+1)}\neq\widehat{M_{a(i)}}^{\text{passive}}$; $M_{a(i+1)}$ is active. By coherency, $M_{a(i+1)}$ is a segment as well, and hence an element of $\widehat{M_{a(i)}}^{\text{passive}}$. But the latter structure is a model of ZFC^-, and so it follows by replacement that $\widehat{M_{a(i+1)}}\in\widehat{M_{a(i)}}^{\text{passive}}$ as well.

Otherwise, $a(i)+1<a(i+1)$. For $j\in[a(i)+1,a(i+1))$, M_j is passive, hence it follows from part (c) of Remark 4.7 that $M_{a(i)}>_1 M_{a(i)+1}>_1 M_{a(i+1)}$. Again, as $M_{a(i)+1}$ is passive, the fact that $M_{a(i+1)}<_1 M_{a(i)+1}$ entails that $M_{a(i+1)}$ is a proper segment of $M_{a(i)+1}$. Now there are two cases: If $M_{a(i)+1}$ is a proper segment of $M_{a(i)}$, then $M_{a(i+1)}$ is a proper segment of $M_{a(i)}$, and it follows as before that $\widehat{M_{a(i+1)}}\in\widehat{M_{a(i)}}$. Otherwise, $\widehat{M_{a(i)}}^{\text{passive}}=M_{a(i)+1}\ni M_{a(i+1)}$, and from this, it follows also that $\widehat{M_{a(i+1)}}\in\widehat{M_{a(i)}}$, as claimed. Thus, $\langle\widehat{M_{a(i)}} \mid i<\omega\rangle$ is a descending $\in$-sequence again, a contradiction.

This shows that $<_1$ is well founded. To see that it is set-like, note that if $P<_1 M$, then $P\in\bigcup_{\alpha\leq\text{ht}(M)}(\widehat{M||\alpha}\cup\{\widehat{M||\alpha}\})$. □

Definition 4.9. The function Λ is defined by $<_1$-recursion as follows.

Case 1: $N=\langle\emptyset,\emptyset,\emptyset\rangle$.

Then $\Lambda(N):=\langle\emptyset,\emptyset,\emptyset,\emptyset\rangle$.

Case 2: $N=\langle \mathrm{J}^E_{\alpha+1},\emptyset\rangle$.

Let $\Lambda(N||\alpha)=\langle \mathrm{J}^{E'}_{\alpha'},F,D\rangle$. Then set

$$\tilde{E}:=E'^\frown\langle\alpha',F\rangle=E'\cup\{\langle\alpha',\beta,X\rangle \mid \beta\in F(X)\},$$

and let $M':=\langle \mathrm{J}^{\tilde{E}}_{\alpha'+1},\emptyset\rangle$. Define $\Lambda(N):=\langle M',D_{M'}\rangle$ if this is a pPλ-structure. Otherwise, $\Lambda(N)$ remains undefined.

Case 3: $N=\langle \mathrm{J}^E_\alpha,F\rangle$*, where* $F\neq\emptyset$.

Then let $\widehat{N}=\langle \mathrm{J}^{E'}_{\alpha'},F'\rangle$ and $\bar{N}=\widehat{N}^{\text{passive}}$. Then $\Lambda(N)$ is defined if $\Lambda(\bar{N})$ is defined, and if the following conditions are satisfied:

(a) $|\bar{N}|=|\Lambda(\bar{N})|$.

(b) $\langle\Lambda(\bar{N}),F'\rangle=\widehat{\langle\Lambda(\bar{N}),F'\rangle}$.

In this case, letting $\Lambda(\bar{N})=\langle \mathrm{J}^{\tilde{E}}_{\tilde{\alpha}},\emptyset,D\rangle$, define $\Lambda(N):=\langle \mathrm{J}^{\tilde{E}}_{\tilde{\alpha}},F',D\rangle$.

Case 4: $N=\langle \mathrm{J}^E_\alpha,\emptyset\rangle$, where α is a limit ordinal.

Then $\Lambda(N)$ is defined, provided the following conditions are met:

(a) For all $\gamma<\alpha$, $\Lambda(N||\gamma)$ is defined and a potential Pseudo-λ-structure.

(b) For $\gamma<\delta<\alpha$, $\Lambda(N||\gamma)$ is a segment of $\Lambda(N||\delta)$.

If this is the case, let $M':=\bigcup_{\gamma<\alpha}\Lambda(N||\gamma)$, in the obvious sense, i.e., $|M'|=\bigcup_{\gamma<\alpha}|\Lambda(N||\gamma)|$, $E^{M'}=\bigcup_{\gamma<\alpha}E^{\Lambda(N||\gamma)}$, and $M':=\langle|M'|,E^{M'}\rangle$. Set: $\Lambda(N):=\langle M',\emptyset,D_{\langle M',\emptyset\rangle}\rangle$.

Remark 4.10. If $\Lambda(N)=M$ is defined, then M is a pPλ-structure.

Lemma 4.11. *Let M be a pPλ-structure for which $\mathtt{S}(M)$ is defined. Then $\Lambda(\mathtt{S}(M))$ is defined, too, and $\Lambda(\mathtt{S}(M)) = M$. In particular, $\mathtt{S}$ is injective.*

Proof. The proof is by $<_0$-induction on M.

Case 1: $M = \langle\emptyset,\emptyset,\emptyset,\emptyset\rangle$.

Then $\mathtt{S}(M) = \langle\emptyset,\emptyset,\emptyset\rangle$, and by definition of Λ, $\Lambda(\langle\emptyset,\emptyset,\emptyset\rangle) = M$.

Case 2: $M = \langle \mathrm{J}^E_{\mu+1}, \emptyset, D_M\rangle$.

Then let $\bar{M} := M||\mu$. Since $\mathtt{S}(M)$ is defined, so is $\mathtt{S}(\bar{M})$. Moreover, by definition of $\mathtt{S}$, $\mathtt{S}(M) = \mathtt{S}(\bar{M}) + 1$. Inductively, $\Lambda(\mathtt{S}(\bar{M})) = \bar{M}$. Hence, $\Lambda(\mathtt{S}(M)) = \Lambda(\mathtt{S}(\bar{M}) + 1) = \bar{M} + 1 = M$, since this clearly is a pPλ-structure.

Case 3: $M = \langle \mathrm{J}^E_{\mu}, F, D_M\rangle$, where $F \neq \emptyset$.

Let $\bar{M} := M^{\text{passive}} <_0 M$. Since $\mathtt{S}(M)$ is defined, so is $\mathtt{S}(\bar{M})$. Noting that $\mathrm{ht}(\bar{M}) = \mathrm{ht}(\mathtt{S}(\bar{M}))$, it follows inductively that $\Lambda(\mathtt{S}(\bar{M})) = \bar{M}$. By Definition 4.1, Case 3, (c), $\widehat{\mathtt{S}(M)} = \langle \mathtt{S}(\bar{M}), F\rangle$. So according to the definition of Λ, $\Lambda(\mathtt{S}(M))$ is defined, and

$$\Lambda(\mathtt{S}(M)) = \langle \Lambda(\widehat{\mathtt{S}(M)}^{\text{passive}})^-, E^{\widehat{\mathtt{S}(M)}}_{\text{top}}, D_{\Lambda(\mathtt{S}(\bar{M}))}\rangle = \langle M, F, D_M\rangle = M.$$

Case 4: $M = \langle \mathrm{J}^E_{\mu}, \emptyset, D_M\rangle$, where μ is a limit.

Then $\mathtt{S}(M||\alpha)$ is defined for every $\alpha < \mathrm{ht}(M)$. Set:

$$\begin{aligned}\tilde{D} := \{\zeta < \mathrm{On}_M \ \mid \ &(\zeta = 0 \vee \mathrm{Lim}(\zeta)) \wedge \\ &\neg(\exists \delta \leq \mathrm{ht}(M) \quad E^M_\delta \neq \emptyset \wedge s^+(\delta)^M \leq \zeta < \delta)\}\end{aligned}$$

Let $\langle \tilde{\eta}_\xi \mid \xi < \mathrm{otp}(\tilde{D})\rangle$ be the monotone enumeration of $\tilde{D}$. For $\xi < \mathrm{otp}(\tilde{D})$, define $\bar{\eta}_\xi$ by $\omega\bar{\eta}_\xi = \tilde{\eta}_\xi$. Then it follows by $<_0$-induction on M that:

- $\mathrm{ht}(\mathtt{S}(M)) = \mathrm{otp}(\tilde{D})$.
- For $\xi < \mathrm{otp}(\tilde{D})$, $\mathtt{S}(M)||\xi = \mathtt{S}(M||\bar{\eta}_\xi)$.
- $\tilde{D}$ is cofinal in On_M (like the proof of Lemma 3.17).

So by induction hypothesis, for all $\xi < \mathrm{ht}(\mathtt{S}(M))$, $\Lambda(\mathtt{S}(M)||\xi) = \Lambda(\mathtt{S}(M||\bar{\eta}_\xi)) = M||\bar{\eta}_\xi$ is defined. Hence $\Lambda(\mathtt{S}(M))$ is defined, and $\Lambda(\mathtt{S}(M)) = \bigcup_{\xi<\mathrm{otp}(\tilde{D})} \Lambda(\mathtt{S}(M)||\xi) = \bigcup_{\xi<\mathrm{otp}(\tilde{D})} M||\bar{\eta}_\xi = M$. □

Towards formulating the converse of this, let's define:

Definition 4.12. For a pPλ- or pPs-structure M, set $I_M := \{\nu \mid E^M_{\omega\nu} \neq \emptyset\}$.

Lemma 4.13. *Let N be a pPs-structure for which $\Lambda(N)$ is defined. Then $\mathtt{S}(\Lambda(N))$ is defined as well, and $\mathtt{S}(\Lambda(N)) = N$. In particular, Λ is injective.*

Proof. The proof is by $<_1$-induction on N, analogous to 4.11. The base and successor cases are as unproblematic as before, so I restrict attention to the limit case. Let $N = \langle \mathrm{J}^E_\alpha, \emptyset\rangle$, where α is a limit. Using a slightly sloppy notation, we then have $\Lambda(N) = \bigcup_{\nu<\alpha} \Lambda(N||\nu)$. Let $M = \Lambda(N)$.

($*$) *If $\mu \in \mathrm{ht}(N) \setminus I_N$, then $\omega \cdot \mathrm{ht}(\Lambda(N||\mu)) \in D_M$.*

Proof of ($*$). $\Lambda(N||\mu) = M||\mu'$ for some $\mu' < \mathrm{ht}(M)$ by Case 4(b) of definition 4.9. By induction hypothesis, $\mathtt{S}(M||\mu') = N||\mu$ is a segment of N. Hence, by Lemma 4.3, there is no $\nu \leq \mathrm{ht}(M)$ with $s^+(M||\nu) \leq \omega\mu' < \nu$. But as $\mu' \notin I_M$, it follows that $\omega\mu' \in D_M$: Otherwise there would be a $\nu' \in I_M$ with $s^+(M||\nu') < \omega\mu' \leq \nu'$, that is, $s^+(M||\nu') < \omega\mu' < \nu'$, and such a ν' doesn't exist, as was pointed out. $\square_{(*)}$

Hence $\mathtt{S}(\Lambda(N)) = \mathtt{S}(M) = \bigcup_{\omega\mu\in D_M} \mathtt{S}(M||\mu^{\text{passive}}) = \bigcup_{\mu\in\mathrm{ht}(N)\setminus I_N} \mathtt{S}(\Lambda(N||\mu)) = N$. □

Definition 4.14. Let $\mathtt{S}(M)$ be defined. Then

$$\mathtt{S}_0(M) := \mathrm{ht}(\mathtt{S}(M)) \quad \text{and} \quad \hat{\mathtt{S}}_0(M) := \mathrm{ht}(\widehat{\mathtt{S}(M)}).$$

Lemma 4.15. *Let $\mathtt{S}(M)$ be defined. Then the following hold:*

(a) If $I_M \cap \mathrm{ht}(M)$ is unbounded in $\mathrm{ht}(M)$, or if M is active, then

$$|\mathtt{S}(M)| = |\mathrm{J}^{E^M}_{\mathtt{S}_0(M)}|, \text{ and } |\widehat{\mathtt{S}(M)}| = |\mathrm{J}^{E^M}_{\hat{\mathtt{S}}_0(M)}| = |M|.$$

(b) If M is passive and I_M is bounded in $\mathrm{ht}(M)$, then either $I_M = \emptyset$ and $|M| = |\mathtt{S}(M)| = |\widehat{\mathtt{S}(M)}|$, or, letting $\bar{\nu} = \sup I_M$,

$$|\mathtt{S}(M||\bar{\nu})| = |\mathtt{S}(M)||\mathtt{S}_0(M||\bar{\nu})| = |\mathrm{J}^{E^M}_{\mathtt{S}_0(M||\bar{\nu})}|.$$

(c) If M is a model of KP, then $|\mathtt{S}(M)| = |M|$.

(d) If $\mathtt{S}(M)$ is a model of KP, then M is $\Sigma_1(\mathtt{S}(M))$.[13] *Moreover, $\Lambda\restriction <_1$"$\{\mathtt{S}(M)\}$ is (uniformly) $\Sigma_1(\mathtt{S}(M))$.*

Proof. The proof is by $<_0$-induction on M again. Let $N = \mathtt{S}(M)$.

If M satisfies (a)-(c), then it is fairly easy to see that (d) holds as well: If $N = \mathtt{S}(M)$ is a model of KP then N is passive, and $\mathrm{ht}(N)$ is a limit (or equal to 1, in which case there is nothing to show). That die the restriction of Λ to the set of $<_1$-predecessors of N can be defined over N in a Σ_1 way follows from the observation that the map $x \mapsto (<_1$"$\{x\})$ is a $\Sigma_1(N)$ function, and from the fact that the recursion in Definition 4.9 is $\Sigma_1(N)$, too. I use the recursion theorem for KP as stated in [Bar75].

Thus, $\Lambda(N)^-$ is $\Sigma_1(N)$, too, which is obvious from Definition 4.1, Case 4. In order to define D_M in N in a Σ_1 way, note that for $\alpha < \mathrm{ht}(N)$ the structure $\Lambda(N||\alpha^{\mathrm{passive}})$ is a segment of M, so that $\omega\mathrm{ht}(\Lambda(N||\alpha^{\mathrm{passive}})) \in D_M$. Hence, using the coherency of enhancements, we get the following Σ_1 definition:

$$\xi \in D_M \iff \exists \alpha < \mathrm{ht}(N) \quad \xi \in D_{\Lambda(N||\alpha^{\mathrm{passive}})}.$$

I now prove that M satisfies (a)-(c) as well, inductively assuming that all $<_1$-predecessors of M satisfy (a)-(d).

Case 1: $M = \langle \emptyset, \emptyset, \emptyset \rangle$.

Trivial.

Case 2: $M = \langle \mathrm{J}^{E^M}_{\bar{\nu}+1}, \emptyset, D_M \rangle$.

Inductively, the claims hold for $M||\bar{\nu}$ already. As I_M is bounded in $\mathrm{ht}(M)$ and M is passive, and since M is not a model of ZF^-, (a) and (c) are vacuously true. So let $\tilde{\nu} = \sup I_M$. If $\tilde{\nu} = 0$, and hence $I_M = \emptyset$, then the claim is obvious again. So let $\tilde{\nu} > 0$. Now $I_{M||\tilde{\nu}}$ is unbounded in $\tilde{\nu}$, or $M||\tilde{\nu}$ is active, and I can make use of the inductive hypothesis that (a) holds for $M||\tilde{\nu}$. If $M||\tilde{\nu}$ is passive, then

$$|\mathtt{S}(M||\tilde{\nu})| = |\widehat{\mathtt{S}(M||\tilde{\nu})}| = |M||\tilde{\nu}|, \text{ and } \mathtt{S}_0(M||\tilde{\nu}) = \tilde{\nu}.$$

If $M||\tilde{\nu}$ is active, then

$$|\mathtt{S}(M||\tilde{\nu})| = |M||s^+(M||\tilde{\nu})|, \text{ and } \mathtt{S}_0(M||\tilde{\nu}) = s^+(M||\tilde{\nu}).$$

[13] This is to say that $|M|$, E^M, E^M_{top}, D_M are $\Sigma_1(\mathtt{S}(M))$.

So I have to show that $|\mathtt{S}(M)||\mathtt{S}_0(M||\tilde{\nu})| = |\mathtt{S}(M||\tilde{\nu})|$. But this is clear, since by Lemma 4.3, $\mathtt{S}(M||\tilde{\nu})$ is a segment of $\mathtt{S}(M)$.

Case 3: $M = \langle \mathrm{J}_\nu^{E^M}, F, D_M \rangle$, where $F \neq \emptyset$.

Then the second part of (a) follows by Remark 4.2, part 1. But the first part follows as well, because $|\mathtt{S}(M)| = |\widehat{\mathtt{S}(M)}||s^+(M)|$, hence $\mathtt{S}_0(M) = s^+(M)$. Let $s^+ := s^+(M)$. Since s^+ is a cardinal in M, this is true in $\widehat{\mathtt{S}(M)}$, too. It follows that $|\widehat{\mathtt{S}(M)}||s^+| = H_{s^+}^{\widehat{\mathtt{S}(M)}} = H_{s^+}^M = |M||s^+|$. Claims (b) and (c) are vacuously true for M.

Case 4: $M = \langle \mathrm{J}_\nu^{E^M}, \emptyset, D_M \rangle$, where ν is a limit ordinal.

If I_M is bounded in ν, (b) is shown just as in Case 2, and for (a), nothing is to be shown. I prove (c). If $I_M = \emptyset$, the claim is trivially true. So let $I_M \neq \emptyset$ and $\tilde{\nu} = \sup I_M$. If $\tilde{\nu} < \mathrm{ht}(M)$, then by (b),

$$|\mathtt{S}(M||\tilde{\nu})| = |M||\mathtt{S}_0(M||\tilde{\nu})|.$$

Since in M there are no extender indices above $\tilde{\nu}$, it's easy to see that for $1 \leq \delta < \nu - \tilde{\nu}$,

$$\mathtt{S}(M||(\tilde{\nu} + \delta)) = \langle \mathrm{J}^{E'}_{\mathtt{S}_0(M||\tilde{\nu})+\delta}, \emptyset \rangle,$$

where I let $\mathtt{S}(M||\tilde{\nu}) = \langle \mathrm{J}^{E'}_{\mathtt{S}_0(M||\tilde{\nu})}, E'_{\omega\mathtt{S}_0(M||\tilde{\nu})} \rangle$.

$(*)$ $|\mathrm{J}^{E'}_{\mathtt{S}_0(M||\tilde{\nu})+\delta}| \subseteq |\mathrm{J}^{E^M}_{\tilde{\nu}+\delta}| \subseteq |\mathrm{J}^{E'}_{\tilde{\nu}+\delta+1}|$.

Proof of $(*)$. The first inclusion follows from the fact that $|\mathtt{S}(M||\gamma)| \subseteq |M||\gamma|$ (see the proof of Lemma 5.15). For the second inclusion, I distinguish two cases:

If $\tilde{\nu} \notin I_M$, then by (a),

$$|\mathtt{S}(M||\tilde{\nu})| = |\widehat{\mathtt{S}(M||\tilde{\nu})}| = |M||\tilde{\nu}|,$$

i.e., $E^M \subseteq |\mathtt{S}(M||\tilde{\nu})|$ and $E^{\mathtt{S}(M)} \subseteq |M||\tilde{\nu}|$. By (d), E^M is even $\Sigma_1(\mathtt{S}(M||\tilde{\nu}))$, hence $E^M \in N||\tilde{\nu}+1$. In fact, it follows by induction on $\delta < \nu$ that $|\mathrm{J}^{E'}_{\tilde{\nu}+\delta}| = |\mathrm{J}^{E^M}_{\tilde{\nu}+\delta}|$.

Now let $\tilde{\nu} \in I_M$. It has to be checked that $E^M \in |N||\tilde{\nu}|$. To this end, let $\beta = \tau(M||\tilde{\nu})^{++M||\tilde{\nu}}$. Then $|M||\beta| = |\mathtt{S}(M||\beta)| = |\mathtt{S}(M||\tilde{\nu})||\beta|$, as $\beta \leq s^+(M||\tilde{\nu})$. So $M||\tau \in |N||\tilde{\nu}|$. By coherency of $M||\tilde{\nu}$, this means that $M||\tilde{\nu}^{\mathrm{passive}} = \mathrm{Ult}(M||\tau, E^M_{\tilde{\nu}}) \in N||\tilde{\nu}+1$, since $E_{\mathrm{top}}^{\mathtt{S}(M||\tilde{\nu})}$ codes $E^M_{\tilde{\nu}}$ and is an element of $|N||\tilde{\nu}|$. The rest of the claim follows by induction on δ. $\square_{(*)}$

Now let M be a model of KP. Then $\nu - \tilde{\nu} = \nu$ and $\mathtt{S}_0(M||\tilde{\nu}) + \nu = \nu$. Hence, $\mathtt{S}_0(M) = \mathrm{ht}(M) = \nu$, since

$$|\mathtt{S}(M)| = \bigcup_{1 \leq \delta < \nu - \tilde{\nu}} |\mathtt{S}(M||\tilde{\nu} + \delta)| = \bigcup_{1 \leq \delta < \nu - \tilde{\nu}} |\mathrm{J}^{E'}_{\mathtt{S}_0(M||\tilde{\nu})+\delta}| = |\mathrm{J}^{E'}_\nu|.$$

So it follows from $(*)$ that $|M| = \mathtt{S}(M)| = |\widehat{\mathtt{S}(M)}|$, which proves (c).

Now let I_M be unbounded in $\mathrm{ht}(M)$. I have to prove (a) and (c). In the current case it suffices to prove (a), though, because M is passive, and so $\mathtt{S}(M) = \widehat{\mathtt{S}(M)} = |M|$ follows from (a). So I have to show $|\mathtt{S}(M)| = |M|$. To this end, I show that for every $\alpha < \nu$ there exists an $\alpha' < \nu$ s.t. $\alpha \leq \alpha'$ and $|M||\alpha'| = |\mathtt{S}(M)||\alpha'|$. So let $\alpha < \nu$ be given. By Lemma 3.17, D_M is unbounded in ν, so let $\omega\beta \in D_M \setminus \alpha$. It follows that

$(+)$ $\forall \gamma \in I_M \setminus (\beta + 1) \quad s^+(\gamma)^M \geq \omega\beta$,

as otherwise $s^+(\gamma)^M < \omega\beta < \gamma$ for some γ, and hence $\omega\beta$ is not in D_M.

Now let $\gamma \in I_M \setminus (\beta + 1)$ be the unique γ with

$$s^+(\gamma)^M = \min\{s^+(\delta)^M \mid \delta \in I_M \setminus (\beta + 1)\}.$$

Then $s^+(\gamma) \in D_M$ by choice of γ, and by (+), $\alpha < \omega\beta \le s^+(\gamma)$. Now $s^+(\gamma)^M$ is a successor cardinal in the ZF^- model $\bar{M} := \langle \mathrm{J}_\gamma^{E^M}, \emptyset, D_{\bar{M}} \rangle$, and hence $M||s^+(\gamma)^M$ is a ZF^- model as well, if $s^+(\gamma)^M < \gamma$. Hence in this case, as (c) is true for $M||s^+(\gamma)^M$, and $s^+(\gamma)^M \in D_M$,

$$|M||s^+(\gamma)^M| = |\mathtt{S}(M||s^+(\gamma)^M)| = |\mathtt{S}(M)||s^+(\gamma)^M|.$$

On the other hand, if $s^+(\gamma) = \gamma$, then the above follows from (a) and from the fact that in this case $\gamma \in D_M$, since then $|\mathtt{S}(M||\gamma)| = |\widehat{\mathtt{S}(M||\gamma)}| = |M||\gamma|$ (the last identity holds by (a)). □

Lemma 4.16. *Let M be a pPλ-structure, for which $N = \mathtt{S}(M)$ exists. If M is active, then $C_M = C_{\widehat{N}} = C_N$. Moreover, $\widehat{N}$ and N satisfy the s'-ISC iff M does.*

Proof. Let's turn to the first part. So let M be active.

(1) *For $\alpha < s(M)$, $|[M]_\alpha| = |[\widehat{N}]_\alpha| = |[N]_\alpha|$.*

Proof of (1). Let $\tau = \tau(M)$. Then $\tau = \tau(N) = \tau(\widehat{N})$, and as $N||\tau = \mathtt{S}(M||\tau)$, it follows by Lemma 4.15 that $|N||\tau| = |\mathtt{S}(M||\tau)| = |M||\tau|$. Since moreover $E^{\widehat{N}}_{\mathrm{top}} = E^M_{\mathrm{top}}$, the first identity is immediate, as for the definition of $[M]_\alpha$ and $[N]_\alpha$ only Σ_0-extender ultrapowers were used. But $(E^N_{\mathrm{top}})^{\mathtt{f}} = E^{\widehat{N}}_{\mathrm{top}}|s^+(M)$, hence the second identity follows as well. $\square_{(1)}$

Now I can show directly that $C_M = C_{\widehat{N}}$: Let $\xi \in C_M$. Then $\tau \le \xi < s(M)$, ξ is a cutpoint of E^M_{top}, and $[M]_\xi$ satisfies the s'-MISC. As $E^M_{\mathrm{top}} = E^{\widehat{N}}_{\mathrm{top}}$, ξ is a cutpoint of $E^{\widehat{N}}_{\mathrm{top}}$, too. It remains to be shown that $[\widehat{N}]_\xi$ satisfies the s'-MISC. To this end, let $\zeta \in [\tau([\widehat{N}]_\xi), \xi)$ be a cutpoint of $E^{[\widehat{N}]_\xi}_{\mathrm{top}} = E^{[M]_\xi}_{\mathrm{top}}$. Since $[M]_\xi$ satisfies the s'-MISC, it follows that $(\zeta^+)^{[M]_\zeta} < (\zeta^+)^M$. But since $|M| = |\widehat{N}|$, keeping (1) in mind, it follows that

$$(\zeta^+)^{[\widehat{N}]_\zeta} = (\zeta^+)^{[M]_\zeta} < (\zeta^+)^M = (\zeta^+)^{\widehat{N}}.$$

The inclusion $C_{\widehat{N}} \subseteq C_M$ is proven entirely analogously, and that $C_N = C_{\widehat{N}}$ can be shown in the same way, making use of acceptability of $\widehat{N}$.

I'm left to show that the s'-ISC carries over from M to N and $\widehat{N}$, and vice versa. Since both directions can be treated in the same way, I just prove the direction from M to N and $\widehat{N}$. To see that it carries over to $\widehat{N}$, assume the contrary. Let M be a counterexample of minimal height. Obviously then M is active. Let $N = \mathtt{S}(M)$. All proper segments of $\widehat{N}$ are $\mathtt{S}$-images of proper segments of M, and thus satisfy the s'-ISC. It suffices therefore to verify those parts of the s'-ISC which refer to $\widehat{N}$. So let $\xi \in C_N$ be given. Then by the first part, $\xi \in C_M$, too. And as M satisfies the s'-ISC, it follows that $[M]_\xi \in |M|$. Hence $E^{\widehat{N}}_{\mathrm{top}}|\xi = E^M_{\mathrm{top}}|\xi \in |M| = |N|$. But in $\widehat{N}^{\mathrm{passive}}$, $[\widehat{N}]_\xi$ is definable from $E^{\widehat{N}}_{\mathrm{top}}|\xi$, and hence, $[\widehat{N}]_\xi \in |\widehat{N}|$. Further, I have to show that $[\widehat{N}]_{\xi'} \in |[\widehat{N}]_\xi|$, given that $\tau \le \xi' < \xi$ and $[\widehat{N}]_{\xi'}$ satisfy the s'-MISC. From (1) it follows that in this case, $[M]_{\xi'}$ satisfies the s'-MISC as well. Thus, $[M]_{\xi'} \in |[M]_\xi|$, i.e., $E^{\widehat{N}}_{\mathrm{top}}|\xi' = E^M_{\mathrm{top}}|\xi' \in |[M]_\xi| = |[\widehat{N}]_\xi|$ (the latter again by (1)), and this means that $[\widehat{N}]_{\xi'} \in |[N]_\xi|$, as before. Hence, $\widehat{N}$ satisfies the s'-ISC. That N does too can be shown analogously. Hence, M was no counterexample after all, so there are none. □

5 Translating Σ_1-formulae

In order to be able to accurately analyze the relationship between projecta and standard parameters in a pPλ structure and its $\mathtt{S}$-image (if existent), a deep understanding of Σ_1-definability in these structures is essential. Such an analysis has to be undertaken in order to derive one of the main results of this work, namely that the (Pseudo)-λ-structure has an $\mathtt{S}$-image, which is a (Pseudo)-s-structure, and vice versa. I shall develop a method for translating Σ_1-formulae in this section.

5.1 Successor levels of premice

If M is a pPλ- or pPs-structure of height $\alpha+1$, then the first step for translating Σ_1-formulae will be to express that formula over the structure $M||\alpha$; the translation procedure will then be defined by recursion. So names for the members of $M||(\alpha+1)$ are needed:

Definition 5.1. Let $\vec{\dot{A}} = \dot{A}_1, \ldots, \dot{A}_l$ be a list of predicate symbols. Since I shall be working with transitive structures that are closed under ordered pairs, one may restrict to *unary predicate symbols*. The set $\mathfrak{C}(\vec{\dot{A}})$ of codes for functions rudimentary in $\vec{\dot{A}}$ is defined by the following clauses.

(a) For all $n \in \omega \setminus \{0\}$ and $k, l < n$, the following symbols are codes for an n-ary function rudimentary in $\vec{\dot{A}}$: $\pi^n_k, p^n_{k,l}, \delta^n_{k,l}$.

(b) The symbol $f_{\dot{A}_k}$ is a code for a 1-ary function rudimentary in $\vec{\dot{A}}$ ($1 \leq k \leq l$).

(c) If f is a code for an n-ary function rudimentary in $\vec{\dot{A}}$, then so is $u^n[f]$.

(d) If h is a code for an m-ary function rudimentary in $\vec{\dot{A}}$ and $h_0, \ldots, h_{m-1}$ are codes for n-ary functions rudimentary in $\vec{\dot{A}}$, then $h \circ (h_0, \ldots, h_{m-1})$ is a code for an n-ary function rudimentary in $\vec{\dot{A}}$ ($m, n \geq 1$).

Let's turn to the interpretation of such codes. Fix sets (or classes) $\vec{A} := A_1, \ldots, A_l$. Given a code t for an n-ary function in $\mathfrak{C}(\vec{\dot{A}})$, I define its interpretation, $\mathtt{val}^{\vec{A}}[t] : \mathrm{V}^n \longrightarrow \mathrm{V}$ by recursion on t as follows.

(a) Let $n \in \omega \setminus \{0\}$, $k, l < n$.

  (1) $\mathtt{val}^{\vec{A}}[\pi^n_k](a_0, \ldots, a_{n-1}) = a_k$.

  (2) $\mathtt{val}^{\vec{A}}[p^n_{k,l}](a_0, \ldots, a_{n-1}) = \{a_k, a_l\}$.

  (3) $\mathtt{val}^{\vec{A}}[\delta^n_{k,l}](a_0, \ldots, a_{n-1}) = a_k \setminus a_l$.

(b) $\mathtt{val}^{\vec{A}}[f_{\dot{A}_k}](a) = A_k \cap a$ ($1 \leq k \leq l$).

(c) Let f be a code for an n-ary function rudimentary in $\vec{\dot{A}}$ for which $\mathtt{val}^{\vec{A}}[f]$ has been defined already. Then $\mathtt{val}^{\vec{A}}[u^n[f]](a_0, \ldots, a_{n-1}) = \bigcup_{b \in a_0} \mathtt{val}^{\vec{A}}[f](b, a_1, \ldots, a_{n-1})$.

(d) Let h be a code for an m-ary function rudimentary in $\vec{\dot{A}}$, and let $h_0, \ldots, h_{m-1}$ be codes for n-ary functions rudimentary in $\vec{\dot{A}}$, such that $\mathtt{val}^{\vec{A}}[h]$ and $\mathtt{val}^{\vec{A}}[h_0], \ldots, \mathtt{val}^{\vec{A}}[h_m]$ have already been defined. Then, for $\vec{a} = a_0, \ldots, a_{n-1}$,

$$\mathtt{val}^{\vec{A}}[h \circ (h_0, \ldots, h_{m-1})](\vec{a}) = \mathtt{val}^{\vec{A}}[h](\mathtt{val}^{\vec{A}}[h_0](\vec{a}), \ldots, \mathtt{val}^{\vec{A}}[h_{m-1}](\vec{a})).$$

In order to avoid a possible confusion, since there are conflicting definitions in the literature, it should be pointed out that **by** $\mathrm{rud}_{\vec{A}}(X)$ **I mean the closure of** $X \cup \{X\}$ **under functions rudimentary in** $\vec{A}$. That's what I refer to as the $\vec{A}$-rudimentary closure of X. So every element of $\mathrm{rud}_{\vec{A}}(X)$ is of the form $f(\vec{a}, X)$, where f is a function rudimentary in $\vec{A}$ and $\vec{a} \in X$. This is the motivation for the following two definitions which basically introduce names for the members of $\mathrm{rud}_{\vec{A}}(X)$.

Definition 5.2. Fix predicate symbols $\dot{\vec{A}}$. The set $\mathfrak{T}(\dot{\vec{A}})$ of terms rudimentary in $\dot{\vec{A}}$ is defined to consist of pairs $t = \langle c, \langle \vec{x} \rangle \rangle$, where $c \in \mathfrak{C}(\dot{\vec{A}})$ is a code for an n-ary function and $\vec{x} = \langle x_0, \ldots, x_{n-1} \rangle$ is an n-tuple, such that, for $i < n$, either x_i is a variable, or $x_i = \Phi$ for a fixed new constant symbol Φ. The set of free variables of t, $\mathrm{Fr}(t)$ is defined to be $\{x_i \mid x_i \neq \Phi\}$.

Evaluations of rudimentary terms are now computed relative to a given interpretation of the predicate symbols and a given interpretation of a universe.

Definition 5.3. I evaluate a rudimentary term $t = \langle c, \langle x_0, \ldots, x_{n-1} \rangle \rangle \in \mathfrak{T}(\dot{\vec{A}})$ in a structure $M = \langle X, \vec{A} \rangle$ as follows. Let a be an assignment in X whose domain contains the free variables of t. Define an extension $\tilde{a}$ of a by setting:

$$\tilde{a}(x) = \begin{cases} a(x) & \text{if } x \neq \Phi, x \in \mathrm{dom}(a), \\ X & \text{if } x = \Phi. \end{cases}$$

Then $\mathtt{val}^M[t](a) := (\mathtt{val}^{\vec{A}}[c])(\tilde{a}(x_0), \ldots, \tilde{a}(x_{n-1}))$. If $\tilde{M} = \langle M, \vec{B} \rangle$ is a structure enhanced by additional predicates, then $\mathtt{val}^{\tilde{M}}[t](a) = \mathtt{val}^M[t](a)$.

The following Lemma is from [Fuc09]. It applies almost immediately if $M||\alpha$ is passive, since in that case, $E^M \subseteq |M||\alpha|$.[14]

Lemma 5.4. *Fix two lists of predicate symbols,* $\dot{\vec{A}}$ *and* $\dot{\vec{B}}$*. Then there is a recursive function* $T = T_{\dot{\vec{A}};\dot{\vec{B}}}$ *with the following property:*

Let $\vec{A}$ *and* $\vec{B}$ *be interpretations of* $\dot{\vec{A}}$ *and* $\dot{\vec{B}}$*. Let* X *be a transitive set closed under functions rudimentary in* $\vec{A}$*, and let* $\vec{A}, \vec{B} \subseteq X$*. Set* $X' = \mathrm{rud}_{\vec{A}}(X)$*, and define* $M := \langle X, \vec{A}, \vec{B} \rangle$*,* $M' := \langle X', \vec{A}, \vec{B} \rangle$*. Let* φ *be a* Σ_0*-formula with free variables* $v_0, \ldots, v_{n-1}$*. Let* $a = \{i_0, \ldots, i_{m-1}\} \in [n]^m$*. For each* $j < m$*, let* $t_j \in \mathfrak{T}(\dot{\vec{A}})$*, such that no free variable of* t_j *occurs as a bound variable in* φ*.*

Then $\psi := T(\varphi, v_{i_0}, t_0, \ldots, v_{i_{m-1}}, t_{m-1})$ *is a* Σ_ω*-formula with the following property: If* $\vec{w} = w_0, \ldots, w_{m'-1}$ *is an enumeration of* $\{v_k \mid k \in n \setminus a\}$*, then the set of free variables of* ψ *is contained in* $\{\vec{w}\} \cup \bigcup_{j<m} \mathrm{Fr}\,(t_j)$ *(here, repetitions may occur). Further, for any assignment* b *of the free variables of* ψ *with values in* X*,*

$$M' \models \varphi[b'] \iff M \models \psi[b],$$

where $b' = b[({}^{v_{i_0}}/_{\mathtt{val}^M[t_0](b)}), \ldots, ({}^{v_{i_{m-1}}}/_{\mathtt{val}^M[t_{m-1}](b)})]$*. Hence, one might very well write:*

$$\psi = \varphi(({}^{v_{i_0}}/_{t_0}), \ldots, ({}^{v_{i_{m-1}}}/_{t_{m-1}})).$$

If α indexes an extender in M, a more specialized translation function, which I shall develop later, is called for. In preparation of a version which is suitable for pPλ-structures, I note the following.

[14] For pPλ-structures M, the predicate D_M deserves extra attention, though.

Lemma 5.5. *Let $M = \mathrm{J}^E_{\alpha+1}$ be a pPλ-structure, so that $F := E_\alpha \neq \emptyset$. Then $|M|$ is the closure of $|M||\alpha| \cup \{|M||\alpha|\}$ under rudimentary functions, the function $f_{E\restriction\alpha}$ and the function f_F. Here, f_Z denotes the function $x \mapsto Z \cap x$.*

Proof. Let A be the closure of $|\mathrm{J}^E_\alpha| \cup \{|\mathrm{J}^E_\alpha|\}$ under rudimentary functions and the function f_E. Hence $A = |M|$. Let B =the closure of $|\mathrm{J}^E_\alpha| \cup \{|\mathrm{J}^E_\alpha|\}$ under rudimentary functions, the function $f_{E\restriction\alpha}$ and the function f_F. It has to be shown that $A = B$.

$\boxed{A \subseteq B}$ Obviously it suffices to show that B is closed under f_E. I show that $E \in B$, from which this follows. We have:

$$E = E\restriction\alpha \cup \underbrace{\{\langle\alpha,\delta,x\rangle \mid \delta \in F(x)\}}_{F'}.$$

Since $E\restriction\alpha \subseteq |M||\alpha|$, it follows that $E\restriction\alpha = f_{E\restriction\alpha}(|\mathrm{J}^E_\alpha|) \in B$. So it suffices to show that $F' \in B$. In order to see this, I will define a series of obviously rudimentary functions, which I will use to get a rud function, which, when applied to the right elements of B, takes the value F'. Set

$$f_0(a,\delta,y,x,c) := \begin{cases} \{\langle\delta,y,x\rangle\} & \text{if } \delta \in y \wedge \langle y,x\rangle \in c, \\ \emptyset & \text{otherwise.} \end{cases}$$

This is a definition by cases, and the relation determining the case is rud, hence so is the function - see [Jen72, p. 234, Properties 1.2.(e) and 1.1.(c),(d)]. Note that a is but a "dummy"-argument. Now set

$$f_1(a,c) := \bigcup_{\delta,y,x\in a} f_0(a,\delta,y,x,c).$$

Again, f_1 is rud by the last scheme in [Jen72, P. 233, Definition in §1]. The function

$$f_2(z) := \langle (z)^3_0, (z)^3_2\rangle$$

is rud by [Jen72, P.234, Properties 1.3.(a) & 1.1.(d)]. Now we have in general that if g is rud, then so is the function $x \mapsto g``x$: $g``x = \bigcup_{y\in x}\{g(y)\}$. Hence, I can define:

$$f_3(x) := f_2``x,$$

in order to get yet another rud function. Now set $f_4 = f_3 \circ f_1$. Then:

$$\begin{aligned} f_4(|\mathrm{J}^E_\alpha|, F) &= f_3(f_1(|\mathrm{J}^E_\alpha|, F)) \\ &= f_2``f_1(|\mathrm{J}^E_\alpha|, F) \\ &= f_2``(\bigcup_{\delta,y,x\in|\mathrm{J}^E_\alpha|} f_0(|\mathrm{J}^E_\alpha|,\delta,y,x,F)) \\ &= f_2``\{\langle\delta,y,x\rangle \mid \delta \in y \wedge \langle y,x\rangle \in F\} \\ &= f_2``\{\langle\delta,y,x\rangle \mid \delta \in y = F(x)\} \\ &= G := \{\langle\delta,x\rangle \mid \delta \in F(x)\}. \end{aligned}$$

Hence $G \in B$ and $E\restriction\alpha \in B$, and hence, $E = (\{\alpha\} \times G) \cup E\restriction\alpha \in B$, which was to be shown.

$\boxed{B \subseteq A}$ For this direction, I am going to show that A is closed under $f_{E\restriction\alpha}$ and f_F. The former is trivial, as $E\restriction\alpha \cap z = E \cap |\mathrm{J}^E_\alpha| \cap z \in A$ for $z \in A$. In order to see the latter, I will show that $F \in A$. Again, I define some functions which are rud in E.

$$g_0(\gamma,\delta,x,u) := \begin{cases} \{\gamma\} & \text{if } \langle\delta,\gamma,x\rangle \in f_E(u), \\ \emptyset & \text{otherwise.} \end{cases}$$

Further:

$$g_1(\delta, u, x) := \bigcup_{\gamma \in u} g_0(\gamma, \delta, x, u).$$

Hence

$$g_1(\alpha, |\mathrm{J}^E_\alpha|, x) = \begin{cases} F(x) & \text{if defined,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Now set:

$$g_2(\delta, u, x) := \{\langle g_1(\delta, u, x), x\rangle\} \text{ and } g_3(\delta, u, v) := \bigcup_{x \in v} g_2(\delta, u, x).$$

Then g_3 is rud in E, and obviously, $g_3(\alpha, |\mathrm{J}^E_\alpha|, |\mathrm{J}^E_\alpha|) = F \in A$, as was to be shown. □

Lemma 5.6. *The function $x \mapsto \mathrm{On} \cap x$ is rud. Fix a rudimentary code c_{On} for this function.*

Proof. The function g, defined by

$$g(z) := \begin{cases} \{z\} & \text{if } z \in \mathrm{On} \\ \emptyset & \text{otherwise.} \end{cases}$$

is rud, since it is defined by cases, and the relation determining the case is Σ_0. But then $\mathrm{On} \cap x = \bigcup_{z \in x} g(z)$, which shows that this function is rud. □

Lemma 5.7. *There is a recursive function T_λ with the following property: Let $M' := \langle \mathrm{J}^E_{\alpha+1}, \emptyset, D_{M'}\rangle$ be a pPλ-structure. Set $D = D_{M'||\alpha}$. Let $M = M'||\alpha$, and set $\tilde{M} = \langle \mathrm{J}^E_{\alpha+1}, \emptyset, D\rangle$. Let φ be a Σ_0-formula in the language of set theory with additional predicate symbols $\dot{E}$, $\dot{F}$ and $\dot{D}$. Let $v_0, \ldots, v_{n-1}$ be the free variables of φ. Let $a = \{i_0, \ldots, i_{m-1}\}$ be an m-element subset of n. For each $j < m$, let c_j be a code for an n_j-ary function rud in $\dot{E}, \dot{F}$, and $x^j_0, \ldots, x^j_{n_j-1}$ a list of symbols so that each x^j_k is either a variable symbol or a fixed constant symbol Φ. Let $\psi := T_\lambda(\varphi, v_{i_0}, c_0, \langle \vec{x}^0\rangle, \ldots, v_{i_{m-1}}, c_{m-1}, \langle \vec{x}^{m-1}\rangle)$. Then ψ is a Σ_ω-formula s.t. the following holds:*

If $\vec{w} = w_0, \ldots, w_{m'-1}$ is an enumeration of $\{v_k \mid k \in n \setminus a\}$, then the set of free variables of ψ is contained in $\{\vec{w}\} \cup \{x^j_k \mid j < m \wedge k < n_j \wedge x^j_k \neq \Phi\}$ (here, repetitions are allowed).

Let b be an assignment of the free variables of ψ with values from $|M|$. Define $b' : \mathrm{Fr}\ (\psi) \cup \{\Phi\} \to |M| \cup \{|M|\}$ by: $b' := b \cup \{\langle |M|, \Phi\rangle\}$. Then

$$\tilde{M} \models \varphi[({}^{v_{i_0}}/_{\mathtt{val}^{E\restriction\alpha,F}[c_0](b'(\vec{x}^0))}), \ldots, ({}^{v_{i_{m-1}}}/_{\mathtt{val}^{E\restriction\alpha,F}[c_{m-1}](b'(\vec{x}^{m-1}))}), ({}^{\vec{w}}/_{b(\vec{w})})]$$
$$\Longleftrightarrow$$
$$M \models \psi[b].$$

Moreover

$$\tilde{M} \models (\exists v_{i_0} \quad \varphi)[({}^{v_{i_1}}/_{\mathtt{val}^{E\restriction\alpha,F}[c_1](b'(\vec{x}^1))}) \ldots ({}^{v_{m-1}}/_{\mathtt{val}^{E\restriction\alpha,F}[c_{m-1}](b'(\vec{x}^{m-1}))})({}^{\vec{w}}/_{b(\vec{w})})]$$
$$\Longleftrightarrow$$

$$\exists c \in \mathfrak{C}(\dot{E}, \dot{F}) \quad \Big(c \text{ is a code for a 2-ary function } \wedge$$
$$M \models \quad (\exists z \quad T_\lambda(\varphi, v_{i_0}, c, \langle z, \Phi\rangle, v_{i_1}, c_{m-1}, \langle \vec{x}^1\rangle, \ldots, v_{m-1}, c_{m-1}, \langle \vec{x}^{m-1}\rangle))[b]\Big),$$

where z is a new variable.

Proof. The starting point is a Σ_0-formula φ in the language of M'. But as $\dot{F}^{M'} = \emptyset$ (the height of M' is a successor ordinal), it is obvious how to transform φ into an equivalent Σ_0-formula φ' in which $\dot{F}$ does not occur anymore - just replace "$\dot{F}(v)$" with "$v \neq v$".

Now I define a preliminary transformation $\bar{T}(\psi)$ by induction on formulae ψ in which $\dot{F}$ does not occur, as follows (I want to express φ, where $\dot{E}$ is interpreted as $\bar{E} := E\restriction\omega\alpha$ and $\dot{F}$ as $F := E_{\omega\alpha}$):

Fix new variables z and z'. Later z will be replaced by $|\mathrm{J}^E_\alpha|$ and z' by $\omega\alpha$.

If $\varphi \equiv \dot{E}(v)$, then set:

$$\bar{T}(\varphi) := \dot{E}(v) \vee (\exists \delta, x, y \in z \quad v = \langle z', \delta, x\rangle \wedge \dot{F}(y, x) \wedge \delta \in y).$$

The other atomic formulae remain unchanged – Note that $\dot{F}$ does not occur. The inductive steps are as usual; this atomic case is the only true change that is made. So far, we have for $\vec{a} \in |\tilde{M}|$:

$$\begin{aligned} \tilde{M} \models \varphi[(\vec{v}/\vec{a})] \quad &\iff \quad \tilde{M} \models \varphi'[\vec{a}] \\ &\iff \quad \langle |\mathrm{J}^E_{\alpha+1}|, \bar{E}, F, D\rangle \models \bar{T}(\varphi')[(\vec{v}/\vec{a}), (z'/\omega\alpha), (z/|\mathrm{J}^E_\alpha|)]. \end{aligned}$$

Now let $c_{\mathrm{On}} \in \mathfrak{C}$ be the code from Lemma 5.6. Let a, v_{i_0}, …, $v_{i_{m-1}}$, c_0, …, c_{m-1}, $\vec{x}^0$, …, $\vec{x}^{m-1}$, $\vec{w}$, b and b' as in the statement of the Lemma. Then

$$\begin{aligned} \tilde{M} \quad &\models \quad \varphi[(v_{i_0}/\mathtt{val}^{\bar{E},F}[c_0](b'(\vec{x}^0))), \ldots, (v_{m-1}/\mathtt{val}^{\bar{E},F}[c_{m-1}](b'(\vec{x}^{m-1}))), \\ &\qquad (\vec{w}/b(\vec{w}))] \\ \iff \quad \langle |\mathrm{J}^E_{\alpha+1}|, \bar{E}, F, D\rangle \quad &\models \quad \bar{T}(\varphi')[(v_{i_0}/\mathtt{val}^{\bar{E},F}[c_0](b'(\vec{x}^0))), \ldots, \\ &\qquad (v_{m-1}/\mathtt{val}^{\bar{E},F}[c_{m-1}](b'(\vec{x}^{m-1}))), \\ &\qquad (z/\mathtt{val}^{\bar{E},F}[\pi^1_0](|\mathrm{J}^E_\alpha|)), (z'/\mathtt{val}^{\bar{E},F}[c_{\mathrm{On}}](|\mathrm{J}^E_\alpha|)), \\ &\qquad (\vec{w}/b(\vec{w}))] \\ \iff \quad \langle |\mathrm{J}^E_\alpha|, \bar{E}, F, D\rangle \quad &\models \quad T_{\dot{E},\dot{F};\dot{D}}\big(\bar{T}(\varphi'), v_{i_0}, c_0, \langle \vec{x}^0\rangle, \ldots, v_{m-1}, c_{m-1}, \langle \vec{x}^{m-1}\rangle, \\ &\qquad z, \pi^1_0, \Phi, z', c_{\mathrm{On}}, \Phi\big)[b] \end{aligned}$$

The last equivalence follows from Lemma 5.4. It is applicable, since $|\mathrm{J}^E_\alpha|$ is closed under functions which are rud in $\bar{E}$ and F, and because by Lemma 5.5, $|\mathrm{J}^E_{\alpha+1}|$ is precisely the closure of $|\mathrm{J}^E_\alpha| \cup \{|\mathrm{J}^E_\alpha|\}$ under all functions that are rud in $\bar{E}$ and F. In the third line, note that π^1_0 is a rudimentary code for the identity.

The second part of the Lemma now follows from the first, making use of the fact that every element of $|\mathrm{J}^E_{\alpha+1}|$ is of the form $g(a, |\mathrm{J}^E_\alpha|)$, for a function g which is rud in $\bar{E}$ and F. This traces back to lemma 5.5 and the fact that a list of arguments $a_0, \ldots, a_{n-1} \in |\mathrm{J}^E_\alpha|$ can be coded by one, namely $\langle a_0, \ldots, a_{n-1}\rangle \in |\mathrm{J}^E_\alpha|$, so that it can be rudimentarily decoded by the component functions $(\cdot)^n_i$. Moreover, $|\mathrm{J}^E_\alpha|$ is needed as an argument at most once, since if $b \in |\mathrm{J}^E_{\alpha+1}|$ is of the form $g'(a, |\mathrm{J}^E_\alpha|, \ldots, |\mathrm{J}^E_\alpha|)$, where g' is rud (in E, F), then the function defined by $g(x, y) = g'(x, y, \ldots, y)$ is rud too, and $g(a, |\mathrm{J}^E_\alpha|) = b$. If, on the other hand, $|\mathrm{J}^E_\alpha|$ does not occur as an argument at all, then one can always add "dummy"-arguments. $\square$

Corollary 5.8. *There is a recursive function $\bar{T}_\lambda$ with the following property:*
Let $M = \langle \mathrm{J}^E_{\alpha+1}, \emptyset\rangle$ be a weak j-ppm. Let $D = D_{M||\alpha}$. Set $F := E_{\omega\alpha}$ and $\bar{E} := E\restriction\omega\alpha$. Let $\varphi(\vec{x})$ be a Σ_n-formula. Then $\bar{T}_\lambda(\varphi)$ is again a Σ_n-formula with two additional free variables, so that for arbitrary elements $\vec{a}$ of $|M|$,

$$\langle M, D\rangle \models \varphi[\vec{a}] \iff \langle |M|, \bar{E}, F, D\rangle \models \bar{T}_\lambda(\varphi)[\vec{a}, \omega\alpha, |\mathrm{J}^E_\alpha|].$$

(In fact, one could replace D by an arbitrary predicate contained in $M||\alpha$.)

Proof. The definition of the transformations $\bar{T}_\lambda(\varphi) = \bar{T}(\varphi')$ from the previous proof can be expanded to arbitrary formulae. □

Similar constructions yield the corresponding results for pPs-structures.

Lemma 5.9. *There is a recursive function T_s with the following property: Let $N' := \langle \mathrm{J}^E_{\alpha+1}, \emptyset\rangle$ be a pPs-structure. Let $N = N'||\alpha$. Let φ be a Σ_0-formula in the language of set theory with additional predicate symbols $\dot{E}$ and $\dot{F}$. Let $v_0, \dots, v_{n-1}$ be the free variables of φ. Let $a = \{i_0, \dots, i_{m-1}\}$ be an m element subset of n. For each $j < m$, let c_j be a code for an n_j-ary function rud in $\dot{E}, \dot{F}$, and $x^j_0, \dots, x^j_{n_j-1}$ a list s.t. each x^j_k is either a variable symbol or a fixed constant symbol Φ. Let $\psi := T_s(\varphi, v_{i_0}, c_0, \langle\vec{x}^0\rangle, \dots, v_{i_{m-1}}, c_{m-1}, \langle\vec{x}^{m-1}\rangle)$.*

Then ψ is a Σ_ω-formula s.t. the following holds:

If $\vec{w} = w_0, \dots, w_{m'-1}$ is an enumeration of $\{v_k \mid k \in n \setminus a\}$, then the set of free variables of ψ is contained in $\{\vec{w}\} \cup \{x^j_k \mid j < m \wedge k < n_j \wedge x^j_k \neq \Phi\}$.

Let b be an assignment of the free variables of ψ with values from $|N|$. Define $b' :$ Fr $(\psi) \cup \{\Phi\} \to |N| \cup \{|N|\}$ by: $b' := b \cup \{\langle |N|, \Phi\rangle\}$. Then

$$N' \models \varphi[({}^{v_{i_0}}/{}_{\mathtt{val}^{E\restriction\alpha,F}[c_0](b'(\vec{x}^0))}), \dots, ({}^{v_{i_{m-1}}}/{}_{\mathtt{val}^{E\restriction\alpha,F}[c_{m-1}](b'(\vec{x}^{m-1}))}), ({}^{\vec{w}}/{}_{b(\vec{w})})]$$
$$\Longleftrightarrow$$
$$N \models \psi[b].$$

Moreover,

$$N' \models (\exists v_{i_0} \quad \varphi)[({}^{v_{i_1}}/{}_{\mathtt{val}^{E\restriction\alpha,F}[c_1](b'(\vec{x}^1))}) \dots ({}^{v_{m-1}}/{}_{\mathtt{val}^{E\restriction\alpha,F}[c_{m-1}](b'(\vec{x}^{m-1}))})({}^{\vec{w}}/{}_{b(\vec{w})})]$$
$$\Longleftrightarrow$$

$$\exists c \in \mathfrak{C}(\dot{E}, \dot{F}) \quad \Big(c \text{ is a code for a 2-ary function } \wedge$$
$$N \models \quad (\exists z \quad T_s(\varphi, v_{i_0}, c, \langle z, \Phi\rangle, v_{i_1}, c_{m-1}, \langle\vec{x}^1\rangle, \dots, v_{m-1}, c_{m-1}, \langle\vec{x}^{m-1}\rangle))[b]\Big),$$

where z is a new variable.

Proof. The proof bears no new ideas. □

Corollary 5.10. *There is a recursive function $\bar{T}_s$ with the following property:*

Let $N = \langle \mathrm{J}^E_{\alpha+1}, \emptyset\rangle$ be a pPs-structure. Let $F := \dot{F}^{\tilde{C}_0(N||\alpha)}$ and $\bar{E} := E\restriction\omega\alpha$. Let $\varphi(\vec{x})$ be a Σ_n-formula. Then $\bar{T}_s(\varphi)$ is again a Σ_n-formula with two additional free variables so that for arbitrary elements $\vec{a}$ of $|N|$,

$$N \models \varphi[\vec{a}] \iff \langle |N|, \bar{E}, F\rangle \models \bar{T}_s(\varphi)[\vec{a}, \omega\alpha, |\mathrm{J}^E_\alpha|].$$

Proof. As before. □

5.2 Σ_1-definability in pPs-structures and their maximal continuations

For the rest of this section, fix an active pPs-structure $N = \langle \mathrm{J}^E_{s^+}, F^c\rangle$ (where F be an extender of length $s^+(F)$ in the functional representation) and set:

τ	$=$	$\tau(F)$	κ	$=$	$\kappa(F)$
s	$=$	$s(F)$	λ	$=$	$\lambda(F)$
π	$=$	π^N_s	$\widehat{N}$	$=$	$\langle \mathrm{J}^{E'}_\nu, \widehat{F}\rangle$
D	$=$	$D(N\|\|\tau, \kappa, s^+)$	$I\!\!D$	$=$	$I\!\!D(N\|\|\tau, F)$

Hence $F = \widehat{F}|s^+$. For the definitions of $D(N||\tau, \kappa, s^+)$ and $I\!\!D(N||\tau, F)$, see Section 2.5.

In order to see the rough approach for translating formulae between N and $\widehat{N}$, let $\varphi = \varphi(\vec{x})$ be a Σ_1-formula, and $\vec{\xi} < s^+$. The idea is to make use of the Łoś theorem, in order to express over N that φ is true in $\widehat{N}$ of $\vec{\xi}$:

$$\begin{aligned} \widehat{N} \models \varphi[\vec{\xi}] \quad &\Longleftrightarrow \quad \exists \beta < \tau \quad \langle \mathrm{J}^{E'}_{\pi(\beta)}, \widehat{F} \cap \mathrm{J}^{E'}_{\pi(\beta)} \rangle \models \varphi[\vec{\xi}] \\ &\Longleftrightarrow \quad N \models \exists \beta < \tau \; \exists t_1, t_2 \in D \quad ([t_1] = \pi(\mathrm{J}^E_\beta) \wedge [t_2] = \widehat{F} \cap \pi(\mathrm{J}^E_\beta) \\ &\qquad\qquad \wedge I\!\!D \models (\langle t_1, t_2 \rangle \models \varphi[\vec{\xi^*}])), \end{aligned}$$

where $\xi^* = \langle \xi, \mathrm{id} \rangle$. I used the notation of Definition 2.5 here, as well as the fact that Σ_0-extender product embeddings are cofinal. I deal with the problem of expressing "$[t] = \widehat{F} \cap \pi(\mathrm{J}^E_\beta)$" over N first.

(1) *Let $\beta < \tau$. Let then $c = c^\beta$ be the $<_{\mathrm{J}^E_\tau}$-minimal surjection from κ onto $\mathcal{P}(\kappa) \cap \mathrm{J}^E_\beta$, and for $x \in \mathcal{P}(\kappa) \cap \mathrm{J}^E_\beta$, let*

$$f_x := \langle \langle x, x \cap \gamma \rangle \mid \gamma < \kappa \rangle.$$

Further, define $Z = Z^\beta : \kappa \longrightarrow |N||\tau|$ by:

$$Z(\nu) = Z^\beta(\nu) := \{ f_{c(\mu)}(\nu) \mid \mu < \nu \}.$$

Then $\pi(Z)(\kappa) = \widehat{F} \cap \pi(\mathrm{J}^E_\beta)$.

Proof of (1). It is obvious that $Z \in |N||\tau|$. In the course of the proof, several functions are going to be defined, for which this is just as obvious. There, like here, a more explicit argument showing this is omitted.

Two directions have to be verified. For the inclusion from left to right, let $\pi(g)(\vec{\gamma}) \in \pi(Z)(\kappa)$, where $\langle \vec{\gamma}, g \rangle \in D$. Let $\vec{\gamma} \in (s^+)^n$. Then

$$\langle \vec{\gamma}, \kappa \rangle \in F(\underbrace{\{ \langle \vec{\mu}, \nu \rangle \mid g(\vec{\mu}) \in Z(\nu) \}}_{A}).$$

Define a function $\delta : \kappa^{n+1} \longrightarrow (\kappa + 1)$ by:

$$\delta(\vec{\mu}, \nu) := \begin{cases} \bar{\nu} & \text{if } \langle \vec{\mu}, \nu \rangle \in A \text{ and } \bar{\nu} \text{ is minimal with } g(\vec{\mu}) = f_{c(\bar{\nu})}(\nu), \\ \kappa & \text{otherwise.} \end{cases}$$

Then $A = \{ \langle \vec{\mu}, \nu \rangle \mid \delta(\vec{\mu}, \nu) < \kappa \wedge g(\vec{\mu}) = f_{c(\delta(\vec{\mu}, \nu))}(\nu) \}$; hence, setting $h(\vec{\zeta}, \xi, \theta) := f_{c(\delta(\vec{\zeta}, \xi))}(\theta)$,

$$A = \{ \langle \vec{\mu}, \nu \rangle \mid \delta(\vec{\mu}, \nu) < \kappa \wedge g(\vec{\mu}) = h(\vec{\mu}, \nu, \nu) \}.$$

As $\langle \vec{\gamma}, \kappa \rangle \in F(A)$, it follows that:

$$\widehat{N}^{\text{passive}} \models \pi(\delta)(\vec{\gamma}, \kappa) < \lambda \wedge \pi(g)(\vec{\gamma}) = \pi(h)(\gamma, \kappa, \kappa).$$

We have:

$$\mathrm{J}^E_\tau \models \forall \vec{\mu}, \nu < \kappa \quad \delta(\vec{\mu}, \nu) < \kappa \longrightarrow \delta(\vec{\mu}, \nu) < \nu,$$

hence

$$\widehat{N}^{\text{passive}} \models \forall \vec{\mu}, \nu < \lambda \quad \pi(\delta)(\vec{\mu}, \nu) < \lambda \longrightarrow \pi(\delta)(\vec{\mu}, \nu) < \nu.$$

Because $\pi(\delta)(\vec{\gamma},\kappa) < \lambda$, we know that $\pi(\delta)(\vec{\gamma},\kappa) < \kappa$. Now let $\zeta = \pi(\delta)(\vec{\gamma},\kappa)$. Then

$$\widehat{N}^{\text{passive}} \models \pi(g)(\vec{\gamma}) = \pi(f)_{\pi(c)(\zeta)}(\kappa).$$

Obviously $\pi(c)$ is a surjection from λ onto $\mathcal{P}(\lambda) \cap \mathrm{J}^{E'}_{\pi(\beta)}$, and for $\theta < \kappa$:

$$\pi(c)(\theta) = \pi(c)(\pi(\theta)) = \pi(c(\theta)), \text{ and hence, } \pi(c)(\theta) \cap \kappa = c(\theta) = \pi(c(\theta)) \cap \kappa.$$

So, $\pi(g)(\vec{\gamma}) = \pi(f)_{\pi(c)(\vec{\zeta})}(\kappa) = \langle \pi(c)(\zeta), \pi(c)(\zeta) \cap \kappa\rangle = \langle \pi(c(\zeta)), c(\zeta)\rangle = \langle \widehat{F}(c(\zeta)), c(\zeta)\rangle$. Thus, $\pi(g)(\vec{\gamma}) \in \widehat{F}$, and because

$$\forall \delta, \theta < \lambda \quad \pi(f)_{\pi(c)(\delta)}(\theta) \in \mathrm{J}^{E'}_{\pi(\beta)},$$

it follows that $\pi(g)(\vec{\gamma}) \in \mathrm{J}^{E'}_{\pi(\beta)}$ as well.

For the other direction, let $\langle a, b\rangle \in \widehat{F} \cap \mathrm{J}^{E'}_{\pi(\beta)}$. Then

(a) $a = \pi(b)$.

(b) $b \in \mathrm{J}^{E}_{\beta} \cap \mathcal{P}(\kappa)$
(because $a = \pi(b) \in \pi(\mathrm{J}^{E}_{\beta} \cap \mathcal{P}(\kappa))$).

Hence $\langle a, b\rangle = \pi(f_b)(\kappa)$. Now let $\gamma < \kappa$ be minimal such that $b = c(\gamma)$. For $\gamma < \nu < \kappa$, it follows by definition of Z that

$$f_{c(\gamma)}(\nu) \in Z(\nu).$$

Hence

$$(\gamma, \kappa) \subseteq \{\nu \mid f_{c(\gamma)}(\nu) \in Z(\nu)\},$$

i.e.,

$$\{\gamma\} \times (\gamma, \kappa) \subseteq \{\langle \beta, \nu\rangle \mid f_{c(\beta)}(\nu) \in Z(\nu)\},$$

and correspondingly,

$$\underbrace{F(\{\gamma\} \times (\gamma, \kappa))}_{=\{\gamma\}\times(\gamma,\lambda)} \subseteq F(\{\langle \beta, \nu\rangle \mid f_{c(\beta)}(\nu) \in Z(\nu)\}).$$

Hence $\langle \gamma, \kappa\rangle \in F(\{\langle \beta, \nu\rangle \mid f_{c(\beta)}(\nu) \in Z(\nu)\})$. This means that $\pi(f)_{\pi(c)(\gamma)}(\kappa) \in \pi(Z)(\kappa)$. But

$$\pi(f)_{\pi(c)(\gamma)}(\kappa) = \langle \pi(c)(\gamma), \pi(c)(\gamma) \cap \kappa\rangle = \langle \pi(c(\gamma)), c(\gamma)\rangle = \langle \pi(b), b\rangle = \langle a, b\rangle.$$

Hence $\langle a, b\rangle \in \pi(Z)(\kappa)$, as claimed. $\square_{(1)}$

(2) *Let $t^\beta = \langle \kappa, Z^\beta\rangle$, where Z^β is defined as in (1). Then the function $\beta \mapsto t^\beta$ is $\Sigma_1(\mathrm{J}^E_\tau)$. According to (1), $[t^\beta] = \widehat{F} \cap \mathrm{J}^{E'}_{\pi(\beta)}$.*

Proof of (2). Obvious. $\square_{(2)}$

The next step that has to be done is to express over N the Σ_1-satisfaction relation of the term model $\mathbb{D}$. So let $t_1, \ldots t_n \in D$ be terms, and let a Σ_1 formula ψ with n free variables be given. Let $t_i = \langle \vec{\gamma^i}, g_i\rangle$. Then by the Łoś theorem for extender products:

$$\big(\mathbb{D} \models \psi[\vec{t}]\big) \iff \Big(\langle \vec{\gamma^1}, \ldots, \vec{\gamma^n}\rangle \in F(\{\langle \vec{\mu^1}, \ldots, \vec{\mu^n}\rangle \mid \mathrm{J}^E_\tau \models \psi[g_1(\vec{\mu^1}), \ldots, g_n(\vec{\mu^n})]\})\Big).$$

(3) *The relation $\{\langle d, b\rangle \mid \exists n < \omega \quad d \in (s^+)^n \wedge b \in \mathcal{P}(\kappa^n) \cap N \wedge d \in F(b)\}$ is $\Sigma_1(N)$.*

Proof of (3). Clearly, $d \in F(b) \iff \exists\gamma\exists\xi \quad \langle\gamma,\xi,d,b\rangle \in F^c$; see Definition 3.3. $\Box_{(3)}$

Since the parameters τ and κ are definable over N (using the predicate F^c) by a Σ_1-formula, (1) can be transformed into a Σ_1-formula. One arrives at:

Lemma 5.11. *There are recursive functions d and $\mathbf{d}$ s.t. the following holds for every active pPs-structure N:*

(a) Let $\varphi(v_1,\ldots,v_n)$ be a Σ_1-formula in the language of N. Then $d(\varphi)$ is a Σ_1-formula in the same language, s.t. for arbitrary $\vec{\xi} \in \mathrm{On}_N$,

$$\widehat{N} \models \varphi[\vec{\xi}] \qquad \iff \qquad N \models d(\varphi)[\vec{\xi}].$$

(b) Let $F = E^N_{\mathrm{top}}$ and $\tau = \tau(F)$. Let $\pi : \mathrm{J}^{E^N}_\tau \longrightarrow_F \mathrm{J}^{E^{\widehat{N}}}_\nu$ be the canonical embedding, and let $\varphi(v_1,\ldots,v_n)$ be a Σ_1-formula in the language of N. Then for arbitrary $\langle\vec{\alpha}^i, f^i\rangle$ s.t. $f^i \in D$ and $\vec{\alpha}^i < s$ $(i = 1,\ldots,n)$:

$$\widehat{N} \models \varphi[\pi(f^1)(\vec{\alpha}^1),\ldots,\pi(f^n)(\vec{\alpha}^n)] \iff N \models \mathbf{d}(\varphi)[\langle\vec{\alpha}^1,f^1\rangle,\ldots,\langle\vec{\alpha}^n,f^n\rangle].$$

(c) Let $\varphi(v_1,\ldots,v_n)$ be a Σ_1-formula in the language $\mathcal{L}$ of $\tilde{\mathcal{C}}_0(N)$. Then $d(\varphi)$ is a Σ_1-formula in the same language, s.t. for arbitrary $\vec{\xi} \in \mathrm{On}_N$,

$$\tilde{\mathcal{C}}_0(\widehat{N}) \models \varphi[\vec{\xi}] \qquad \iff \qquad \tilde{\mathcal{C}}_0(N) \models d(\varphi)[\vec{\xi}].$$

(d) If N is a ps-structure and $\varphi(v_1,\ldots,v_n)$ is a Σ_1-formula in the language $\mathcal{L}^*$ of $\mathcal{C}_0(N)$[15]*, then $d(\varphi)$ is a Σ_1-formula in the same language s.t. for arbitrary $\vec{\xi} \in \mathrm{On}_N$,*

$$\mathcal{C}_0(\widehat{N}) \models \varphi[\vec{\xi}] \qquad \iff \qquad \mathcal{C}_0(N) \models d(\varphi)[\vec{\xi}].$$

Proof. Part (a) follows from (1)-(3) and the remark preceding them. For part (b), one just has to change the beginning of the above argument:

$$\begin{aligned}
&\widehat{N} \models \varphi[\pi(f^1)(\vec{\alpha}^1),\ldots,\pi(f^n)(\vec{\alpha}^n)] \\
\iff\quad &\exists\beta<\tau \quad \langle \mathrm{J}^{E'}_{\pi(\beta)}, \widehat{F} \cap \mathrm{J}^{E'}_{\pi(\beta)}\rangle \models \varphi[\pi(f^1)(\vec{\alpha}^1),\ldots,\pi(f^n)(\vec{\alpha}^n)] \\
\iff\quad &N \models \exists\beta<\tau\ \exists t_1,t_2 \in D \quad ([t_1] = \pi(\mathrm{J}^E_\beta) \wedge [t_2] = \widehat{F} \cap \pi(\mathrm{J}^E_\beta) \\
&\qquad \wedge \mathbb{D} \models (\langle t_1,t_2\rangle \models \varphi[\langle\vec{\alpha}^1,f^1\rangle,\ldots,\langle\vec{\alpha}^n,f^n\rangle])),
\end{aligned}$$

Part (c) follows from (a), since one can just use $\dot{s}^{\tilde{\mathcal{C}}_0(N)} = \dot{s}^{\tilde{\mathcal{C}}_0(\widehat{N})}$ as an additional parameter on both sides. For part (d), one makes use of the fact that in the case of a ps-structure N with $\dot{q}^N = \langle a,f\rangle$ we have that $\dot{q}^{\mathcal{C}_0(\widehat{N})} = \pi(f)(a)$, where π is defined as in (b). Then it's clear, using (b), how to define $d(\varphi)$ for Σ_1-formulae in the language $\mathcal{L}^*$. $\Box$

Now I turn to the translation in the opposite direction.

Lemma 5.12. *There is a recursive function c, taking Σ_1-formulae of $\mathcal{L}^*$ to Σ_1-formulae of $\mathcal{L}^*$, with the following properties:*

1. *If $\varphi \in \mathcal{L}$, then $c(\varphi) \in \mathcal{L}$.*

[15] For the definition of $\mathcal{L}$ and $\mathcal{L}^*$, see Def. 3.33 and 3.38

2. *Let $\varphi(\vec{x})$ be a Σ_1-formula in $\mathcal{L}$, N a pPs-structure and $\vec{a}$ arbitrary. Then*

$$\tilde{\mathcal{C}}_0(N) \models \varphi[\vec{a}] \iff \tilde{\mathcal{C}}_0(\widehat{N}) \models c(\varphi)[\vec{a}].$$

3. *The corresponding statement holds for $\mathcal{L}^*$-formulae. I.e., letting $\varphi(\vec{x})$ be a Σ_1-formula in $\mathcal{L}^*$, N a ps-structure and $\vec{a}$ arbitrary,*

$$\mathcal{C}_0(N) \models \varphi[\vec{a}] \iff \mathcal{C}_0(\widehat{N}) \models c(\varphi)[\vec{a}].$$

If N is a model of some language, $\varphi(\vec{x})$ a formula of this language and $\vec{a} \notin |N|$, then I let $\neg(N \models \varphi[\vec{a}])$ here.

Proof. If N is passive or active of type III, then nothing has to be shown, since then $\tilde{\mathcal{C}}_0(N) = \tilde{\mathcal{C}}_0(\widehat{N})$. I.e., in this case, one could set $c(\varphi) = \varphi$. But this definition by cases must be incorporated in one uniform definition of c which works uniformly for all pPs-structures. This will be done at the end of the proof. I first construct the restriction of c to $\mathcal{L}$-formulae. $\mathcal{L}^*$-formulae will be treated at the end of the proof. So let $N = \langle \mathrm{J}_\alpha^{E^N}, (F|\alpha)^c \rangle$ be active of type I or II, where F be the top extender of $\widehat{N}$. So $\alpha = s^+(\alpha)^{\widehat{N}}$. I am going to derive a transformation of φ that behaves as desired in this case.

Let $\kappa = \mathrm{crit}(F)$, $\lambda = F(\kappa)$ and $\tau = (\kappa^+)^{\widehat{N}}$. Then

$$\tilde{\mathcal{C}}_0(N) \models \varphi[\vec{a}] \iff \exists \delta < \alpha \exists \bar{F} \quad \bar{F} = (F|\alpha)^c \cap |\mathrm{J}_\delta^{E^N}| \wedge \langle \mathrm{J}_\delta^{E^N}, \bar{F}, \kappa, s \rangle \models \varphi[\vec{a}].$$

Due to the coherency of N with $\widehat{N} = \langle \mathrm{J}_{\alpha'}^{E^{\widehat{N}}}, F \rangle$, obviously, $\mathrm{J}_\alpha^{E^N} = \mathrm{J}_\alpha^{E^{\widehat{N}}}$. But it must be expressed over $\tilde{\mathcal{C}}_0(\widehat{N})$ that $\xi < \alpha$ (and this will not be difficult, since in $\tilde{\mathcal{C}}_0(N)$, the constant s is available, and $\alpha = s^{+\widehat{N}}$), and we have to express "$\bar{F} = (F|\alpha)^c \cap |\mathrm{J}_\delta^{E^N}|$", of course.

One remark is due here: From the proof of the fact that a coherent structure in the sense of Jensen is always amenable, it can be seen that for $\xi < \tau$ necessarily $F{\restriction}|\mathrm{J}_\xi^{E^{\widehat{N}}}| \in |\widehat{N}|$ – see the proof of [Jen97, chapter 1, p. 11, Lemma 4].

Now let F' be the hypermeasure representation of $F|\alpha$. Keeping definition 3.3, items 3. and 4., in mind, I first define a Σ_0-formula $\bar{\varphi}_1(w, f, z)$, such that for all $u, v \in |\widehat{N}|$,

$$u = F' \cap ([s]^{<\omega} \times v) \iff \tilde{\mathcal{C}}_0(\widehat{N}) \models \bar{\varphi}_1[u, F{\restriction}c, v],$$

for some superset c of v with $F{\restriction}c \in |N|$ (and then for every such superset). Set:

$$\begin{array}{rcl}\bar{\varphi}_1 & := & (w \subseteq \mathrm{V}^2 \wedge \forall \langle b, y \rangle \in w \quad (b \in [s]^{<\omega} \wedge y \in z \wedge b \in f(y)) \wedge \\ & & \wedge \forall \langle b, y \rangle \in [s]^{<\omega} \times z \quad (b \in f(y) \longrightarrow \langle b, y \rangle \in w)).\end{array}$$

Obviously, $\bar{\varphi}_1$ has the desired properties.

In order to be able to express item 3. of Definition 3.3 rigorously, I introduce the following function from [MS94, P. 9, 2^{nd} Remark]: For $\xi < \tau$, γ_ξ is the least ordinal γ s.t.

$$F' \cap ([s]^{<\omega} \times |\mathrm{J}_\xi^{E^N}|) \in |\mathrm{J}_\gamma^{E^N}|.$$

At the place cited above, it is shown that $\langle \gamma_\xi \mid \xi < \tau \rangle$ is cofinal in $s^+ := \mathrm{ht}(N)$.

I now want to define a Σ_0-formula $\bar{\varphi}_2(f, g, v_1, v_2)$ with the following property: Let $\xi, \delta \in \mathrm{On}_{\widehat{N}}$ and $G = \mathrm{J}^{E^{\widehat{N}}}{\restriction}\lambda$. Then

$$\delta = \gamma_\xi \iff \tilde{\mathcal{C}}_0(\widehat{N}) \models \bar{\varphi}_2[F{\restriction}G(\xi), G, \delta, \xi].$$

If such a formula is true, it follows in particular that $\xi < \tau$. It will be defined by:

$$\bar{\varphi}_2(f, g, v_1, v_2) := \exists w \in g(v_1) \quad (\bar{\varphi}_1(w, f, g(v_2)) \wedge \forall \bar{\delta} < \delta \quad w \notin |g(\bar{\delta})|).$$

Thus, one finally sees that for $\bar{F} \in |\widehat{N}|$, $\bar{F} = (F|\alpha)^c \cap |\mathrm{J}_\mu^{E^{\widehat{N}}}| \in |\widehat{N}|$ iff

$$\tilde{\mathcal{C}}_0(\widehat{N}) \models \qquad \exists\theta\ \exists\delta\ \exists f\ \exists g$$

(1) $\gamma_\theta \geq \omega\mu$, so that $f \supset F{\restriction}\theta$ knows enough about F.

$$\left\{ \begin{array}{l} \Big(g = \mathrm{J}^{E^N}{\restriction}\lambda \wedge f = F{\restriction}|g(\theta)| \wedge \\ \quad \wedge \delta \geq \omega\mu \wedge \underbrace{\bar{\varphi}_2(f, g, \delta, \theta)}_{\gamma_\theta = \delta} \wedge \end{array} \right.$$

(2) $\bar{F} \subseteq (F|\alpha)^c \cap |\mathrm{J}_\mu^{E^{\widehat{N}}}|$.

$$\left\{ \begin{array}{l} \wedge\Big(\forall y \in \bar{F}\ \exists \gamma, \xi, a, x \in C_n(y) \quad \big(y = \langle \gamma, \xi, a, x\rangle \wedge \\ \qquad \wedge \gamma > s \wedge \mathrm{crit}(F) < \xi < \tau \wedge \\ \qquad \wedge \exists w \in |g(\gamma)| \qquad \underbrace{\bar{\varphi}_1(w, f, g(\xi))}_{w = F' \cap ([s]^{<\omega} \times \mathrm{J}_\xi^{E^{\widehat{N}}})} \quad \wedge \\ \qquad \wedge a \in [\gamma]^{<\omega} \wedge x \in |g(\xi)| \wedge a \in f(x)\big)\Big) \end{array} \right.$$

(3) $(F|\alpha)^c \cap |\mathrm{J}_\mu^{E^{\widehat{N}}}| \subseteq \bar{F}$.

$$\left\{ \begin{array}{l} \wedge \forall \langle \gamma, \xi, a, x\rangle \in |g(\mu)| \\ \quad \Big(\big(\gamma > \dot{s} \wedge \mathrm{crit}(F) < \xi < \tau \wedge \\ \quad\ \wedge \exists w \in |g(\gamma)|\ \bar{\varphi}_1(w, f, g(\xi)) \wedge \\ \quad\ \wedge a \in [\gamma]^{<\omega} \wedge x \in g(\xi) \wedge a \in F(x)\big) \longrightarrow \\ \qquad \longrightarrow \langle \gamma, \xi, a, x\rangle \in \bar{F}\Big)\Big). \end{array} \right.$$

I leave the verification that this formula works to the reader. Call it $\psi(\bar{F}, \mu)$, and set:

$$\begin{aligned} c(\varphi(\vec{x})) \quad := \quad & ((\dot{s} \neq \emptyset \wedge \exists \bar{F} \exists \mu \quad \psi(\bar{F}, \mu) \wedge \varphi_{\langle \mathrm{J}_\mu^{E^{\widehat{N}}}, \bar{F}, \kappa, s\rangle}(\vec{x})) \\ & \vee(\dot{s} = \emptyset \wedge \varphi(\vec{x}))). \end{aligned}$$

The question whether $\dot{s} = \emptyset$ decides here whether $\tilde{\mathcal{C}}_0(N) = \tilde{\mathcal{C}}_0(\widehat{N})$. If this formula is true in $\tilde{\mathcal{C}}_0(\widehat{N})$, any μ making it true will automatically be less than α. But instead of verifying this, it would do no harm to demand in addition: $\exists h \quad \mathrm{Funk}\ (h) \wedge \mathrm{dom}(h) = \dot{s} \wedge \mathrm{ran}(h) = \mu$. It is obvious that this definition of c behaves as wished for p$\mathrm{P}s$-structures.

In the following, I show how to expand c to Σ_1 formulae in the language $\mathcal{L}^*$. Corresponding to the case of p$\mathrm{P}s$-structures, for an active ps-structure N of type II (this is the only case in which $\mathcal{C}_0(N)$ differs substantially from $\tilde{\mathcal{C}}_0(N)$) of height α, and a Σ_1 formula φ in $\mathcal{L}^*$, we have:

$$\begin{aligned} \mathcal{C}_0(N) \models \varphi[\vec{a}] \quad \Longleftrightarrow \quad & \exists \delta < \alpha \exists \bar{F} \exists \bar{q} \quad \bar{F} = \dot{F}^{\mathcal{C}_0(N)} \cap |\mathrm{J}_\delta^{E^N}| \wedge \bar{q} = \dot{q}^{\mathcal{C}_0(N)} \wedge \\ & \wedge \langle \mathrm{J}_\delta^{E^N}, \bar{F}, \kappa, s, \bar{q}\rangle \models \varphi[\vec{a}]. \end{aligned}$$

If it is clear how to express "$x = \dot{q}^{\mathcal{C}_0(N)}$" uniformly by a Σ_1 formula over $\mathcal{C}_0(\widehat{N})$, it is obvious how to define c.

For $\langle a, f\rangle, \langle b, g\rangle \in \Gamma'(N)$ let

$$a_{\langle a,f\rangle,\langle b,g\rangle} := \{c \in [\kappa]^n \mid f_{a,a\cup b}(c) = g_{b,b\cup c}(c)\},$$

where $f_{a,a\cup b}$ and $g_{b,b\cup c}$ arise from adding the right "dummy"-variables to f and g; see [Ste00, S. 4f]. Let n be the cardinality of $a \cup b$. Then (in the terminology from definition 3.37):

(a) *Let $\langle a, f\rangle \in \Gamma'(N)$, and let θ be minimal s.t. $f \in |\mathrm{J}_\theta^{E^N}|$ (hence $\theta < \tau(N)$). Let $\langle b, g\rangle \prec_N \langle a, f\rangle$. Then $a_{\langle a,f\rangle,\langle b,g\rangle} \in |\mathrm{J}_{\theta+1}^{E^N}|$.*

Proof of (a). As $\langle b, g\rangle \prec_N \langle a, f\rangle$, $g \leq_N f$, and hence $g \in |\mathrm{J}_\theta^{E^N}|$. For the definition of $a_{\langle a,f\rangle,\langle b,g\rangle}$ no exact knowledge of a and b is necessary, it suffices to know how a lies in b. There are only finitely many possibilities for this. Hence $a_{\langle a,f\rangle,\langle b,g\rangle}$ are definable from f and g, and so they are elements of $\mathrm{J}_{\theta+1}^{E^N}$. $\square_{(a)}$

(b) *Let $\dot{q}^{\mathcal{C}_0(\widehat{N})}$ be the ζ-th element of $|\widehat{N}|$ with respect to $<_N$. Then $\zeta < \lambda$.*

Let $f : \kappa \longrightarrow |\mathrm{J}_\kappa^{E^N}|$ be defined by:

$$f(\gamma) = \text{ the } \gamma\text{ -th element of } |\mathrm{J}_\kappa^{E^N} \text{ wrt. } <_{\mathrm{J}_\kappa^{E^N}} .$$

Then $\pi_\lambda^{\widehat{N}}(f)(\zeta) = \dot{q}^{\mathcal{C}_0(\widehat{N})}$.

Proof of (b). $\dot{q}^{\mathcal{C}_0(\widehat{N})}$ can be coded as an element of $\mathcal{P}(\mathcal{P}(\xi))$, where $\xi = \max C_{\widehat{N}}$. But $\mathcal{P}(\mathcal{P}(\xi)) \cap |\widehat{N}| \subseteq |\mathrm{J}_{\xi^{++}}^{E}|^{\widehat{N}}$. Since $\xi < s(N) < \lambda$, and λ is a limit cardinal in $\widehat{N}$, it follows that $\dot{q}^{\mathcal{C}_0(N)} \in |\widehat{N}||\lambda|$, from which we can deduce that $\zeta < \lambda$.

The second part of the claim is obvious. $f \in |N||\tau|$, since it is definable in $N||\kappa$. $\square_{(b)}$

Hence $\langle g, b\rangle$ is the $\prec_N$-minimal element of $\Gamma'(N)$ s.t. $\pi_s^N(g)(b) = \dot{q}^{\mathcal{C}_0(\widehat{N})}$ (i.e.: $\langle g, b\rangle = \dot{q}^{\mathcal{C}_0(N)}$) iff ξ, H, R, f, s^+ and ζ exist in $|\widehat{N}|$, so that the conjunction of the following statements is true in $\mathcal{C}_0(\widehat{N})$:

1. $s^+ = (\dot{s}^{\mathcal{C}_0(\widehat{N||\lambda})})^{+\widehat{N}}$ and ζ, f are defined as in (b).
2. There is an $n < \omega$, s.t. $g : [\kappa]^n \longrightarrow |\widehat{N}||\tau|$ and $b \in [s^+]^n$.
3. $\xi < \tau$ and $f, g \in |\widehat{N}||\xi|$.
4. $H = \dot{F}^{\mathcal{C}_0(\widehat{N})} \restriction |\widehat{N}||\xi + 2|$.
5. $R =<_{\mathrm{J}_\tau^{E^{\widehat{N}}}}$.
6. $b \cup \{\zeta\} \in H(a_{\langle b,g\rangle,\langle\{\zeta\},f\rangle})$.
7. for all functions $h \in |\widehat{N}||\tau|$ with $\mathrm{dom}(h) = [\kappa]^n$ for some $n < \omega$, and for all $c \in [s^+]^{<\omega}$, we have:
$$((h = g \wedge c <_{\mathrm{lex}} b) \vee gRh) \longrightarrow c \cup b \notin H(a_{\langle c,h\rangle,\langle b,g\rangle}).$$

This corresponds to a Σ_1 formula in $\mathcal{L}^*$. Thus it is clear how to expand c to Σ_1 formulae in $\mathcal{L}^*$, and the proof is complete. $\square$

Remark 5.13. The proof made extensive use of the fact that in the active type I- or II-case the constant $s(E_{\mathrm{top}}^N)$ is available in $\tilde{\mathcal{C}}_0(\widehat{N})$.

5.3 Σ_1-definability from N to $\Lambda(N)$

Definition 5.14. For an ordinal α, we set:

$$\alpha \dot{-} 1 = \begin{cases} \bar{\alpha} & \text{if } \alpha = \bar{\alpha}+1, \\ 0 & \text{if } \alpha \text{ is a limit or } \alpha = 0. \end{cases}$$

Lemma 5.15. *There are Σ_1 formulae $\varphi_{\mathrm{V}}(x,y), \varphi_E(x,y), \varphi_F(x)$ such that for every pPλ-structure $M = \langle \mathrm{J}^E_\alpha, F, D_M\rangle$ with $\alpha > 1$, for which $\mathtt{S}(M)$ is defined, we have:*

(a) $|\widehat{\mathtt{S}(M)}| = \{z \mid M \models \varphi_{\mathrm{V}}[z, \alpha\dot{-}1]\}$.

(b) $E^{\widehat{\mathtt{S}(M)}} = \{z \mid M \models \varphi_E[x, \alpha\dot{-}1])\}$.

(c) $E^{\widehat{\mathtt{S}(M)}}_{\mathrm{top}} = \{z \mid M \models \varphi_F[z]\}$.

Here, let $\widehat{\mathtt{S}(M)} = \langle |\widehat{\mathtt{S}(M)}|, E^{\widehat{\mathtt{S}(M)}}, E^{\widehat{\mathtt{S}(M)}}_{\mathrm{top}}\rangle$.

Moreover, $\langle E^{\widehat{\mathtt{S}(M||\gamma)}} \mid \gamma < \mathrm{ht}(M)\rangle$ *and* $\langle |\widehat{\mathtt{S}(M||\gamma)}| : \gamma < \mathrm{ht}(M)\rangle$ *are uniformly* $\Sigma_1(M)$.

Proof. Define for $\beta \le \alpha$:

$$F(\beta) := \widehat{\mathtt{S}(M||\beta)}.$$

Let $F(\beta) = \langle |F(\beta)|, E^{F(\beta)}, E^{F(\beta)}_{\mathrm{top}}\rangle$.

(1) *There are Σ_1 formulae $\chi(z,w,y)$ and $\psi(z,w,x,y)$, so that for $\beta < \alpha$ and $F(\beta) \in |\mathrm{J}^E_{\beta+1}|$,*

(a) $|F(\beta+1)| = \{z \mid \mathrm{J}^E_{\beta+1} \models \psi[z, \beta, |F(\beta)|, E^{F(\beta)}]\}$.

(b) $E^{F(\beta+1)} = \{z \mid \mathrm{J}^E_{\beta+1} \models \chi[z, \beta, E^{F(\beta)}]\}$.

Proof of (1). I will just present formulae that work. The verifications are standard.

$$\begin{aligned}
\psi \;\equiv\;& \exists n \exists s^+ \exists \tilde{e} \exists e \exists f \\
& (e = E_{\omega w} \wedge ((e \neq \emptyset \wedge s^+ = s^+(\langle \mathrm{J}^E_w, e\rangle)) \vee (e = \emptyset \wedge s^+ = \mathrm{ht}(x))) \wedge \\
& \wedge \tilde{e} = y \cup \{\langle s^+, b, a\rangle \mid \exists n < \omega \quad b \subseteq [\mathrm{crit}(e)]^n \wedge a \in [s^+]^n \wedge \dot{E}(\langle \omega w, a, b\rangle)\} \\
& \wedge f \text{ is a function } \wedge \mathrm{dom}(f) = s^+ + n + 1 \wedge f(0) = \emptyset \wedge \\
& \wedge \forall \gamma + 1 \in \mathrm{dom}(f) \quad f(\gamma+1) = S^{\tilde{e}}(f(\gamma)) \wedge \\
& \wedge \forall \lambda \in \mathrm{dom}(f) \quad (\mathrm{Lim}(\lambda) \longrightarrow f(\lambda) = \bigcup_{\xi<\lambda} f(\xi)) \wedge \\
& \wedge z \in f(s^+ + n)).
\end{aligned}$$

$$\begin{aligned}
\chi \;\equiv\;& z \in y \vee (\exists s^+ \exists e \quad (e = E_{\omega w} \wedge e \neq \emptyset \wedge s^+ = s^+(\langle \mathrm{J}^E_w, e\rangle) \wedge \\
& \qquad \exists n < \omega \; \exists b \; \exists a \quad b \subseteq [\mathrm{crit}(e)]^n \wedge a \in [s^+]^n \wedge a \in e(b) \wedge z = \langle s^+, b, a\rangle))
\end{aligned}$$

(2) $E^{F(\beta)}$ *and* $|F(\beta)|$ *are elements of* $|M||\beta+1|$*, and there are Σ_1 formulae ψ' and χ', so that*

(a) In J^E_β, $\langle |F(\gamma)| \;:\; \gamma < \beta\rangle$ *is defined by* ψ'.

(b) In J^E_β, $\langle E^{F(\gamma)} \mid \gamma < \beta\rangle$ *is defined by* χ'.

Proof of (2). For $\gamma < \beta$,

$$u = |F(\gamma)| \iff \mathrm{J}^E_\beta \models \underbrace{\exists f \exists g \quad (\tilde{\psi}(f) \wedge \tilde{\chi}(g) \wedge u = f(\gamma))}_{\psi'(u,\gamma)} \text{ and}$$

$$t = E^{F(\gamma)} \iff \mathrm{J}^E_\beta \models \underbrace{\exists f \exists g \quad (\tilde{\psi}(f) \wedge \tilde{\chi}(g) \wedge t = g(\gamma))}_{\chi'(t,\gamma)},$$

where $\tilde{\psi}$ and $\tilde{\chi}$ are the following formulae:

$$\begin{aligned}
\tilde{\psi}(f) \quad \equiv \quad & (f \text{ is a function } \wedge \\
& \wedge \mathrm{dom}(f) \in \mathrm{On} \wedge f(0) = \emptyset \wedge \\
& \wedge \forall \nu + 1 \in \mathrm{dom}(f) \quad (f(\nu+1) = \{z \mid \mathrm{J}^E_{\nu+1} \models \psi(z, \nu, f(\nu), g(\nu)\}) \wedge \\
& \wedge \forall \lambda \in \mathrm{dom}(f) \quad (\mathrm{Lim}(\lambda) \longrightarrow f(\lambda) = \bigcup_{\omega\xi \in D_{\langle \mathrm{J}^E_\lambda, \emptyset \rangle}} f(\xi))),
\end{aligned}$$

$$\begin{aligned}
\tilde{\chi}(g) \quad \equiv \quad & (g \text{ is a function } \wedge \\
& \wedge \mathrm{dom}(g) \in \mathrm{On} \wedge g(0) = \emptyset \wedge \\
& \wedge \forall \nu + 1 \in \mathrm{dom}(g) \quad (g(\nu+1) = \{z \mid \mathrm{J}^E_{\nu+1} \models \chi(z, \nu, g(\nu)\}) \wedge \\
& \wedge \forall \lambda \in \mathrm{dom}(g) \quad (\mathrm{Lim}(\lambda) \longrightarrow g(\lambda) = \bigcup_{\omega\xi \in D_{\langle \mathrm{J}^E_\lambda, \emptyset \rangle}} g(\xi))).
\end{aligned}$$

The proof that these formulae behave as desired proceeds by induction on β. For the successor step, one uses (1), and the limit step is obvious. $\square_{(2)}$

Now the formulae φ_{V}, φ_E and φ_F can be defined. Built into these formulae is a distinction between the case that α is a limit and the case that it is a successor. This can be seen from the parameter $\alpha \dot{-} 1$ which equals 0 precisely when α is a limit - note that $\alpha > 1$. This parameter is substituted for the free variable y. Again, I will just present the formulae. It is obvious that they are as wished.

$$\begin{aligned}
\varphi_{\mathrm{V}} \equiv \quad & (y = \emptyset \wedge \exists \gamma \exists u \quad (\dot{D}(\omega\gamma) \wedge \psi'(u,\gamma) \wedge z \in u)) \vee \\
\vee \quad & (y \neq \emptyset \wedge \exists e \exists u \quad \chi'(e,y) \wedge \psi'(u,y) \wedge \psi(z,y,u,e)),
\end{aligned}$$

$$\begin{aligned}
\varphi_E \equiv \quad & (y = \emptyset \wedge \exists \gamma \exists e \quad (\dot{D}(\omega\gamma) \wedge \chi'(e,\gamma) \wedge z \in e) \vee \\
\vee \quad & (y \neq \emptyset \wedge \exists e \quad \chi'(e,y) \wedge \chi(z,y,e)),
\end{aligned}$$

$$\varphi_F \equiv \quad \dot{F}(z).$$

$\square$

Lemma 5.16. *There are functions $\hat{g}$ and g with the following property: If $M = \langle \mathrm{J}^E_\alpha, F, D \rangle$ ($\alpha > 1$) is a pPλ-structure for which $N = \mathtt{S}(M)$ is defined, and if φ is a Σ_1 formula, then $\hat{g}(\varphi)$ and $g(\varphi)$ are Σ_1 formulae such that for arbitrary $\vec{x}$, the following holds:*

(a) If φ is a formula in the language of $\widehat{N}$, then $\hat{g}(\varphi)$ is a formula in the language of M, and

$$\widehat{N} \models \varphi[\vec{x}] \iff M \models \hat{g}(\varphi)[\vec{x}, \alpha \dot{-} 1].$$

(b) *If φ is a formula in the language of $\tilde{\mathcal{C}}_0(\widehat{N})$, then $\hat{g}(\varphi)$ is a formula in the language of $\tilde{\mathcal{C}}_0(M)$, and*

$$\tilde{\mathcal{C}}_0(\widehat{N}) \models \varphi[\vec{x}] \iff \tilde{\mathcal{C}}_0(M) \models \hat{g}(\varphi)[\vec{x}, \alpha \dot{-} 1].$$

(c) *If M is a pλ-structure, and φ is a formula in the language of $\mathcal{C}_0(\widehat{N})$, then $\hat{g}(\varphi)$ is a formula in the language of $\mathcal{C}_0(M)$, and*

$$\mathcal{C}_0(\widehat{N}) \models \varphi[\vec{x}] \iff \mathcal{C}_0(M) \models \hat{g}(\varphi)[\vec{x}, \alpha \dot{-} 1].$$

(d) *If φ is a formula in the language of $\tilde{\mathcal{C}}_0(N)$, then $g(\varphi)$ is a formula in the language of $\tilde{\mathcal{C}}_0(M)$, and*

$$\tilde{\mathcal{C}}_0(N) \models \varphi[\vec{x}] \iff \tilde{\mathcal{C}}_0(M) \models g(\varphi)[\vec{x}, \alpha \dot{-} 1].$$

(e) *If M is a pλ-structure and φ is a formula in the language of $\mathcal{C}_0(N)$, then $g(\varphi)$ is a formula in the language of $\mathcal{C}_0(M)$, and*

$$\mathcal{C}_0(N) \models \varphi[\vec{x}] \iff \mathcal{C}_0(M) \models g(\varphi)[\vec{x}, \alpha \dot{-} 1].$$

Proof. I will first deduce how to define $\hat{g}(\varphi)$ in the case that φ is a Σ_1-formula in the language of $\widehat{N}$. If $\mathrm{ht}(M)$ is a limit, then

$$\begin{aligned} \widehat{N} \models \varphi[\vec{x}] \quad \iff \quad & \vec{x} \in |\widehat{N}| \wedge \\ & \exists \omega\alpha \in D \quad \langle |\widehat{\mathtt{S}(M||\alpha)}|, E^{\widehat{\mathtt{S}(M||\alpha)}}, F \cap |\widehat{\mathtt{S}(M||\alpha)}| \rangle \models \varphi[\vec{x}], \end{aligned}$$

and if $\alpha = \bar{\alpha} + 1$ is a successor ordinal, then $E^{\widehat{\mathtt{S}(M||\alpha)}} = E^{\mathtt{S}(M||\alpha)} = E^{\mathtt{S}(M||\bar{\alpha})} \subseteq E^{\widehat{\mathtt{S}(M||\bar{\alpha})}} \in |\mathrm{J}^E_\alpha|$, and so in this case,

$$\begin{aligned} \widehat{N} \models \varphi[\vec{x}] \quad \iff \quad & \vec{x} \in |\widehat{N}| \wedge \exists u, e, \tilde{e} \in |M| \quad u \in |\widehat{N}| \wedge u \text{ is transitive } \wedge \\ & \wedge \tilde{e} = E^{\widehat{\mathtt{S}(M||\bar{\alpha})}} \wedge e = \tilde{e} \cap u \wedge \langle u, e, \emptyset \rangle \models \varphi[\vec{x}]. \end{aligned}$$

Using the formulae φ_{V}, χ' and ψ' from the previous lemma 5.15, this is expressible over M, as desired – again, I use the parameter $\alpha \dot{-} 1$, to decide whether α is a limit or not:

$$\begin{aligned} \widehat{N} \models \varphi[\vec{x}] \quad \iff \quad & M \models (\alpha \dot{-} 1 = \emptyset \wedge \exists e \exists \tilde{e} \exists f \exists u \exists \gamma \\ & \quad (\dot{D}(\omega\alpha) \wedge \psi'(u, \alpha) \wedge \chi'(\tilde{e}, \alpha) \wedge f = F \cap u \wedge e = \tilde{e} \cap u \\ & \quad \langle u, e, f \rangle \models \varphi[\vec{x}]) \\ & \vee(\alpha \dot{-} 1 \neq \emptyset \wedge \exists e \exists \tilde{e} \exists u \\ & \quad \chi'(\tilde{e}, \alpha \dot{-} 1) \wedge \varphi_{\mathrm{V}}(u, \alpha \dot{-} 1) \wedge u \text{ is transitive} \\ & \quad e = \tilde{e} \cap u \wedge \langle u, e, \emptyset \rangle \models \varphi[\vec{x}]). \end{aligned}$$

The definition of $\hat{g}(\varphi)$ can be read off this formula.

The additional constants in the languages of the structures $\tilde{\mathcal{C}}_0(M)$ and $\tilde{\mathcal{C}}_0(\widehat{N})$, or $\mathcal{C}_0(M)$ and $\mathcal{C}_0(\widehat{N})$, are interpreted in these structures in the same way, so that they can be treated like additional parameters. It is obvious how to expand $\hat{g}$ to act on the larger class of formulae.

Now the function c from lemma 5.12 can be used:

$$g := \hat{g} \circ c.$$

The so-defined function g does what we asked for. Note that c always yields formulae in the language of $\tilde{\mathcal{C}}_0(\widehat{N})$ or $\mathcal{C}_0(\widehat{N})$. □

5.4 Σ_1-Definability from M to $\mathtt{S}(M)$

The formulation of the following is a bit technical, in order to set up everything for its proof. What is applied later on for the most part are the Corollaries 5.18, 5.19 and 5.20.

Lemma 5.17. *Let $M = \langle \mathrm{J}^E_\nu, E_{\omega\nu}, D_M\rangle$ be a pPλ-structure, for which $\mathtt{S}(M)$ is defined. For $\alpha \le \nu$ let*

$$o_\alpha = \mathrm{ht}(\mathtt{S}(M||\alpha)) \text{ and } \hat{o}_\alpha = \mathrm{ht}(\widehat{\mathtt{S}(M||\alpha)}).$$

Then there are sequences $\langle e^\mu_\alpha \mid \langle\mu,\alpha\rangle \in S\rangle$ and $\langle \hat{e}^\mu_\alpha \mid \langle\mu,\alpha\rangle \in \hat{S}\rangle$ with the following properties:

(a) $S = \{\langle\mu,\alpha\rangle \mid \mu \le \nu \wedge \alpha \le o_\mu\}$ and $\hat{S} = \{\langle\mu,\alpha\rangle \mid \mu \le \nu \wedge \alpha \le \hat{o}_\mu\}$. In order to simplify the notation, I shall write, for $\mu \le \nu$:

$$\begin{array}{rcl|rcl} e^\mu & = & \langle e^\mu_\alpha \mid \alpha \le o_\mu\rangle & \hat{e}^\mu & = & \langle \hat{e}^\mu_\alpha \mid \alpha \le \hat{o}_\mu\rangle \\ \underline{e}^\mu & = & e^\mu \restriction o_\mu & \underline{\hat{e}}^\mu & = & \hat{e}^\mu \restriction \hat{o}_\mu \\ e^\mu_{\mathrm{top}} & = & e^\mu_{o_\mu} & \hat{e}^\mu_{\mathrm{top}} & = & \hat{e}^\mu_{\hat{o}_\mu} \\ \mathtt{S}(M||\mu)^+ & := & \langle \mathtt{S}(M||\mu), \underline{e}^\mu\rangle & \widehat{\mathtt{S}(M||\mu)}^+ & := & \langle \widehat{\mathtt{S}(M||\mu)}, \underline{\hat{e}}^\mu\rangle. \end{array}$$

(b) $e^\mu_\alpha, \hat{e}^\mu_\alpha \in \omega$, if defined. Identifying natural numbers with recursive functions, and presupposing a recursive coding of formulae by natural numbers, for every formula φ with free variables $x_1, \ldots, x_n$ in the language of M, and for all $\mu \le \nu$, $\vec{\xi} < \omega \cdot o_\mu$,

$$(M||\mu) \models \varphi[\xi_1, \ldots, \xi_n] \iff \mathtt{S}(M||\mu)^+ \models e^\mu_{\mathrm{top}}(\varphi)[\xi_1, \ldots, \xi_n, p_\mu],$$

and for $\vec{\xi} < \omega \cdot \hat{o}_\alpha$,

$$(M||\mu) \models \varphi[\xi_1, \ldots, \xi_n] \iff \widehat{\mathtt{S}(M||\mu)}^+ \models \hat{e}^\mu_{\mathrm{top}}(\varphi)[\xi_1, \ldots, \xi_n, p_\mu],$$

where, $p_\mu = o_\mu \dot{-} 1$. In particular, $e^\mu_{\mathrm{top}}(\varphi)$ is a formula in the language of $\mathtt{S}(M||\mu)^+$, and the corresponding applies to $\hat{e}^\mu_\alpha$. Also, $\hat{e}^\mu_{\mathrm{top}}$ and e^μ_{top} map Σ_1-formulae to Σ_1-formulae.

(c) For $\mu + 1 \le \mathrm{ht}(M)$, $\underline{e}^{\mu+1} = e^\mu = \underline{\hat{e}}^{\mu+1}$ and $e^{\mu+1} = \hat{e}^{\mu+1}$. For limits $\mu \le \mathrm{ht}(M)$,

$$\underline{e}^\mu = \bigcup_{\omega\alpha \in D^*_{M||\mu}} \underline{e}^\alpha, \quad \text{and} \quad \underline{\hat{e}}^\mu = \bigcup_{\omega\alpha \in D_{M||\mu}} \underline{e}^\alpha.$$

(d) $\underline{e}^\mu$, and $\underline{\hat{e}}^\mu$ are $\Sigma_1(\mathtt{S}(M||\mu))$, and $\Sigma_1(\widehat{\mathtt{S}(M||\mu)})$, respectively. e^μ_{top} and $\hat{e}^\mu_{\mathrm{top}}$ are uniformly $\Sigma_\omega(\mathtt{S}(M||\mu))$, $\Sigma_\omega(\widehat{\mathtt{S}(M||\mu)})$, respectively.

Before beginning the proof of this lemma, let's note a useful consequence:

Corollary 5.18. *Let M be a pPλ-structure, whose $N = \mathtt{S}(M)$ exists. Then there is a sequence $F^N = \langle f^N_\mu \mid \mu \le \mathrm{ht}(N)\rangle$ of functions from ω to ω with the following properties (in the following, we write f_μ for f^N_μ):*

(a) $\Lambda(N||\mu) \models \varphi[\vec{\xi}] \iff N||\mu \models f_\mu(\varphi)[\vec{\xi}, \mu \dot{-} 1]$, where $\vec{\xi} < \omega\mu$.

(b) $f_\mu(\varphi)$ is a Σ_1-formula, if φ is.

(c) f_μ is uniformly $\Sigma_\omega(N||\mu)$.

(d) $F = \{\langle n, m, \gamma\rangle \mid n = f_\gamma(m) \wedge \gamma < \mathrm{ht}(N)\}$ *is uniformly* $\Sigma_1(N)$.

Analogously, there exists a sequence $\hat{F}^N = \langle \hat{f}^N_\mu \mid \mu \le \mathrm{ht}(\widehat{N})\rangle$ *of functions from* ω *to* ω *with the properties*

(a') $\Lambda(N||\mu) \models \varphi[\vec{\xi}] \iff \widehat{N||\mu} \models \hat{f}_\mu(\varphi)[\vec{\xi}, \mu \dot{-} 1]$, *where* $\vec{\xi} < \mathrm{On}_{\widehat{N||\mu}}$.

(b') *If* φ *is a* Σ_1*-formula, then so is* $\hat{f}_\mu(\varphi)$.

(c') $\hat{f}_\mu$ *is uniformly* $\Sigma_\omega(\widehat{N||\mu})$.

(d') $\widehat{F} = \{\langle n, m, \gamma\rangle \mid n = \hat{f}_\gamma(m) \wedge \gamma < \mathrm{ht}(\widehat{N})\}$ *is uniformly* $\Sigma_1(\widehat{N})$.

Here, $\hat{f}_\mu$ *stands for* $\hat{f}^N_\mu$.

Proof. I construct the sequence F^N; the construction of $\hat{F}^N$ is analogous. Let $\mu \le \mathrm{ht}(N)$, and let $N||\mu = \mathtt{S}(M||\beta)$. In the notation of the previous Lemma 5.17, define $f_\mu(\varphi)$, by replacing every occurrence of e^β_γ $(\gamma < \mu)$ in the formula $e^\beta_{\mathrm{top}}(\varphi)$ by its Σ_1-definition over $N||\gamma$. This yields a Σ_1-formula by Lemma 5.17(d), from which the uniform definability of f_μ follows as well. So (a)-(c) are obviously satisfied. Note that $p_\beta = \mu \dot{-} 1$.

Now let χ be the uniform definition from (c), i.e., for $\mu \le \mathrm{ht}(N)$ and $m, n \in \omega$,

$$n = f_\mu(m) \longleftrightarrow N||\mu \models \chi[n, m].$$

In order to see (d), note that the above means:

$$N \models (N||\mu \models \chi[n, m]),$$

i.e. $N \models \chi_{N||\mu}[n, m]$, and this means that the relation F is $\Sigma_1(N)$. □

Proof of Lemma 5.17. I construct the sequences e^μ and $\hat{e}^\mu$ by recursion on μ.

Case 0: $\mu = 0$.

This case is trivial, and the definition of e^0_0, $\hat{e}^0_0$ is irrelevant.

Case 1: $\mu = 1$.

In this case, $D_M = \{\emptyset\}$, and it suffices to define $e^\mu_1 = \hat{e}^\mu_1$ by setting:

$$e^\mu_1(\varphi(\vec{x})) = \varphi^*(\vec{x}),$$

where φ^* results from replacing every occurrence of $\dot{D}(v)$ in φ with $v = \emptyset$; thus, the constant e^μ_0 does not occur in $e^\mu_1(\varphi)$.

Case 2: $\mu = \bar{\mu} + 1$.

According to (c), in this case $\hat{e}^\mu = e^\mu$ and $\underline{e}^\mu = e^{\bar{\mu}}$. So one merely has to define $e^{\bar{\mu}+1}$, and this means one has to define $e^{\bar{\mu}+1}_{\mathrm{top}}$, because $o_\mu = o_{\bar{\mu}} + 1$. For then the domain of $e^{\bar{\mu}+1}$ is $o_\mu + 1$, as demanded; $\underline{e}^{\bar{\mu}+1} = e^{\bar{\mu}}$.

Case 2.A: $M||\bar{\mu}$ *is active and* $s^+(\bar{\mu})^M < \bar{\mu}$.

Let $F = E_{\bar{\mu}}$, $\bar{E} = E\restriction\omega\bar{\mu}$. First, let φ be a Σ_1-formula. Then $D_{M||\mu} = D_{M||\bar{\mu}} \setminus (s^+(\bar{\mu})^M, \bar{\mu})$. For the sake of readability, I carry out the following construction only for formulae with but one free variable x, i.e., $\varphi = \varphi(x)$; the general case is not more complicated in principle, writing it down explicitly causes some additional trouble, though. So let $\xi < \omega o_\mu = \omega(o_{\bar{\mu}}+1) = s^+(\bar{\mu})^M + \omega$. Setting $\tilde{M} := \langle \mathrm{J}^E_\mu, \emptyset, D_{M||\bar{\mu}}\rangle$,

$$(M||\mu) \models \varphi[\xi] \iff \tilde{M} \models \varphi'[({}^x/_\xi), ({}^y/_{\bar{\mu}})],$$

where

$$\varphi' = (\exists\gamma \quad (\gamma = s^+(\bar\mu))_{M||\bar\mu} \wedge \tilde\varphi)$$

and $\tilde\varphi$ results from replacing every occurrence of „$D(\zeta)$“ in φ with „$(D(\zeta) \wedge \zeta \leq \gamma)$“. Obviously then φ' is still a Σ_1-formula, albeit in the parameter $\bar\mu$. To be precise, φ' still has to be transformed into prenex normal form, but this can be done effectively.

In the case that $\xi \in [s^+(M||\bar\mu), s^+(M||\bar\mu) + \omega)$, I introduce a new variable as follows:

$$\varphi^* := \varphi'({}^x/_{\gamma+w}).$$

I.e., every occurrence of x in φ' is replaced with $\gamma + w$, where w is a new variable. More precisely, φ^* is a Σ_1-formula which is equivalent to this substitution. It is clear that this transformation can be carried out effectively. The purpose of this transformation is that now, setting $\xi = s^+(M||\bar\mu) + n$,

$$M||\mu \models \varphi[\xi] \iff \tilde M \models \varphi^*[({}^y/_{\bar\mu}), ({}^w/_n)];$$

this is only true in case $\xi \geq s^+(M||\bar\mu)$, though.

Now let

$$\varphi' = \exists z \quad \varphi_0' \quad \text{and} \quad \varphi* = \exists z \quad \varphi_0^*,$$

where φ_0' and φ_0^* are Σ_0-formulae. Using Lemma 5.7, we get:

$$\begin{array}{rcl}
M||\mu & \models & \varphi[\xi] \\
\iff & & (\xi < s^+(M||\bar\mu) \wedge \tilde M \models \varphi'[({}^x/_\xi), ({}^y/_{\bar\mu})]) \\
& \vee & (\xi = s^+(M||\bar\mu) + n \wedge \tilde M \models \varphi^*[({}^y/_{\bar\mu}), ({}^w/_n)]) \\
\iff & & (\xi < s^+(M||\bar\mu) \wedge \tilde M \models \exists z \quad \varphi_0'[({}^x/_\xi), ({}^y/_{\mathtt{val}^{\bar E, F}[c_{\mathrm{On}}](|M||\bar\mu|)})]) \\
& \vee & (\xi = s^+(M||\bar\mu) + n \wedge \tilde M \models \exists z \quad \varphi^*[({}^y/_{\mathtt{val}^{\bar E, F}[c_{\mathrm{On}}](|M||\bar\mu|)}), ({}^w/_n)]) \\
\iff & & \exists c \in \mathfrak{C}(\dot E, \dot F) \quad (c \text{ codes a 2-ary function } \wedge \\
& & ((\xi < s^+(M||\bar\mu) \wedge M||\bar\mu \models \exists x_0 \quad T_\lambda(\varphi_0', z, c, \langle x_0, \Phi\rangle, y, c_{\mathrm{On}}, \Phi)[({}^x/_\xi)]) \\
& \vee & (\xi = s^+(M||\bar\mu) + n \wedge M||\bar\mu \models \exists x_0 \quad T_\lambda(\varphi_0^*, z, c, \langle x_0, \Phi\rangle, y, c_{\mathrm{On}}, \Phi)[({}^w/_n)]))).
\end{array}$$

But this is equivalent to:

$$\begin{array}{ll}
\exists\psi & \exists c \in \mathfrak{C}(\dot E, \dot F) \quad \big(c \text{ codes a 2-ary function} \wedge \\
& ((\xi < o_{\bar\mu} \wedge \\
& \quad \psi = T_\lambda(\varphi_0', z, c, \langle x_0, \Phi\rangle, y, c_{\mathrm{On}}, \Phi) \wedge \mathtt{S}(M||\bar\mu)^+ \models e^{\bar\mu}_{\mathrm{top}}(\exists x_0 \quad \psi)[({}^x/_\xi)]) \vee \\
& (\xi = o_{\bar\mu} + n \wedge \\
& \quad \psi = T_\lambda(\varphi_0^*, z, c, \langle x_0, \Phi\rangle, y, c_{\mathrm{On}}, \Phi) \wedge \mathtt{S}(M||\bar\mu)^+ \models e^{\bar\mu}_{\mathrm{top}}(\exists x_0 \quad \psi)[({}^w/_n)]))\big),
\end{array}$$

which can be written as

$$\begin{array}{rcl}
\mathtt{S}(M||\mu)^+ & \models & \exists\psi \, \exists c \in \mathfrak{C}(\dot E, \dot F) \quad \big(c \text{ codes a 2-ary function } \wedge \\
& & ((\xi < o_{\bar\mu} \wedge \\
& & \quad \psi = T_\lambda(\varphi_0', z, c, \langle x_0, \Phi\rangle, y, c_{\mathrm{On}}, \Phi) \wedge \\
& & \quad (\mathtt{S}(M||\mu)^+||o_{\bar\mu}) \models e^{\bar\mu}_{\mathrm{top}}(\exists x_0 \quad \psi)[({}^x/_\xi)]) \vee \\
& & (\xi \geq o_{\bar\mu} \wedge \\
& & \quad \psi = T_\lambda(\varphi_0^*, z, c, \langle x_0, \Phi\rangle, y, c_{\mathrm{On}}, \Phi) \wedge \\
& & \quad (\mathtt{S}(M||\mu)^+||o_{\bar\mu}) \models e^{\bar\mu}_{\mathrm{top}}(\exists x_0 \quad \psi)[({}^w/_{\xi - o_{\bar\mu}})]))\big).
\end{array}$$

As $e^{\bar{\mu}}_{\text{top}} = e^{\mu}_{o_{\bar{\mu}}}$, this formula can now be used as the definition of e^{μ}_{top} for Σ_1-formulae. The model $\mathtt{S}(M||\mu)^+||o_{\bar{\mu}}$ is to be understood as $\langle \mathrm{J}^{E'}_{o_{\bar{\mu}}}, E'_{\omega o_{\bar{\mu}}}, e^{\mu}\restriction o_{\bar{\mu}}\rangle$, if $\mathtt{S}(M||\mu) = \langle \mathrm{J}^{E'}_{o_{\mu}}, \emptyset\rangle$. Hence, $\mathtt{S}(M||\mu)^+||o_{\bar{\mu}} = \mathtt{S}(M||\bar{\mu})^+$.

Note that no parameters are used in $e^{\bar{\mu}}_{\text{top}}(\chi)$, since in the current case, $\bar{\mu}$ is a limit – see the definition of p_μ in (b). Moreover, $\mathtt{S}(M||\mu)^+||o_{\bar{\mu}}$ obviously is $\Sigma_1(\mathtt{S}(M||\mu)^+)$ in $o_{\bar{\mu}}$, as is the definition of the satisfaction relation for elements of $\mathtt{S}(M||\mu)$. The reason why $\mathtt{S}(M||\mu)^+||o_{\bar{\mu}} = \mathtt{S}(M||\bar{\mu})^+ \in \mathtt{S}(M||\mu)$ is that $\underline{e}^{\bar{\mu}}$ is $\Sigma_1(S(M||\bar{\mu}))$, by (d). This way, it follows that $\mathtt{S}(M||\bar{\mu})^+$ is amenable. This is because $\underline{e}^{\alpha}$ is $\Sigma_1(\mathtt{S}(M||\alpha))$, for $\alpha < \mu$. Hence $e^{\bar{\mu}}_{\text{top}}(\chi)$ is a $\mathbf{\Sigma}_1$-formula if χ is. Note also that $p_\mu = o_{\bar{\mu}}$, so that the right parameter is used in the formula $e^{\mu}_{\text{top}}(\varphi)$.

Now let's turn to the general case that $\psi(\vec{x})$ is a Σ_n-formula, for $n > 1$. Let ψ have the form

$$\psi(\vec{x}) \equiv Q_1 y_1 \dots Q_n y_n \varphi(\vec{\xi}, \vec{y}),$$

where φ is Σ_1 and every Q_i is either $\forall$ or $\exists$. Then

$$\begin{aligned}
(M||\mu) \models \psi[\xi] \quad &\iff \quad Q_1 a_1 \in |M||\mu| \dots Q_n a_n \in |M||\mu| \quad (M||\mu) \models \varphi[\vec{\xi}, \vec{a}] \\
&\iff \quad Q_1 \vec{\zeta}^1 < \operatorname{otp}(D_{M||\mu}) Q_1 p_1 < \omega \dots \\
&\qquad\quad \dots Q_n \vec{\zeta}^n < \operatorname{otp}(D_{M||\mu}) Q_n p_n < \omega \\
&\qquad\quad (M||\mu) \models \varphi[\vec{\xi}, h^1_{M||\mu}(\vec{\zeta}^1, p_1), \dots, h^1_{M||\mu}(\vec{\zeta}^n, p_n)] \\
&\iff \quad Q_1 \vec{\zeta}^1 < o_\mu Q_1 p_1 < \omega \dots Q_n \vec{\zeta}^n < o_\mu Q_n p_n < \omega \\
&\qquad\quad (M||\mu) \models \bar{\varphi}[\vec{\xi}, \vec{\vec{\zeta}}, \vec{p}] \\
&\iff \quad Q_1 \vec{\zeta}^1 < o_\mu Q_1 p_1 < \omega \dots Q_n \vec{\zeta}^n < o_\mu Q_n p_n < \omega \\
&\qquad\quad \mathtt{S}(M||\mu)^+ \models e^{\mu}_{\text{top}}(\bar{\varphi})[\vec{\xi}, \vec{\vec{\zeta}}, \vec{p}, o_{\bar{\mu}}] \\
&\iff \quad \mathtt{S}(M||\mu)^+ \models Q_1 \vec{\zeta}^1 < o_{\bar{\mu}} + 1 Q_1 p_1 < \omega \dots \\
&\qquad\quad \dots Q_n \vec{\zeta}^n < o_{\bar{\mu}} + 1 Q_n p_n < \omega \quad e^{\mu}_{\text{top}}(\bar{\varphi})[\vec{\xi}, \vec{\vec{\zeta}}, \vec{p}, o_{\bar{\mu}}] \\
&\overset{\text{def}}{\iff} \quad \mathtt{S}(M||\mu)^+ \models e^{\mu}_{\text{top}}(\psi)[\vec{\xi}, o_{\bar{\mu}}].
\end{aligned}$$

Here, Lemma 4.5 was essential. The map e^{μ}_{top} is then recursive. Now define $c_0 \in \omega$ by

$$c_0 := e^{\mu}_{\text{top}},$$

where I again identify recursive functions with natural numbers, so $c_0 \in \omega$.

Case 2.B: $M||\bar{\mu}$ is active and $s^+(\bar{\mu})^M = \bar{\mu}(= o_{\bar{\mu}})$, or $M||\bar{\mu}$ is passive and $o_{\bar{\mu}} = \omega\bar{\mu}$.

Again, let $F = E_{\bar{\mu}}$ and $\bar{E} = E\restriction\omega\bar{\mu}$. In this case, $D_{M||\mu} = D_{M||\bar{\mu}} \cup \{\omega\bar{\mu}\}$ and $o_\mu = o_{\bar{\mu}} + 1$. Let $\xi < \omega o_\mu$, and let $\varphi(x)$ be a Σ_1-formula. Let $\tilde{M} := \langle \mathrm{J}^E_\mu, \emptyset, D_{M||\bar{\mu}}\rangle$. Then

$$(M||\mu) \models \varphi[\xi] \iff \tilde{M} \models \varphi'[\xi, \omega\bar{\mu}],$$

where φ' results from replacing every occurrence of „$D(\gamma)$“ in φ with „$D(\gamma) \vee \gamma = \omega\bar{\mu}$“. Note that in the current case, $\omega o_{\bar{\mu}} = o_{\bar{\mu}} = \bar{\mu} = \mathtt{val}^{\bar{E},F}[c_{\mathrm{On}}](|M||\bar{\mu}|)$. The rest of the construction works as in case 2.A. Set:

$$c_1 := e^{\mu}_{\text{top}}.$$

Case 2.C: $M||\bar{\mu}$ is passive and $o_{\bar{\mu}} < \omega\bar{\mu}$.

Just as in case 2.B, we have $D_{M||\mu} = D_{M||\bar{\mu}} \cup \{\omega\bar{\mu}\}$. Again, let $\varphi(x)$ be a Σ_1-formula, and let $\xi < \omega o_\mu = \omega o_{\bar{\mu}} + \omega$. Then

$$(M||\mu) \models \varphi[\xi] \iff \langle \mathrm{J}^E_\mu, \emptyset, D_{M||\bar{\mu}}\rangle \models \varphi'[\xi, \omega\bar{\mu}],$$

where φ' is defined as in case 2.B. But now, $\omega o_{\bar{\mu}} = \mathtt{val}^{\bar{E},F}[\pi_0^1](\omega o_{\bar{\mu}})$. The rest of the construction works as in case 2.A. Set:

$$c_2 := e^{\mu}_{\mathrm{top}}.$$

Case 3: μ is a limit.

Then the definition of $\underline{e}^{\mu}$ and $\underline{\hat{e}}^{\mu}$ is prescribed by (c). It remains to define e^{μ}_{top} and $\hat{e}^{\mu}_{\mathrm{top}}$. In the definition of $e^{\mu}_{\mathrm{top}}(\varphi)$, I will stick to the case that φ is a Σ_1-formula. The expansion to arbitrary formulae is as in case 2.A. It is essential here that $|M||\mu| = h^1_{M||\mu}(D^*_{M||\mu})$.

Case 3.A: $M||\mu$ is passive.

Then $D_{M||\mu} = D^*_{M||\mu}$ and $o_\mu = \mathrm{otp}(D_{M||\mu})$. In this case, $\mathtt{S}(M||\mu) = \mathtt{S}\widehat{(M||\mu)}$, and one can set: $\hat{e}^{\mu}_{\mathrm{top}} := e^{\mu}_{\mathrm{top}}$. I concentrate on the definition of e^{μ}_{top}. Let $\varphi(\vec{x})$ be a Σ_1-formula and $\vec{\xi} < \omega o_\mu$. Since, by Lemma 3.17, D_M is unbounded in $\mathrm{On}_{M||\mu}$, one can argue as follows (I will give additional explanations below):

$$\begin{aligned}
& (M||\mu) \models \varphi[\vec{\xi}] \\
\iff\quad & \exists\zeta < \mathrm{otp}(D_{M||\mu})\exists\delta \quad \omega\delta = \eta^{M||\mu}_{\zeta+1} \wedge (M||\mu)|(|M||\delta|) \models \varphi[\vec{\xi}] \\
\iff\quad & \exists\zeta < \mathrm{otp}(D_{M||\mu})\exists\delta \quad \omega\delta = \eta^{M||\mu}_{\zeta+1} \wedge (M||\delta) \models \varphi[\vec{\xi}] \\
\iff\quad & \exists\zeta < \mathrm{otp}(D_{M||\mu})\exists\delta \quad \omega\delta = \eta^{M||\mu}_{\zeta+1} \wedge \mathtt{S}(M||\delta)^+ \models e^{\delta}_{\mathrm{top}}(\varphi)[\vec{\xi}, p_\delta] \\
\iff\quad & \exists\zeta < \mathrm{otp}(D_{M||\mu})\exists\delta \quad \omega\delta = \eta^{M||\mu}_{\zeta+1} \wedge \mathtt{S}(M||\delta)^+ \models e^{\mu}_{\zeta+1}(\varphi)[\vec{\xi}, \zeta] \\
\iff\quad & \exists\zeta < \mathrm{otp}(D_{M||\mu})\exists\delta \quad \omega\delta = \eta^{M||\mu}_{\zeta+1} \wedge (\mathtt{S}(M||\mu)^+||(\zeta+1) \models e^{\mu}_{\zeta+1}(\varphi)[\vec{\xi}, \zeta] \\
\iff\quad & \mathtt{S}(M||\mu)^+ \models \underbrace{(\exists\zeta \quad \mathtt{S}(M||\mu)^+||(\zeta+1) \models e^{\mu}_{\zeta+1}(\varphi)[\vec{\xi}, \zeta])}_{e^{\mu}_{\mathrm{top}}(\varphi)}
\end{aligned}$$

Moving from the second line to the third line, note that by coherency, $D_{M||\mu} \cap \omega\delta = D_{M||\delta}$, where I let $\omega\delta := \eta^{M||\mu}_{\zeta+1}$. Moreover, $M||\delta$ is passive, for otherwise it would follow that $\delta = \omega\delta = \eta^{M||\mu}_{\zeta+1}$ and $s^+(\delta)^M = \delta$ (or else $\delta \notin D_{M||\mu}$), which cannot be, by Lemma 3.20, as then δ would have to be a limit point of $D_{M||\mu}$. When moving from line four to line five, I used that $e^{\mu}_{\zeta+1} = e^{\delta}_{\mathrm{top}}$, where, again, $\omega\delta = \eta^{M||\mu}_{\zeta+1}$. To see this, note a property of $\langle e^\alpha \mid \alpha \le \nu\rangle$, which can be verified by induction:

(∗) *Let $\mu \le \nu$ and $\underline{e}^{\mu}$ already defined, so that conditions (a)-(d) are satisfied. Let $\omega\alpha, \omega\beta \in D_{M||\mu}$. Then $\underline{e}^{\alpha} \subseteq \underline{e}^{\beta} \subseteq \underline{e}^{\mu}$.*

Since by Lemma 3.20, $\omega\delta \neq s^+(\gamma)^M$, for all $\gamma \le \omega\mu$, it follows by Lemma 3.19 that $\eta^{M||\mu}_{\zeta+2} = \omega\delta + \omega$, hence that $\omega(\delta+1) \in D_{M||\mu}$. Thus by (∗):

$$\underline{e}^{\delta+1} \subseteq e^{\mu}.$$

But by (c), $\underline{e}^{\delta+1} = e^{\delta}$, and hence,

$$e^{\mu}_{\zeta+1} = e^{\delta}_{\mathrm{top}},$$

as wished, because $e^{\delta}_{\mathrm{top}} = e^{\delta}_{o_\delta}$, and, due to Corollary 4.5,

$$o_\delta = \mathrm{otp}(D_{M||\delta}) = \mathrm{otp}(D_{M||\mu} \cap \omega\delta) = \zeta + 1;$$

note that $\eta^{M||\mu}\restriction\zeta+1$ is the monotone enumeration of $D_{M||\mu} \cap \eta^{M||\mu}_{\zeta+1}$. For the same reason, $p_\delta = o_{\delta-1} = \zeta$; it is obvious that δ is a successor ordinal, as otherwise $o_\delta = \zeta+1$ would have to be a limit ordinal.

Finally, let me justify the transition from the fifth to the sixth line. Firstly,

$$(\mathtt{S}(M||\mu)^+)||(\zeta+1) := \langle \mathtt{S}(M||\mu)||\zeta+1, e^\mu \restriction \zeta+1\rangle.$$

That this structure is the same as $\mathtt{S}(M||\zeta+1)^+$ is a consequence of $(*)$. Since the expansion of e^μ_{top} to arbitrary Σ_ω-formulae is unproblematic, this case is complete. Set:

$$c_3 := e^\mu_{\text{top}},$$

so $c_3 \in \omega$.

Case 3.B: $M||\mu$ is active.

In this case, e^μ and $\hat{e}^\mu$ have to defined one by one. I start with $\hat{e}^\mu$. Let $F = E^M_\mu$. The definition of $\underline{\hat{e}}^\mu$ is determined by (c). It remains to define $\hat{e}^\mu_{\text{top}}$. Let $\mathtt{S}((M||\mu)^{\text{passive}}) = \langle \mathrm{J}^{E'}_\mu, \emptyset\rangle$. Then by definition 4.1, case 3, (a) and (c), $\widehat{\mathtt{S}(M||\mu)} = \langle \mathrm{J}^{E'}_\mu, F\rangle$. By (b), $|\widehat{\mathtt{S}(M||\mu)}| = |M||\mu|$. Let $\varphi(\vec{x})$ be a Σ_1-formula. Then, letting $\widehat{N} := \widehat{\mathtt{S}(M||\mu)}$,

$$\begin{array}{rrl} & M||\mu \models & \varphi[\vec{a}] \\ \iff & \widehat{N} \models & \exists\alpha\exists f\exists d\exists Q \\ & & \quad(\widehat{N}||\alpha \text{ is passive } \wedge \text{ ``} Q = \Lambda(\widehat{N}||\alpha)^- \text{''} \wedge f = \dot{F}\cap|Q| \wedge \\ & & \quad\wedge d = D_Q \wedge \langle Q^{\text{passive}}, f, d\rangle \models \varphi[\vec{x}]) \\ \stackrel{\text{def}}{\iff} & \widehat{N} \models & \varphi'[\vec{a}], \end{array}$$

since $\{\mathrm{ht}(\Lambda(\widehat{N}||\alpha)) \mid \alpha < \mathrm{ht}(\widehat{N})\}$ is unbounded in μ. Moreover, for $\alpha < \mu$, if $\widehat{N}||\alpha$ is passive, then $D_{\Lambda(\widehat{N}||\alpha)^-} = D_M \cap \omega\alpha$. By Lemma 4.15 (part d), "$Q = \Lambda(\widehat{N}||\alpha)$" is $\Sigma_1(\widehat{N})$, hence φ' $\Sigma_1(\widehat{N})$. Hence define:

$$\hat{e}^\mu_{\text{top}}(\varphi) = \varphi'.$$

This defines $\hat{e}^\mu_{\text{top}}\restriction\{\Sigma_1-\text{ formulae }\}$.

For formulae of arbitrary complexity, I now define inductively a preliminary function t that has all the desired properties, except the preservation of Σ_1-formulae. Finally, it is only applied to formulae of higher complexity, in order to complete the definition of $\hat{e}^\mu_{\text{top}}$. The definition of $t(\varphi)$ for the case that φ is atomic:

$$\begin{array}{rcll} t(\dot{D}(x)) & := & \exists\alpha & (\widehat{N}||\alpha \text{ is passive } \wedge \dot{D}_{\Lambda(\widehat{N}||\alpha)}(x)), \\ t(\dot{E}(x)) & := & \exists\alpha & (\dot{E}^{\Lambda(\widehat{N}||\alpha)}(x)), \\ t(\dot{F}(x)) & := & \dot{F}(x), & \\ t(xRy) & := & xRy & \quad\text{for } R \in \{\dot{=}, \dot{\in}\}. \end{array}$$

The expansion of t to Boolean combinations of and quantifications over formulae whose t-images are defined already, is standard. This defines t. Now define:

$$\hat{e}^\mu_{\text{top}}(\varphi) := t(\varphi), \text{ if } \varphi \text{ is not } \Sigma_1 .$$

Note that in the current case, the structure $\widehat{N}^+$ is not needed at all, and that the translation doesn't only work for ordinal parameters. I.e.,

$$M||\mu \models \varphi[\vec{a}] \iff \widehat{\mathtt{S}(M||\mu)} \models \hat{e}^\mu_{\text{top}}(\varphi)[\vec{a}].$$

Now e^μ has to be defined, which, by (c), reduces to defining e^μ_{top}. Let $N := \mathtt{S}(M||\mu)$, $F = \dot{F}^{\tilde{\mathcal{C}}_0(N)}$ and $\kappa = \mathrm{crit}(F)$. First, I am going to define e^μ_{top} only for Σ_1-formulae. To this end, I use the function d from Lemma 5.11, and make use of the fact there is a transformation of Σ_1-formulae from $M||\mu$ to $\widehat{N}$ already (the fact that the additional predicate of $\widehat{\Lambda(M||\mu)}^+$ is not used simplifies the construction to follow). So

$$\begin{array}{rll} & M||\mu & \models \varphi[\vec{\xi}] \\ \iff & \widehat{\Lambda(M||\mu)} & \models \hat{e}^\mu_{\mathrm{top}}(\varphi)[\vec{\xi}] \\ \iff & \Lambda(M||\mu) & \models \underbrace{(d \circ \hat{e}^\mu_{\mathrm{top}})}_{:=e^\mu_{\mathrm{top}}}(\varphi)[\vec{\xi}] \end{array}$$

for $\vec{\xi} < s^+(M||\mu)$. Let $\pi : N||\tau \longrightarrow_{F^{\mathfrak{f}}} \widehat{N}^{\mathrm{passive}}$. I define another function $\tilde{d}$, transforming Σ_ω-formulae into Σ_ω-formulae (but not Σ_1-formulae into Σ_1-formulae), so that

$$\begin{array}{rll} & \widehat{N} & \models \varphi[\pi(f_1)(a_1), \ldots, \pi(f_n)(a_n)] \\ \iff & N & \models \tilde{d}(\varphi)[\langle f_1, a_1\rangle, \ldots, \langle f_n, a_n\rangle]. \end{array}$$

A definition that does this will be arrived at in the following. First, define for $\langle a, f\rangle, \langle b, g\rangle \in \Gamma'(N, F)$:

$$\begin{array}{lll} a^{=}_{\langle a,f\rangle,\langle b,g\rangle} & := & \{c \mid N||\tau \models f_{a,a\cup b}(c) = g_{b,a\cup b}(c)\} \\ a^{\in}_{\langle a,f\rangle,\langle b,g\rangle} & := & \{c \mid N||\tau \models f_{a,a\cup b}(c) \in g_{b,a\cup b}(c)\}. \end{array}$$

Moreover, for $x \in \mathcal{P}(\kappa) \cap |N||\tau|$ define a function $x^* \in \Gamma'(N, F)$ by:

$$x^* := \langle\langle x, x \cap \alpha\rangle \mid \alpha < \kappa\rangle.$$

Here I use (as at several places before) a somewhat sloppy notation, by identifying a function $f \in ({}^\kappa|N||\tau|) \cap |N||\tau|$ with the function $f' : [\kappa]^1 \longrightarrow |N||\tau|$ which takes $\{\alpha\}$ to $f(\alpha)$.

(+) *For* $\langle a, f\rangle \in \Gamma'(N, F)$, $\widehat{N} \models \dot{F}(\pi(f)(a))$ *iff there is an* $x \subseteq \kappa$ *such that* $a \cup \{\kappa\} \in F^{\mathfrak{f}}(a^{=}_{\langle\{\kappa\},x^*\rangle,\langle a,f\rangle})$.

Proof of (+). Note that

$$\pi(x^*)(\kappa) = \langle\pi(x), x\rangle = \langle\widehat{F}(x), x\rangle$$

for $x \in \mathcal{P}(\kappa)$. The claim follows immediately, applying Łoś's theorem. $\Box_{(+)}$

This observation enables us to define $\tilde{d}(\varphi)$ by induction on φ as follows. If φ is atomic: Let $\varphi = \dot{F}(x)$. Then set:

$$\begin{array}{rcl} \tilde{d}(\varphi)(v) & = & \exists f \exists a \exists c \exists \bar{x} \exists x \\ & & (v = \langle a, f\rangle \wedge \bar{x} \subseteq \kappa \wedge x = \bar{x}^* \wedge c = a^{=}_{\langle a,f\rangle,\langle\{\kappa\},x\rangle} \wedge \\ & & \wedge \exists\gamma\exists\xi \quad \dot{F}(\langle\gamma, \xi, a \cup \{\kappa\}, c\rangle)). \end{array}$$

Note that $\dot{F}^N = (\widehat{F}|s^+(\widehat{N}))^c$. By observation (+), this definition works. For the other atomic formulae, the definition is analogous. Let $\varphi(x, y) = xRy$, where $R \in \{\dot{=}, \dot{\in}\}$. Then set:

$$\begin{array}{rcl} \tilde{d}(\varphi)(x, y) & = & \exists f \exists a \exists g \exists b \exists c \\ & & (x = \langle a, f\rangle \wedge y = \langle b, g\rangle \wedge c = a^R_{\langle a,f\rangle,\langle b,g\rangle} \wedge \\ & & \wedge \exists\gamma\exists\xi \quad \dot{F}(\langle\gamma, \xi, a \cup b, c\rangle)). \end{array}$$

It is obvious how to treat Boolean combinations. I deal with quantifications as follows:

$$\tilde{d}(\substack{\forall \\ \exists}v \quad \varphi) := \substack{\forall \\ \exists}t \in \Gamma'(N,F) \quad \tilde{d}(\varphi)({}^{v}/{}_{t}).$$

This finishes the definition of $\tilde{d}$. Finally, define $e^{\mu}_{\mathrm{top}}(\varphi)$, in the case that φ is not a Σ_1-formula, by:

$$e^{\mu}_{\mathrm{top}}(\varphi(\vec{x})) := \exists \vec{z} \quad (\bigwedge_{j_1}^{n} \text{“}z_j = x_j^{\circ}\text{”} \wedge \tilde{d}(\varphi)({}^{\vec{x}}/{}_{\vec{z}})).$$

Here, “$z_j = x_j^{\circ}$” is supposed to mean: x_j is an ordinal and $z_j = \langle x_j, \mathrm{id}\restriction\kappa\rangle$. Set:

$$c_4 := e^{\mu}_{\mathrm{top}}.$$

This defines $\langle e^{\mu} \mid \mu \leq \nu\rangle$. It remains to verify that the construction doesn't terminate prematurely, i.e., that condition (d) is always satisfied. It is easy to check that for $\alpha \leq \mu$ and $\xi < o_{\alpha}$,

$$e^{\alpha}_{\xi} = \begin{cases} c_0 & \text{if } \xi = \bar{\xi}+1 \text{ and } \mathtt{S}(M||\alpha)||\bar{\xi} \text{ is active and } \bar{\xi} < \lambda(E_{\bar{\xi}}^{\mathtt{S}(M||\alpha)}) , \\ c_1 & \text{if } \xi = \bar{\xi}+1 \text{ and } \mathtt{S}(M||\alpha)||\bar{\xi} \text{ is active and } \lambda(E_{\bar{\xi}}^{\mathtt{S}(M||\alpha)}) < \bar{\xi} , \\ & \text{or } \mathtt{S}(M||\alpha)||\bar{\xi} \text{ is passive and } o_{\bar{\xi}} = \omega\bar{\xi} , \\ c_2 & \text{if } \xi = \bar{\xi}+1 , \mathtt{S}(M||\alpha)||\bar{\xi} \text{ is passive and } o_{\bar{\xi}} < \omega\bar{\xi} , \\ c_3 & \text{if } \xi \text{ is a limit and } \mathtt{S}(M||\alpha)||\xi \text{ is passive}, \\ c_4 & \text{if } \xi \text{ is a limit and } \mathtt{S}(M||\alpha)||\xi \text{ is active}. \end{cases}$$

In order to see that $\underline{e}^{\alpha}$ is uniformly $\Sigma_1(\mathtt{S}(M||\alpha))$, one has to make sure that it can be decided in $\mathtt{S}(M||\alpha)$ by a Σ_1-formula whether $o_{\xi} = \omega\xi$ for $\xi + 1 < o_{\alpha}$. The other conditions that must be checked in order to decide which case we're in, are obviously $\Sigma_1(\mathtt{S}(M||\alpha))$ – e.g., $\bar{\xi} < \lambda(E_{\bar{\xi}}^{\mathtt{S}(M||\alpha)})$ iff $s := s(E_{\bar{\xi}}^{\mathtt{S}(M||\alpha)}) < \lambda(E_{\bar{\xi}}^{\mathtt{S}(M||\alpha)})$, and the latter is the case iff $\pi_s^{\mathtt{S}(M||\alpha)||\bar{\xi}}(\mathrm{id})(s) < \pi_s^{\mathtt{S}(M||\alpha)||\bar{\xi}}(\mathrm{crit}(E_{\bar{\xi}}^{\mathtt{S}(M||\alpha)}))$, which can be expressed easily, using Łoś's theorem, over $\mathtt{S}(M||\alpha)$.

Firstly, note that it suffices to decide whether $o_{\xi} = \xi$, for:

(1) *Let $\omega\xi = o_{\xi}$. Then $o_{\xi} = \xi$.*

Proof of (1). $\omega\xi = o_{\xi} \leq \xi \leq \omega\xi$. $\square_{(1)}$ So $\omega\xi = o_{\xi} \iff (\xi = o_{\xi} \wedge \xi = \omega\xi)$. The following claim yields a criterion which enables us to decide whether or not $o_{\xi} = \xi$.

(2) $o_{\xi} = \xi \longleftrightarrow |\mathtt{S}(M||\xi)| = |(M||\xi)|$.

Proof of (2). The direction from right to left is trivial. So let $o_{\xi} = \xi$. If $M||\xi$ is active, then $o_{\xi} = s^{+}(\xi)^{M} = \xi$, and it is obvious that the claim is true. So let I_M be the set of extender indices of M. If $I_M \cap \xi$ is unbounded in ξ, then the claim is true, by Lemma 4.15. So let $\delta := \sup(I_M \cap \xi) < \xi$, and let $M||\xi$ be passive. It is easy to see:

$$|\mathrm{J}^{E^M}_{o_\delta}| = |\mathrm{J}^{E'}_{o_\delta}|,$$

where I let $\mathtt{S}(M||\xi) = \langle \mathrm{J}^{E'}_{o_\xi}, \emptyset\rangle$.

Case 1: $\delta \notin I_M$.

Then $o_{\delta} = \delta$, and by induction on $\alpha \leq \xi - \delta$ it follows that:

$$|M||(\delta+\alpha)| = |\mathrm{J}^{E^M}_{\delta+\alpha}| = |\mathrm{J}^{E'}_{\delta+\alpha}| = |\mathtt{S}(M||\delta+\alpha)|,$$

which yields the claim.

Case 2: $\delta \in I_M$.

Then $o_\delta = s^+(\delta)^M$. So we're done if $s^+(\delta)^M = \delta$ (see case 1). So let $o_\delta < \delta$. Then for $\alpha \leq \xi - o_\delta$,

$$o_{\delta+\alpha} = o_\delta + \alpha,$$

as there are no extender indices between δ and ξ. Hence, for $\alpha_0 := \xi - \delta$,

$$o_{\delta+\alpha_0} = o_\delta + \alpha_0 = \xi = o_\xi.$$

So $\alpha_0 = \xi - o_\delta$. Now let $\beta < o_\xi - o_\delta (= \xi - o_\delta = \alpha_0)$. Then

$$|\mathrm{J}^{E'}_{o_\delta+\beta}| \subseteq |\mathrm{J}^{E}_{\delta+\beta}| \subseteq |\mathrm{J}^{E}_{\xi}|.$$

On the other hand, for $\beta \leq \xi - \delta = \alpha_0$,

$$|\mathrm{J}^{E}_{\delta+\beta}| \subseteq |\mathrm{J}^{E'}_{\delta+\beta}| \subseteq |\mathrm{J}^{E'}_{o_\xi}|,$$

as $\delta + \beta < \xi = o_\xi$. This shows that $|\mathrm{J}^{E}_{\xi}| = |\mathrm{J}^{E'}_{\xi}|$. $\square_{(2)}$

(3) *There is a Σ_1-formula $\varphi(x)$, so that for every $\xi < o_\alpha$,*

$$\mathtt{S}(M||\mu) \models \varphi[\xi] \iff \xi = \omega o_\xi.$$

Proof of (3). The formula φ expresses: ξ is a limit ordinal, $\xi = \omega\xi$, and there is an x, so that x is a pPλ-structure, $|x| = |\mathrm{J}^{E'}_{\xi}|$ and $\langle \mathrm{J}^{E'}_{\xi}, E'_{\omega\xi}\rangle = \mathtt{S}(x)$. The last part can be expressed by:

$$\exists f(\text{“}f \text{ is a function”} \wedge f = \mathtt{S}\restriction <_0 \text{“}\{x\} \wedge \langle \mathrm{J}^{E'}_{\xi}, E'_{\omega\xi}\rangle = \bigcup_{\omega\alpha \in D_x} f(x||\alpha)).$$

That $\mathtt{S}\restriction x$ is definable in x (even Σ_1-definable) was shown already, and that this x, if existent, is precisely $M||\xi$, follows since $\mathtt{S}$ is injective. The above definition of e^α_ξ by cases hence yields a Σ_1-definition in $\mathtt{S}(M||\alpha)$, as desired. $\square_{(3)}$

The same definition, carried out in $\widehat{\mathtt{S}(M||\alpha)}$, defines $\hat{e}^\alpha$. In order to define e^α_{top}, $\hat{e}^\alpha_{\mathrm{top}}$, a similar definition is used, which doesn't have to be Σ_1, though. More information on this can be found in the following Corollary 5.19. $\square$

The proof of the previous lemma yields another corollary, which will be of importance later:

Corollary 5.19. *For every pPs-structure N, for which $M = \Lambda(N)$ exists, let F^N and $\hat{F}^N$ be the sequences from Corollary 5.18. Then $f_N := f^N_{\mathrm{ht}(N)}$ and $\hat{f}_N = \hat{f}^N_{\mathrm{ht}(\widehat{N})}$ are uniformly $\Sigma_1(N)$ and $\Sigma_1(\widehat{N})$, respectively, for pPs-structures to which the same case of the definition applies. I.e,, there are Σ_1-formulae $\varphi_0, \ldots, \varphi_4$ and $\hat{\varphi}_0, \ldots, \hat{\varphi}_4$ with two free variables, so that for every pPs-structure N with $\mathrm{ht}(N) = \mu$, whose Λ-image exists, do we have:*

(0) If $\mu = \bar{\mu} + 1$, $N||\bar{\mu}$ is active and $\bar{\mu} < \lambda(E^N_{\bar{\mu}})$, φ_0 defines f_N over N, and $\hat{\varphi}_0$ defines $\hat{f}_N$ over $\widehat{N}$.

(1) If $\mu = \bar{\mu} + 1$ and either $N||\bar{\mu}$ is active and $\lambda(E^N_{\bar{\mu}}) < \bar{\mu}$, or $N||\bar{\mu}$ is passive and $\bar{\mu} = \omega\bar{\mu}$, then φ_1 defines f_N over N, and $\hat{\varphi}_1$ defines $\hat{f}_N$ over $\widehat{N}$.

(2) If $\mu = \bar{\mu} + 1$, $N||\bar{\mu}$ is passive and $\bar{\mu} < \omega\bar{\mu}$, then φ_2 defines f_N over N, and $\hat{\varphi}_2$ defines $\hat{f}_N$ over $\widehat{N}$.

(3) If μ is a limit ordinal and $N||\mu$ is passive, then φ_3 defines f_N over N, and $\hat{\varphi}_3$ defines $\hat{f}_N$ over $\widehat{N}$.

(4) If $N||\mu$ is active, then φ_4 defines f_N over N, and $\hat{\varphi}_4$ defines $\hat{f}_N$ over $\widehat{N}$.

The entire constructions of the translations of Σ_1-formulae from M to $\mathtt{S}(M)$ can be carried out for the (Pseudo-)Σ_0-Codes of the structures involved as well.

Corollary 5.20. *For every pPs-structure N whose Λ-image M exists, there is a sequence $F^{\mathrm{p}} = \langle f^{\mathrm{p}}_\mu \mid \mu \leq \mathrm{ht}(\tilde{\mathcal{C}}_0(N))\rangle$, so that for $\mu \leq \mathrm{ht}(N)$, the following holds:*

(a) Let φ be a formula in the language of $\tilde{\mathcal{C}}_0(M)$. Then $f^{\mathrm{p}}_\mu(\varphi)$ is a formula in the language of $\tilde{\mathcal{C}}_0(N)$ such that

$$\tilde{\mathcal{C}}_0(\Lambda(N||\mu)) \models \varphi[\vec{\xi}] \quad\Longleftrightarrow\quad \tilde{\mathcal{C}}_0(N||\mu) \models f^{\mathrm{p}}_\mu(\varphi)[\vec{\xi}, \mu \dot{-} 1],$$

where $\vec{\xi} < \omega\mu$.

(b) Let $N||\mu$, $\Lambda(N||\mu)$ be Ps, Pλ-structures, respectively. Let φ be a formula in the language of $\mathcal{C}_0(M)$. Then $f^{\mathrm{p}}_\mu(\varphi)$ is a formula in the language of $\mathcal{C}_0(N)$ such that

$$\mathcal{C}_0(\Lambda(N||\mu)) \models \varphi[\vec{\xi}] \quad\Longleftrightarrow\quad \mathcal{C}_0(N||\mu) \models f^{\mathrm{p}}_\mu(\varphi)[\vec{\xi}, \mu \dot{-} 1],$$

where $\vec{\xi} < \omega\mu$.

(c) If φ is a Σ_1 formula, then so is $f^{\mathrm{p}}_\mu(\varphi)$.

(c) f^{p}_μ is uniformly $\Sigma_\omega(N||\mu)$.

(d) The relation $R^{\mathrm{p}} = \{\langle n, m, \gamma\rangle \mid n = f^{\mathrm{p}}_\gamma(m) \wedge \gamma < \mathrm{ht}(N)\}$ is uniformly $\Sigma_1(N)$.

(e) The function $f^{\mathrm{p}}_N := f^{\mathrm{p}}_{\mathrm{ht}(N)}$ is uniformly $\Sigma_1(\tilde{\mathcal{C}}_0(N))$ for structures N, that are of the same type (0)-(4) of Corollary 5.19.

Proof. As the structures $\tilde{\mathcal{C}}_0(N)$ and $\mathcal{C}_0(N)$ (and, analogously, $\tilde{\mathcal{C}}_0(M)$ and $\mathcal{C}_0(M)$) only differ substantially from N (analogously, M) if they are active, this is the only case in which there is something to be shown. So I am going to derive a suitable definition of f^{p}_μ in case $\bar{N} = N||\mu$ is active. Let $\bar{M} = \Lambda(\bar{N})$.

I will first define a transformation $\hat{t}_1$ of formulae, which translates Σ_1-formulae in the language of $\tilde{\mathcal{C}}_0(\bar{M})$ into Σ_1-formulae in the language of $\tilde{\mathcal{C}}_0(\widehat{\bar{N}})$ in such a way that

$$\tilde{\mathcal{C}}_0(\bar{M}) \models \varphi[\vec{a}] \iff \tilde{\mathcal{C}}_0(\widehat{\bar{N}}) \models \hat{t}_1(\varphi)[\vec{a}].$$

To this end, let's recapitulate an argument from the proof of Lemma 5.17 (see the definition of $\hat{e}^\mu_{\mathrm{top}}$ in case 3.B):

$$\begin{aligned}
& \tilde{\mathcal{C}}_0(\bar{M}) \models && \varphi[\vec{a}] \\
\iff\ & \tilde{\mathcal{C}}_0(\widehat{\bar{N}}) \models && \exists\alpha\exists f\exists d\exists Q \\
& && \quad(\widehat{\bar{N}}||\alpha \text{ is passive } \wedge \text{“}Q = \Lambda(\widehat{\bar{N}}||\alpha)\text{”} \wedge f = \dot{F} \cap |Q| \wedge \\
& && \quad\wedge d = D_Q \wedge \langle Q^{\text{passive}}, f, d, \dot{s}\rangle \models \varphi[\vec{x}]) \\
\overset{\text{def}}{\iff}\ & \widehat{\bar{N}} \models && \hat{t}_1(\varphi)[\vec{a}].
\end{aligned}$$

This works precisely for the reasons stated before, and because $\dot{s}^{\tilde{\mathcal{C}}_0(\hat{\bar{N}})} = \dot{s}^{\tilde{\mathcal{C}}_0(\bar{M})}$.

Analogously, in case $\bar{M}$ and $\bar{N}$ are potential λ- and s-structures, respectively, and φ is a Σ_1-formula in the language associated to $\mathcal{C}_0(\bar{M})$ in which the symbol $\dot{q}$ occurs (I demand the latter in order to insure the unambiguity of the definition of $\hat{t}_1$):

$$\begin{array}{rrl} & \mathcal{C}_0(\bar{M}) \models & \varphi[\vec{a}] \\ \iff & \tilde{\mathcal{C}}_0(\hat{\bar{N}}) \models & \exists\alpha\exists f\exists d\exists Q \\ & & \quad (\hat{\bar{N}}||\alpha \text{ is passive } \wedge \text{“}Q = \Lambda(\hat{\bar{N}}||\alpha)\text{”} \wedge f = \dot{F} \cap |Q| \wedge \\ & & \quad \wedge d = D_Q \wedge \langle Q^{\text{passive}}, f, d, \dot{s}, \dot{q}\rangle \models \varphi[\vec{x}]) \\ \overset{\text{def}}{\iff} & \hat{\bar{N}} \models & \hat{t}_1(\varphi)[\vec{a}]. \end{array}$$

This works, since by Lemma 4.16, $C_{\bar{M}} = C_{\hat{\bar{N}}}$, and thus $\dot{q}^{\mathcal{C}_0(\hat{\bar{N}})} = \dot{q}^{\mathcal{C}_0(\bar{M})}$ (this Lemma is only relevant in case $\bar{M}$ is of type II, as otherwise, $\bar{N}$ is not of type II either, and thus, by definition, $\dot{q}^{\mathcal{C}_0(\hat{\bar{N}})} = \dot{q}^{\mathcal{C}_0(\bar{M})} = \emptyset$).

Now one has to move from $\tilde{\mathcal{C}}_0(\hat{\bar{N}})$ and $\mathcal{C}_0(\hat{\bar{N}})$ to $\tilde{\mathcal{C}}_0(\bar{N})$ and $\mathcal{C}_0(\bar{N})$, respectively.

If $\bar{M}$ is of type III, then so is $\bar{N}$, and this means that $\tilde{\mathcal{C}}_0(\bar{N}) = \tilde{\mathcal{C}}_0(\hat{\bar{N}})$ (see Def. 3.33). Moreover, $\mathcal{C}_0(\bar{N}) = \mathcal{C}_0(\hat{\bar{N}})$, by Def. 3.38. Hence, in this case, one can set: $f^{\mathtt{p}}_{\mu}(\varphi) = \hat{t}_1(\varphi)$, for Σ_1-formulae φ. Note that in $\tilde{\mathcal{C}}_0(\bar{N})$ it can be decided very easily whether or not $\bar{N}$ is of type III, as this is the case iff $\bar{N}$ is active and $\dot{s}^{\tilde{\mathcal{C}}_0(\bar{N})} \neq \emptyset$. This is essential for condition (e).

Now consider the case that $\bar{M}$ is not of type III. Then I make use of the function d from Lemma 5.11 and define $f^{\mathtt{p}}_{\mu}(\varphi) = d(\hat{t}_1(\varphi))$, for Σ_1-formulae φ in the language of $\mathcal{C}_0(\bar{N})$.

Thus far, $f^{\mathtt{p}}$ was defined for Σ_1-formulae. In order to expand the definition to arbitrary Σ_ω-formulae, I first define an auxiliary function t as in the proof of Lemma 5.17, Case 3.B., and get:

$$\tilde{\mathcal{C}}_0(\bar{M}) \models \varphi[\vec{a}] \iff \tilde{\mathcal{C}}_0(\hat{\bar{N}}) \models t(\varphi)[\vec{a}],$$

and the corresponding for $\mathcal{C}_0(\bar{M})$ and $\mathcal{C}_0(\hat{\bar{N}})$. One can use as definition of t exactly the one made in the proof quoted above, augmented by the following trivial clauses in the case of atomic formulae:

$$\begin{array}{rcll} t(xR\dot{s}) & := & xR\dot{s} & \quad \text{for } R \in \{\dot{=}, \dot{\in}\} \\ t(\dot{q}(x)) & := & \dot{q}(x). & \end{array}$$

Again, this works because the constants involved are interpreted the same way in the structures involved.

In order to get from $\tilde{\mathcal{C}}_0(\hat{\bar{N}})$ or $\mathcal{C}_0(\hat{\bar{N}})$ to $\tilde{\mathcal{C}}_0(\bar{N})$ and $\mathcal{C}_0(\bar{N})$, resp., one can proceed as before in the proof of 5.17, case 3.B. The function $\tilde{d}$ is expanded in the obvious way, and finally, one can define: $f^{\mathtt{p}}_{\mu}(\varphi(\vec{x})) := \exists\vec{z} \ \ (\bigwedge_{j=1}^{n} \text{“}z_j = x^{\circ}_j\text{”} \wedge \tilde{d}(\varphi)(^{\vec{x}}/_{\vec{z}}))$ for formulae φ which are not Σ_1, as before. □

6 Projecta

The aim of this chapter is to show that for $1 \leq n < \omega$, the n-th projecta of M and $\mathtt{S}(M)$ (and those of their corresponding Σ_0-codes) are the same. Not surprisingly, the proof proceeds by induction on n. The next section is the base case of the induction. Before beginning, as a reminder, and to fix the notation, let's quote the following definition from [Zem02, P. 34].

Definition 6.1. Let $<^*$ be the well-ordering of $[\mathrm{On}]^{<\omega}$ defined by:

$$a <^* b \iff a \neq b \wedge \max(a \Delta b) \in b,$$

where $a\Delta b$ is the symmetric difference $(a \setminus b) \cup (b \setminus a)$.

Let M be an acceptable J-model. Then set, for $a \in [\mathrm{On}_M]^{<\omega}$ and $i < \omega$:

$$a^i := a \cap [\omega\rho_M^{i+1}, \omega\rho_M^i);$$

I suppress any mention of M in the notation when it's clear from context which M is meant. The *standard parameter* of M, p_M, is defined by:

$$p_M :=<^* -\min(P_M^* \cap [\mathrm{On}]^{<\omega}),$$

and for $n < \omega$ let $p_{M,n}$, the *standard parameter above* $\omega\rho_M^n$, is defined by:

$$p_{M,n} :=<^* -\min(P_M^n \cap [\mathrm{On}]^{<\omega}).$$

6.1 The first projecta of M and $\mathtt{S}(M)$

Lemma 6.2. *Let M be a pPλ-structure s.t. $N = \mathtt{S}(M)$ exists. Then $P_M \cap [\mathrm{ht}(N)]^{<\omega} \neq \emptyset$. In particular, $p_{M,1} \subseteq \mathrm{ht}(N)$. The same is true for p_M^0 (note: $p_M^0 = p_M \setminus \omega\rho_M^1$).*

Proof. By Lemma 4.5, $h_M^1(\mathrm{ht}(N)) = |M|$, so there is a $p \in P_M \cap [\mathrm{ht}(N)]^{<\omega}$. As $p_{M,1}$ is $<^*$-minimal in P_M, it follows that $p_{M,1} \subseteq \mathrm{ht}(N)$. To see that $p_M^0 \subseteq \mathrm{ht}(N)$, note that if $p \in P_M^*$ is such that $p' := p^0 \setminus \mathrm{ht}(N) \neq \emptyset$, then I can set $\bar{p} := (p \cap \mathrm{ht}(N)) \cup s$, where $s \in [\mathrm{ht}(N)]^{<\omega}$ is such that every $\gamma \in p'$ has the form $h_M^1(\langle i, r\rangle)$, where i, r are s.t. $\{i\} \cup r \subseteq s$. Then $\bar{p} \in P_M^*$, and obviously, $\bar{p} <^* p$. As p_M is $<^*$-minimal in P_M^*, it follows that p_M, and hence p_M^0, is contained in $\mathrm{ht}(N)$. □

Remark 6.3. Let M be a pPλ-structure for which $\mathtt{S}(M)$ is defined. Then $\rho_M^1 \leq \mathrm{ht}(\mathtt{S}(M))$.

Proof. By Lemma 3.25, $\rho_M^1 \leq \mathrm{otp}(D_M^*)$, and by Lemma 4.5,

$$\mathrm{ht}(\mathtt{S}(M)) = \begin{cases} \mathrm{otp}(D_M) & \text{if } M \text{ is passive,} \\ \cup\mathrm{otp}(D_M^*) & \text{otherwise.} \end{cases}$$

So if M is passive, we're done. Otherwise, either $s^+(M) = \mathrm{ht}(M) = \mathrm{ht}(\mathtt{S}(M))$, or $s^+(M) < \mathrm{ht}(M)$ and $\mathrm{otp}(D_M^*)$ is a successor ordinal, and hence $\rho_M^1 \leq \cup\mathrm{otp}(D_M^*)$. □

Lemma 6.4. *Let $\mathtt{S}(M)$ be defined. Then:*

(a) $\omega\rho_M^1 = \omega\rho^1_{\widehat{\mathtt{S}(M)}} = \omega\rho^1_{\mathtt{S}(M)}$,

(b) $\mathbf{\Sigma}_1(M) \cap \mathcal{P}(H_M^1) = \mathbf{\Sigma}_1(\widehat{\mathtt{S}(M)}) \cap \mathcal{P}(H^1_{\widehat{\mathtt{S}(M)}}) = \mathbf{\Sigma}_1(\mathtt{S}(M)) \cap \mathcal{P}(H^1_{\mathtt{S}(M)})$.

Hence, $\omega\rho_N^1$, and $\mathbf{\Sigma}_1(N) \cap \mathcal{P}(H_N^1)$, are the same for $N \in \{M, \mathtt{S}(M), \widehat{\mathtt{S}(M)}, \tilde{\mathcal{C}}_0(M), \tilde{\mathcal{C}}_0(\mathtt{S}(M)), \tilde{\mathcal{C}}_0(\widehat{\mathtt{S}(M)}), \mathcal{C}_0(M), \mathcal{C}_0(\mathtt{S}(M)), \mathcal{C}_0(\widehat{\mathtt{S}(M)})\}$.

Proof. First of all, we may assume that $\mathrm{otp}(D_M) \geq \omega$, since if this is not the case, $\mathtt{S}(M) = \widehat{\mathtt{S}(M)} = M^-$, so that the lemma trivializes in this case, because then D_M, being a finite set, can be treated as a parameter.

So let's show the first identity in claim (a). Two directions must be verified.

$\boxed{\omega\rho^1_M \geq \omega\rho^1_{\widehat{\mathtt{S}(M)}}, \omega\rho^1_{\mathtt{S}(M)}}$ I start by showing the first part. Assume the contrary, i.e., that $\omega\rho^1_M < \omega\rho^1_{\widehat{\mathtt{S}(M)}}$. Then let A be $\Sigma_1(M)$ in $p_{M,1}$ s.t. $A \cap \omega\rho^1_M \notin M$. Let φ be a Σ_1-formula defining A, so that for arbitrary $x \in M$:

$$x \in A \iff M \models \varphi[x, p_{M,1}].$$

I want to use Corollary 5.18 in order to transfer this to $\widehat{\mathtt{S}(M)}$. To be able to do that, note that $p_{M,1} \subseteq \mathrm{ht}(\mathtt{S}(M))$, by Lemma 6.2. Moreover, $\omega\rho^1_M \leq \mathrm{otp}(D_M) = \mathrm{ht}(\widehat{\mathtt{S}(M)})$: If $\rho^1_M > 1$, then $\omega\rho^1_M = \rho^1_M \leq \mathrm{otp}(D^*_M) \leq \mathrm{otp}(D_M)$, and if $\rho^1_M = 1$, then $\omega\rho^1_M = \omega \leq \mathrm{otp}(D_M)$ by assumption.

So for $\gamma < \omega\rho^1_M$,

$$M \models \varphi[\gamma, p_{M,1}] \iff \widehat{\mathtt{S}(M)} \models \widehat{f}(\varphi)[\gamma, q^1_M, \mathrm{ht}(\widehat{\mathtt{S}(M)}) \dot{-} 1],$$

where $\widehat{f} = \widehat{f}^{\mathtt{S}(M)}_{\mathrm{ht}(\widehat{\mathtt{S}(M)})}$ is defined as in 5.18 – the above equivalence in fact holds for arbitrary $\gamma < \mathrm{On}_{\widehat{\mathtt{S}(M)}}$. So $A \cap \omega\rho^1_M$ is $\mathbf{\Sigma}_1(\widehat{\mathtt{S}(M)})$, hence, since it was assumed that $\omega\rho^1_M < \omega\rho^1_{\widehat{\mathtt{S}(M)}}$, it follows that $A \cap \omega\rho^1_M \in \widehat{\mathtt{S}(M)} \subseteq M$, a contradiction.

In order to show that $\omega\rho^1_M \geq \omega\rho^1_{\mathtt{S}(M)}$, I use the same argument. By Lemma 6.2, $q^1_M \subseteq \mathrm{ht}(\mathtt{S}(M))$, and instead of $\widehat{f}$ I now use $f^{\mathtt{S}(M)}_{\mathrm{ht}(\mathtt{S}(M))}$ from Corollary 5.18.

$\boxed{\omega\rho^1_M \leq \omega\rho^1_{\widehat{\mathtt{S}(M)}}, \omega\rho^1_{\mathtt{S}(M)}}$ Here, make essential use of Lemma 5.16. Suppose the contrary. Then let A be a set that's defined over $\widehat{\mathtt{S}(M)}$ in the parameter q by a $\mathbf{\Sigma}_1$-formula $\varphi(x,y)$, s.t. $A \cap \omega\rho^1_{\widehat{\mathtt{S}(M)}} \notin \widehat{\mathtt{S}(M)}$. So,

$$x \in A \longleftrightarrow \mathtt{S}(M) \models \varphi[x, q].$$

Now it follows from part (a) of the abovementioned Lemma, that the set $Z = \{\langle a,b\rangle \mid \mathtt{S}(M) \models \varphi[x,y]\}$ is $\Sigma_1(M)$ in $\mathrm{ht}(M) \dot{-} 1$. In particular, $A = \{x \mid \langle x, q\rangle \in Z\}$ is $\Sigma_1(M)$ in the parameter $q \cup \{\mathrm{ht}(M) \dot{-} 1\}$. Hence, $\bar{A} := A \cap \omega\rho^1_{\widehat{\mathtt{S}(M)}} \in M$, since, by assumption, $\omega\rho^1_M > \omega\rho^1_{\widehat{\mathtt{S}(M)}}$. By acceptability of M, this entails that $\bar{A} \in \mathrm{J}^{E^M}_{\rho^1_M}$, since ρ^1_M is a cardinal of M. For the same reason, $\rho^1_M \in D_M$. Now if M is active, then $|M| = |\widehat{\mathtt{S}(M)}|$, and so $\bar{A} \in |\widehat{\mathtt{S}(M)}|$, a contradiction. If, on the other hand, M is passive, then, by Lemma 4.3, the fact that $\rho^1_M \in D_M$ implies that $\mathtt{S}(M||\rho^1_M)$ is a segment of $LS(M)$. Hence, $|\mathtt{S}(M||\rho^1_M)| \subseteq |\widehat{\mathtt{S}(M)}|$. But by Lemma 4.15, $|M||\rho^1_M| = |\mathtt{S}(M||\rho^1_M)|$. Hence $|M||\rho^1_M| \subseteq |\widehat{\mathtt{S}(M)}|$, and this means that $\bar{A} \in \widehat{\mathtt{S}(M)}$, which contradicts the choice of A.

To see that $\omega\rho^1_M \leq \omega\rho^1_{\mathtt{S}(M)}$, again assume the contrary. So let A be $\mathbf{\Sigma}_1(\tilde{\mathcal{C}}_0(\mathtt{S}(M)))$ s.t. $A \cap \omega\rho^1_{\mathtt{S}(M)} \notin |\mathtt{S}(M)|$. By part (d) of Lemma 5.16, it then follows that A is $\mathbf{\Sigma}_1(\tilde{\mathcal{C}}_0(M))$ in the same parameters and $\mathrm{ht}(M) \dot{-} 1$. Hence, A is $\mathbf{\Sigma}_1(M)$. The contradiction is now reached as before – only in order to see that $|\mathtt{S}(M||\rho^1_M)| \subseteq |\mathtt{S}(M)|$, an additional argument is needed:

If M is passive, then $\mathtt{S}(M||\rho^1_M)$ even is a segment of $\mathtt{S}(M)$: By Lemma 4.3, it suffices to see that $\omega\rho^1_M \notin [s^+(M||\mu), \mu)$, for every μ which indexes an extender in M. As $\omega\rho^1_M$ is a cardinal in M, it even follows that $\omega\rho^1_M \notin (s(M||\mu), \mu]$, for every such μ (it is essential here that $\mu < \mathrm{ht}(M)$, as M is passive).

But if M is active, then $\rho^1_M \leq s^+(M) = \cup\mathrm{otp}(D^*_M)$, and

$$|\mathtt{S}(M||\rho^1_M)| = |M||\rho^1_M| \subseteq |M||s^+(M)| = |\mathtt{S}(M||s^+(M)| = |\mathtt{S}(M)|.$$

The rest of the argument works as before. $\square_{(a)}$

Let's turn to the proof of (b): From the proof of (a) it follows that $H^1_M = H^1_{\widehat{\mathtt{S}(M)}}$. To see that $\mathbf{\Sigma}_1(M) \cap \mathcal{P}(\omega\rho^1_M) = \mathbf{\Sigma}_1(\mathtt{S}(M)) \cap \mathcal{P}(\omega\rho^1_M)$, the arguments from the proof of (a) work. One only has to check that one can work with parameters from $|\mathtt{S}(M)|$. But this follows from the fact that $|M| = h^1_M(\mathrm{ht}(\mathtt{S}(M)))$. It's now easy to see how to get the desired claim by working with a surjection from $\omega\rho^1_M$ onto H^1_M that's Σ_1-definable over $H^1_M = H^1_{\mathtt{S}(M)}$, and re-using the arguments from the proof of (a); this is only necessary for the direction from left to right. The other direction is immediate, by Lemma 5.16. The part of the claim concerning $\widehat{\mathtt{S}(M)}$ is shown analogously. $\square_{(b)}$

The consequence of (a) and (b) that's drawn in the statement of the Lemma follows, because the additional constants that are available in the Σ_0-codes of the structures involved can be viewed as additional parameter. $\square$

7.2 The n-th projecta of M and $\mathtt{S}(M)$

From the results of the previous section, I want to deduce:

Lemma 7.5. *Let $\mathtt{S}(M)$ be defined. Then for $n \geq 1$:*

(a) $\omega\rho^n_M = \omega\rho^n_{\mathtt{S}(M)}$,

(b) $\mathbf{\Sigma}^{(n-1)}_1(M) \cap \mathcal{P}(H^n_M) = \mathbf{\Sigma}^{(n-1)}_1(\mathtt{S}(M)) \cap \mathcal{P}(H^n_{\mathtt{S}(M)})$.

Again, it even follows that $\omega\rho^n_N$, $\mathbf{\Sigma}^{(n-1)}_1(N) \cap \mathcal{P}(H^n_N)$ are the same for every $N \in \{M, \mathtt{S}(M), \widehat{\mathtt{S}(M)}, \tilde{\mathcal{C}}_0(M), \tilde{\mathcal{C}}_0(\mathtt{S}(M)), \tilde{\mathcal{C}}_0(\widehat{\mathtt{S}(M)}), \mathcal{C}_0(M), \mathcal{C}_0(\mathtt{S}(M)), \mathcal{C}_0(\widehat{\mathtt{S}(M)})\}$.

But before proving this, I need some fine structural basics. First, let's quote the following fundamental Lemma on $\Sigma^{(n)}_l$-relations that can be found in [Zem97, 1.1.25] or [Zem02, 1.6.3].

Lemma 7.6. *Let M be an acceptable J-model. Let $n, l < \omega$. $R(\vec{x}^{n+1}, \ldots, \vec{x}^0)$ is $\Sigma^{(n+1)}_l(M)$ iff the relation $R_{\vec{x}} := \{\langle\vec{x}^{n+1}\rangle \mid R(\vec{x}^{n+1}, \vec{x})\}$ (here, $\vec{x} = \vec{x}^n, \ldots, \vec{x}^0$) is uniformly $\Sigma_l(\langle H^{n+1}_M, Q^1_{\vec{x}}, \ldots, Q^m_{\vec{x}}\rangle)$, where every $Q^i_{\vec{x}}$ $(i = 1, \ldots, m)$ is of the form*

$$Q^i_{\vec{x}} = \{\langle\vec{z}^{n+1}\rangle \mid Q^i(\vec{z}^{n+1}, \vec{x})\},$$

and $Q^i(\vec{z}^{n+1}, \vec{x})$ $\Sigma^{(n)}_1(M)$.

I will use this lemma in order to show the following:

Lemma 7.7. *Let M be an acceptable J-model, and let $B(\vec{y}^n, \vec{x}^n, \ldots, \vec{x}^0)$ be $\Sigma^{(n)}_1(M)$. Then there is some $i < \omega$ with the following property:*

For all $\vec{y}^{n+1}$ and all $\vec{r} = \vec{r}^n, \ldots, \vec{r}^0$,

$$B(\vec{y}^{n+1}, \vec{r}) \iff A^{n+1,\langle\vec{r}\rangle}_M(i, \langle\vec{y}^{n+1}\rangle).$$

Proof. Induction on n.

$\boxed{\text{n=0}}$ In this case, B is $\Sigma_1(M)$. So let

$$B(\vec{y}^0, \vec{x}^0) \iff M \models \varphi_i[\langle \vec{y}^0 \rangle, \langle \vec{x}^0 \rangle].$$

We now have, for all $\vec{r}^0$ and $\vec{y}^1$ ($\vec{y}^1 \in H^1_M$):

$$A^{1,\langle \vec{r}^0 \rangle}_M(i, \langle \vec{y}^1 \rangle) \iff M \models \varphi_i[\langle \vec{y}^1 \rangle, \langle \vec{r}^0 \rangle] \iff B(\vec{y}^1, \vec{r}^0),$$

by definition of $A^{1,\langle \vec{r}^0 \rangle}_M$. Hence, i is as wished.

$\boxed{n \to n+1}$ Let $B(\vec{y}^{n+1}, \vec{x}^{n+1}, \vec{x}^n, \ldots, \vec{x}^0)$ be $\Sigma_1^{(n+1)}(M)$. By lemma 7.6, $B_{\vec{x}^n,\ldots,\vec{x}^0}$ is uniformly $\Sigma_1(\langle H^{n+1}_M, Q^1_{\vec{x}^n,\ldots,\vec{x}^0}, \ldots, Q^m_{\vec{x}^n,\ldots,\vec{x}^0} \rangle)$ for suitable relations $Q^j(\vec{z}^{n+1}, \vec{x}^n, \ldots, \vec{x}^0)$, that are $\Sigma_1^{(n)}(M)$ (for $1 \leq j \leq m$). We can substitute the variables $\vec{z}^{n+1}$ in Q^j by $\vec{z}^n$, in order to get relations $\tilde{Q}^j(\vec{z}^n, \vec{x}^n, \ldots, \vec{x}^0)$, so that Q^j is a specialization of the $\Sigma_1^{(n)}(M)$ relation $\tilde{Q}^j$.

Now the inductive hypothesis can be applied to the $\tilde{Q}^j(\vec{z}^n, \vec{x}^n, \ldots, \vec{x}^0)$. We get i_j such that for all $\vec{z}^{n+1}$ and all $\vec{x} = \vec{x}^n, \ldots, \vec{x}^0$,

$$Q^j(\vec{z}^{n+1}, \vec{x}) \iff \tilde{Q}^j(\vec{z}^{n+1}, \vec{x}) \iff A^{n+1,\langle \vec{x} \rangle}_M(i_j, \langle \vec{z}^{n+1} \rangle).$$

Now let φ be a Σ_1 formula in the language with additional predicate symbols $\dot{Q}^1, \ldots, \dot{Q}^m$, so that

$$\begin{array}{rl} & \forall \vec{y}^{n+1}, \vec{x}^{n+1}, \vec{x}^n, \ldots, \vec{x}^0 \quad (B(\vec{y}^{n+1}, \vec{x}^{n+1}, \vec{x}^n, \ldots, \vec{x}^0) \\ \iff & \langle H^{n+1}_M, Q^1_{\vec{x}^n,\ldots,\vec{x}^0}, \ldots, Q^m_{\vec{x}^n,\ldots,\vec{x}^0} \rangle \models \varphi[\vec{y}^{n+1}, \vec{x}^{n+1}]). \end{array}$$

Let $\tilde{\varphi}$ denote the formula that results from φ by replacing every occurrence of $\dot{Q}^j(w)$ with $\dot{A}(i_j, w)$. Then $\tilde{\varphi}$ is a Σ_1 formula in the language of $\langle H^{n+1}, A^{n+1,\langle \vec{x}^n,\ldots,\vec{x}^0 \rangle}_M \rangle$, and it follows that for all $\vec{y}^{n+1}, \vec{x}^{n+1}, \vec{x}^n, \ldots, \vec{x}^0$,

$$\begin{array}{rcl} & \langle H^{n+1}_M, Q^1_{\vec{x}^n,\ldots,\vec{x}^0}, \ldots, Q^m_{\vec{x}^n,\ldots,\vec{x}^0} \rangle & \models \varphi[\vec{y}^{n+1}, \vec{x}^{n+1}] \\ \iff & \langle H^{n+1}, A^{n+1,\langle \vec{x}^n,\ldots,\vec{x}^0 \rangle}_M \rangle & \models \tilde{\varphi}[\vec{y}^{n+1}, \vec{x}^{n+1}] \\ \iff & M^{n+1,\langle \vec{x}^n,\ldots,\vec{x}^0 \rangle} & \models \varphi_i[\vec{y}^{n+1}, \vec{x}^{n+1}], \end{array}$$

Here, i is the number of $\tilde{\varphi}$ in some canonical enumeration of the Σ_1 formulae of the language with additional constant symbol $\dot{A}$ with two free variables.

Hence, the above holds, in particular, for $\vec{y}^{n+2}$ instead of $\vec{y}^{n+1}$. Letting $\vec{r'} = \vec{x}^n, \ldots, \vec{x}^0$,

$$\begin{array}{rl} & B(\vec{y}^{n+2}, \vec{x}^{n+1}, \vec{r'}) \\ \iff & M^{n+1,\langle \vec{r'} \rangle} \models \varphi_i[\vec{y}^{n+2}, \vec{x}^{n+1}] \\ \iff & A^{n+2,\langle \vec{x}^{n+1}, \vec{r'} \rangle}_M(i, \langle \vec{y}^{n+2} \rangle). \end{array}$$

hence setting $\vec{r} = \langle \vec{x}^{n+1}, \vec{r'} \rangle$:

$$B(\vec{y}^{n+2}, \vec{r}) \iff A^{n+2,\langle \vec{r} \rangle}_M(i, \langle \vec{y}^{n+2} \rangle),$$

so i has the desired properties. □

Instead of proceeding directly to a proof of Lemma 7.5, I am going to attack a more general result – mainly so in order to emphasize that no special properties of pPλ-structures or $\mathtt{S}$ are needed here.

Lemma 7.8. *Let M and N be acceptable J-models with:*

(i) $\mathbf{\Sigma}_1(M) \cap \mathcal{P}(H^1_M) = \mathbf{\Sigma}_1(N) \cap \mathcal{P}(H^1_N)$.

(ii) for $\alpha \in \mathrm{Card}^M \cap \mathrm{Card}^N$ $H^M_\alpha = H^N_\alpha$.

Then for every $n \geq 1$*:*

(a) $\omega\rho^n_M = \omega\rho^n_N$,

(b) $\mathbf{\Sigma}_1^{(n-1)}(M) \cap \mathcal{P}(H^n_M) = \mathbf{\Sigma}_1^{(n-1)}(N) \cap \mathcal{P}(H^n_N)$.

Proof. Induction on n.

For $n = 1$, what has to be shown is just the assumption (that $\omega\rho^1_M = \omega\rho^1_N$ follows immediately). So let $n \geq 1$, and assume (a) and (b) hold. First I am going to show that (a) holds for $n + 1$ as well:

Really there are two directions to prove, but since the roles played by M and N are completely symmetric, it suffices to show that, e.g., $\omega\rho^{n+1}_M \leq \omega\rho^{n+1}_N$. To this end, let B be $\Sigma_1(N^{n,q})$ in some parameter $r \in [\omega\rho^n_N]^{<\omega}$, for a $q \in P^n_N$, so that $B \cap \omega\rho^{n+1}_N \notin |N|$. Now $N^{n,q} = \langle H^n_N, A^{n,q}_N\rangle$, and

$$A := A^{n,q}_N \in \mathbf{\Sigma}_1^{(n-1)}(N) \cap \mathcal{P}(H^n_N) = \mathbf{\Sigma}_1^{(n-1)}(M) \cap \mathcal{P}(H^n_M)$$

by the inductive assumption (b). So let A be $\Sigma_1^{(n-1)}(M)$ in $p \in \Gamma^{n-1}_M$ (for an explanation of the notation, I refer to [Zem02, p. 18]). Then by Lemma 7.7:

$$\forall y^n \quad y^n \in A \longleftrightarrow A^{n,p}_M(i, y^n),$$

for an adequate $i < \omega$. Let B be defined over $N^{n,q}$ by the Σ_1-formula φ, i.e., let

$$x \in B \iff \langle H^n_N, A\rangle \models \varphi[x, r].$$

Let $\tilde{\varphi}$ be the Σ_1-formula obtained by replacing every occurrence of $\dot{A}(j, z)$ in φ with $\dot{A}(i, \langle j, z\rangle)$. Then

$$\langle H^n_N, A\rangle \models \varphi[x, r] \iff \langle H^n_M, A^{n,p}_M\rangle \models \tilde{\varphi}[x, r].$$

This shows that B is $\mathbf{\Sigma}_1(M^{n,p})$. So if it were the case that $\omega\rho^{n+1}_N < \omega\rho^{n+1}_M$, then it would follow that $B \cap \omega\rho^{n+1}_N \in M$. But by acceptability of M, this would mean that $B \cap \omega\rho^{n+1}_N \in H^n_M = H^n_N$, a contradiction.

This shows (a) for $n + 1$.

To see that (b) holds as well, first note that it follows from the fact that the $(n + 1)$-st projecta of M and N coincide that the $(n+1)$-st reducts coincide as well. This is because either $\omega\rho^{n+1} < \omega\rho^1$, in which case $\omega\rho^{n+1}$ is a cardinal both in M and in N, so that (ii) can be applied, or $\omega\rho^{n+1} = \omega\rho^1$, and then, according to (i), $H^{n+1}_M = H^1_M = H^1_N = H^{n+1}_N$ (I wrote $\omega\rho^m$ for $\omega\rho^m_M = \omega\rho^m_N$ here, when $m \leq n + 1$).

Every $\bar{B} \in \mathbf{\Sigma}_1^{(n)}(N) \cap \mathcal{P}(H^{n+1}_N)$ can be rendered as in the proof of (a), i.e., as the intersection of an element of B in $\mathbf{\Sigma}_1(N^{n,q})$ with H^{n+1}_N, for some $q \in \Gamma^n_N$. But $H^{n+1}_N = H^{n+1}_M$, and the proof of (a) then yields that $\bar{B} = B' \cap H^{n+1}_M$ for some $B' \in \mathbf{\Sigma}_1(M^{n,p})$ and a suitable $p \in \Gamma^n_M$. This means precisely that $\bar{B} \in \mathbf{\Sigma}_1^{(n)}(M) \cap \mathcal{P}(H^{n+1}_M)$.

Again, due to symmetry, it suffices to prove this inclusion. □

Proof of Lemma 7.5. The claim follows in an obvious way from Lemma 6.4 and Lemma 7.8 – here, set $N = \mathtt{S}(M)$. The preconditions are then obviously satisfied. □

8 Soundness and Solidity

In this section, I am going to show that soundness and 1-solidity carry over from M to $\mathtt{S}(M)$.

8.1 Iterated Standard Parameters

Since up to now the focus was on Σ_1-definability, it is advantageous to work with a slightly modified version of the usual standard parameters that I call iterated standard parameters.

Definition 8.1. Let M be an acceptable J-structure. Define a sequence $\langle q^n_M \mid n < \omega\rangle$ by recursion on n by:

$$q^n_M = \text{ the } <^* \text{ -minimal member of } P_{M^{n,\langle q^0,\ldots,q^{n-1}\rangle}}.$$

Set: $q_{M,n} := q^0_M \cup \cdots \cup q^{n-1}_M$.

Lemma 8.2. *Let $1 \le n < \omega$ with $q_{M,n} \in P^n_M$. Then $q_{M,n} = p_{M,n}$.*

Proof. Fix $n \in \omega \setminus 1$.

$\boxed{p_{M,n} \le^* q_{M,n}}$ By assumption, $q_{M,n} \in P^n_M$. But $p_{M,n}$ is the $<^*$-minimal member of P^n_M, hence $p_{M,n} \le^* q_{M,n}$.

$\boxed{q_{M,n} \le^* p_{M,n}}$ Assume the contrary. Then let $m < n$ be minimal so that $q^m_M = q^m_{M,n} >^* p^m_{M,n}$. So $r := \langle q^0_M, \ldots, q^{m-1}_M\rangle = \langle p^0_{M,n}, \ldots, p^{m-1}_{M,n}\rangle$; if $m = 0$, then $r = \emptyset$. By assumption, $r \in P^m_M$ (or, if $m = 0$, then $r = \emptyset \in P^0_M = \{\emptyset\}$). But $\langle p^0_{M,n}, \ldots, p^{m-1}_{M,n}\rangle \in P^m_M$, so

$$p^m_{M,n} \in P_{M^{m,\langle p^0_{M,n},\ldots,p^{m-1}_{M,n}\rangle}} = P_{M^{m,r}}.$$

But this is a contradiction, since by definition, q^m_M is the $<^*$-minimal member of $P_{M^{m,r}}$, hence $q^m_M \le^* p^m_{M,n}$. □

Lemma 8.3. *Let M be an acceptable J-structure whose good parameters can be lengthened (see [Zem02, p. 36]). Then for $1 \le n < \omega$, $q_{M,n} = p_{M,n}$.*

Proof. Induction on $n \ge 1$.

$\boxed{n = 1}$ In this case, the definitions of $q_{M,n}$ and $p_{M,n}$ coincide.

$\boxed{n \longrightarrow n+1}$ Assume the claim is proven for n. Since the good parameters of M can be lengthened, by [Zem02, Lemma 1.9.7]:

$$q_{M,n} = p_{M,n} = p_{M,n+1}\restriction n.$$

It remains to show that $q^n_M = p^n_{M,n+1}$.

$\boxed{q^n_M \le^* p^n_{M,n+1}}$ $p^n_{M,n+1} \in P_{M^{n,\langle p_{M,n+1}\restriction n\rangle}} = P_{M^{n,q_{M,n}}}$. But q^n_M is the $<^*$-minimal element of $P_{M^{n,q_{M,n}}}$, hence $q^n_M \le^* p^n_{M,n+1}$.

$\boxed{p^n_{M,n+1} \le^* q^n_M}$ We know that $q_{M,n} = p_{M,n} \in P^n_M$. Since $q^n_M \in P_{M^{n,q_{M,n}}}$, it follows that $q_{M,n+1} \in P^{n+1}_M$. But $p_{M,n+1}$ is the $<^*$-least member of P^{n+1}_M, hence $p_{M,n+1} \le^* q_{M,n+1}$. As $p_{M,n+1}\restriction n = p_{M,n} = q_{M,n} = q_{M,n+1}\restriction n$, it follows that $p^n_{M,n+1} \le^* q^n_M$. □

Corollary 8.4. *Let M be sound. Then for $n < \omega$, $q^n_M = p^n_M$.*

Proof. By [Zem02, Lemma 1.9.4], the good parameters of M can be lengthened, so Lemma 8.3 can be applied. □

Lemma 8.5. *Let M be an acceptable J-structure such that for every $n \in \omega$,*

$$q^n_M \in R_{M^{n,q_{M,n}}}.$$

Then for every such n,

$$q_{M,n} = p_{M,n}.$$

Moreover, M is then sound.

Proof. I show by induction on $n \geq 1$ that $q_{M,n} \in R^n_M$.

$\boxed{n = 1}$ by assumption $q^0_M \in R_M$.

$\boxed{n \longrightarrow n+1}$ by our inductive hypothesis, $q_{M,n} \in R^n_M$. By [Zem02, Lemma 1.9.3], $q_{M,n}$ can be lengthened by some q' in such a way that $q_{M,n}{}^\frown q' \in P^{n+1}_M$. This means that for every $q' \in P_{M^{n,q_{M,n}}}$, $q_{M,n}{}^\frown q' \in P^{n+1}_M$. In particular, this is true for q^n_M. Hence, $q_{M,n+1} \in P^{n+1}_M$. By assumption, $q^n_M \in R_{M^{n,q_{M,n}}}$, and by inductive hypothesis, $q_{M,n} \in R^n_M$, hence $q_{M,n+1} = q_{M,n}{}^\frown q^n_M \in R^{n+1}_M$.

This concludes the inductive proof.

Now in particular, for every $1 \leq n < \omega$, $q_{M,n} \in P^n_M$. So Lemma 8.2 can be applied, which gives us that $q_{M,n} = p_{M,n}$. That M is sound now follows from [Zem02, Lemma 1.9.6], since it was shown that $q_{M,n} = p_{M,n} \in R^n_M$ for every $n \in \omega \setminus 1$. □

Lemma 8.6. *Let M be a pPλ-structure. Then $q^0_M \subseteq D_M$.*

Proof. I use the following characterization of q^0_M:

$q^0_M = \{\nu_0, \ldots, \nu_{n-1}\}$, where for $i < n$, ν_i is the least ordinal α with the following property:

$(*)_i$ *There are a $u \in [\alpha+1]^{<\omega}$ and a set A which is $\Sigma_1(M)$ in the parameter $\{\nu_m \mid m < i\} \cup u$, so that $A \cap \omega p^1_M \notin M$.*

So n is minimal so that no α fulfilling $(*)_n$ exists.

Proof. I want to show that $q^0_M \subseteq D_M$. Assume the contrary. Let $i < n$ be least such that $\nu_i \notin D_M$. One then can choose some $\mu \in M$, so that

$$s^{+M}(\mu) < \nu_i \leq \mu.$$

Let $\kappa = \mathrm{crit}(E^M_\mu)$, $\tau = (\kappa^+)^{M||\mu}$ and $s = s^M(\mu)$, $s^+ = s^{+M}(\mu)$. Finally, let

$$\pi = \pi^{M||\mu}_s : \mathrm{J}^{E^M}_\tau \longrightarrow_{E^M_\mu} (M||\mu)^{\mathrm{passive}}.$$

Obviously π is $\Sigma_1(M)$ in the parameter s^+, because μ is uniquely determined by s^+ and definable from s^+ in a Σ_1 way, as are the parameters κ, τ and s. Now I show:

$$\nu_i \text{ is } \Sigma_1(M) \text{ in the parameters } \beta_0, \ldots, \beta_{l-1} \leq s^+.$$

Case 1: $\nu_i = \mu$.

Then $\nu_i = \sup \pi``\tau$, and hence ν_i is $\Sigma_1(M)$ in the parameter s^+, because this is true of π and τ.

Case 2: $\nu_i < \mu$.

Since s is the support of E^M_μ, there are a function $f : \kappa^m \to M||\tau$, $f \in M||\tau$ and ordinals $\gamma_0, \ldots, \gamma_{m-1} < s$, so that

$$\nu_i = \pi(f)(\gamma_0, \ldots, \gamma_{m-1}).$$

Now let h be a $\Sigma_1(M||\tau)$ surjection from τ onto $M||\tau$, and let $h(\zeta) = f$. Then $\nu_i = \pi(h(\zeta))(\gamma_0, \ldots, \gamma_{m-1})$, and hence, ν_i is Σ_1-definable over M in the parameters $\zeta, \gamma_0, \ldots, \gamma_{m-1}, s^+$, which all are $\leq s^+$.

This proves the claim, which in turn leads to a contradiction: According to the above characterization of q^0_M, choose $u \subseteq \nu_i + 1$ and a set A, so that A is $\Sigma_1(M)$ in $u \cup \{\nu_0, \ldots, \nu_{i-1}\}$ and $A \cap \omega\rho^1_M \notin M$. Set $\bar{u} = (u \setminus \{\nu_i\}) \cup \{\beta_0, \ldots, \beta_{l-1}\}$. Then A is obviously $\Sigma_1(M)$ in $\bar{u} \cup \{\nu_0, \ldots, \nu_{i-1}\}$, but $\bar{u} \subseteq \nu_i$, contradicting the minimality of ν_i satisfying $(*)_i$. $\square$

Lemma 8.7. *Let M and N be acceptable J-models with:*

(i) M is sound.

(ii) For all $q \in H^1_M$, $\{a \mid a \text{ is } \Sigma_1(M^{q^0_M}) \text{ in } q\} = \{a \mid a \text{ is } \Sigma_1(N^{q^0_N}) \text{ in } q\}$.

(iii) $\boldsymbol{\Sigma}_1(M) \cap \mathcal{P}(H^1_M) = \boldsymbol{\Sigma}_1(N) \cap \mathcal{P}(H^1_N)$.

(iv) For $\alpha \in \text{Card}^M \cap \text{Card}^N$, $H^M_\alpha = H^N_\alpha$.

(v) $q^0_M = q^0_N \in R_N$.

Then N is sound as well. For the definition of q^n_M, see definition 8.1.

Proof. By Lemma 7.8, it's clear that (iii) and (iv) together entail the following for arbitrary $n \geq 1$:

(a) $\omega\rho^n_M = \omega\rho^n_N$,

(b) $\boldsymbol{\Sigma}^{(n-1)}_1(M) \cap \mathcal{P}(H^n_M) = \boldsymbol{\Sigma}^{(n-1)}_1(N) \cap \mathcal{P}(H^n_N)$.

So in the following I'll write H^n for $H^n_M = H^n_N$ and $\omega\rho^n$ for $\omega\rho^n_M = \omega\rho^n_N$ – thus the asymmetry in (ii) vanishes.

By Corollary 8.4 we have for $n \geq 1$: $p_{M,n} = q_{M,n}$.

First, I am going to prove by induction on $n \geq 1$:

(a) For all $q \in H^n$,

$\{a \mid a \text{ is } \Sigma_1(M^{n,\langle q^0_M,\ldots,q^{n-1}_M\rangle}) \text{ in } q\} = \{a \mid a \text{ is } \Sigma_1(N^{n,\langle q^0_N,\ldots,q^{n-1}_N\rangle}) \text{ in } q\}$.

(b) $q^n_M = q^n_N \in R_{N^{n,\langle q^0_N,\ldots,q^{n-1}_N\rangle}}$.

$\boxed{n=1}$ Part (a) holds by assumption (ii).

For (b): It obviously follows from (a) that $P_{M^{q^0_M}} = P_{N^{q^0_N}}$, as $H^1_M = H^1_N$ and $\omega\rho^2_M = \omega\rho^2_N$. Hence, both sets have the same $<^*$-least member, which shows that $q^1_M = q^1_N$, which is the first part of (b). In order to see that $q^1_N \in R_{N^{q^0_N}}$, note that by Lemma 8.4, $q^1_N = q^1_M = p^1_{M,2} \in R_{M^{1,q^0_M}}$, since M is sound. Hence, there is a surjection from $\omega\rho^2_M$ onto H^1_M that is $\Sigma_1(M^{q^0_M})$ in q^1_M, and by (a), this surjection is also $\Sigma_1(N^{1,q^0_N})$ in $q^1_M = q^1_N$. Hence, $q^1_N \in R_{N^{1,q^0_N}}$, and (b) is proven for $n = 1$.

$\boxed{n \to n+1}$ Let A be $\Sigma_1(M^{n+1,\langle q^0_M,\ldots,q^n_M\rangle})$ in $q \in H^{n+1}$. Let φ be a Σ_1 formula defining A over $M^{n+1,\langle q^0_M,\ldots,q^n_M\rangle}$, i.e., for $a \in H^{n+1}$,

$$\begin{aligned} a \in A \quad &\Longleftrightarrow \quad M^{n+1,\langle q^0_M,\ldots,q^n_M\rangle} \models \varphi[a,q] \\ &\Longleftrightarrow \quad \langle H^{n+1}, A^{q^n_M}_{M^{n,\langle q^0_M,\ldots,q^{n-1}_M\rangle}}\rangle \models \varphi[a,q]. \end{aligned}$$

Set

$$\tilde{A} := \{\langle i, b\rangle \mid M^{n,\langle q_M^0,\ldots,q_M^{n-1}\rangle} \models \varphi_i[b, q_M^n]\}.$$

Hence $A^{q_M^n}_{M^{n,q_{M,n}}} = \tilde{A} \cap H^{n+1}$. Obviously $\tilde{A}$ is $\Sigma_1(M^{n,\langle q_M^0,\ldots,q_M^{n-1}\rangle})$ in q_M^n, hence $\tilde{A}$ is, by our inductive assumption (a) for n, also $\Sigma_1(N^{n,\langle q_N^0,\ldots,q_N^{n-1}\rangle})$ in $q_M^n = q_N^n$. So let an index $i < \omega$ be chosen in such a way that for $a \in H^{n+1}$,

$$\begin{aligned} a \in \tilde{A} &\iff N^{n,\langle q_N^0,\ldots,q_N^{n-1}\rangle} \models \varphi_i[a, q_N^n] \\ &\iff A^{q_M^n}_{N^{n,\langle q_N^0,\ldots,q_N^{n-1}\rangle}}(\langle i, a\rangle). \end{aligned}$$

Thus, for $a \in H^{n+1}$:

$$A^{q_M^n}_{M^{n,\langle q_M^0,\ldots,q_M^{n-1}\rangle}}(\langle j, a\rangle) \iff A^{q_N^n}_{N^{n,\langle q_N^0,\ldots,q_N^{n-1}\rangle}}(\langle i, \langle j, a\rangle\rangle).$$

Hence for $a \in H^{n+1}$:

$$\begin{aligned} a \in A &\iff \langle H^{n+1}, A^{q_M^n}_{M^{n,\langle q_M^0,\ldots,q_M^{n-1}\rangle}}\rangle \models \varphi[a, q] \\ &\iff \langle H^{n+1}, A^{q_N^n}_{N^{n,\langle q_N^0,\ldots,q_N^{n-1}\rangle}}\rangle \models \varphi'[a, q], \end{aligned}$$

where φ' results from substituting each occurrence of $\dot{A}(\langle j, x\rangle)$ in φ by $\dot{A}(\langle i, \langle j, x\rangle\rangle)$. φ' is then also a Σ_1 formula. The latter means:

$$N^{n+1,\langle q_N^0,\ldots,q_N^n\rangle} \models \varphi'[a, q],$$

since by (b), $\langle q_N^0, \ldots, q_N^n\rangle \in R_N^{n+1}$, in particular $\rho_N^{n+1} = \rho_{N^{n,\langle q_N^0,\ldots,q_N^{n-1}\rangle}}$. So A is $\Sigma_1(N^{n+1,\langle q_N^0,\ldots,q_N^n\rangle})$ in q. Since I didn't use that M is sound, one can argue in the same way for the opposite direction of (a). So this shows (a) in the inductive step from n to $n+1$.

For (b): It follows from (a) that $P_{M^{n+1,\langle q_M^0,\ldots,q_M^n\rangle}} = P_{N^{n+1,\langle q_N^0,\ldots,q_N^n\rangle}}$, since $H_M^{n+1} = H_N^{n+1}$. So, both sets have the same $<^*$-least member, which shows that $q_M^{n+1} = q_N^{n+1}$, which is the first part of (b). In order to see that $q_N^{n+1} \in R_{N^{n+1,\langle q_N^0,\ldots,q_N^n\rangle}}$, note that by Lemma 8.4, $q_N^{n+1} = q_M^{n+1} = p_M^{n+1} \in R_{M^{n+1,\langle q_M^0,\ldots,q_M^n\rangle}}$, as M is sound. Hence, there is a surjection from $\omega\rho_M^{n+2}$ onto H_M^{n+1}, which is $\Sigma_1(M^{n+1,q_{M,n+1}})$ in q_M^{n+1}, and by (a), this surjection is $\Sigma_1(N^{n+1,q_{N,n+1}})$ in $q_M^{n+1} = q_N^{n+1}$, too. So $q_N^{n+1} \in R_{N^{n+1,\langle q_N^0,\ldots,q_N^n\rangle}}$, and (b) is proven for $n+1$.

So this concludes the inductive proof of (a) and (b).

But the rest follows immediately, since (b) allows us to apply Lemma 8.5 to N. This gives for $1 \leq n < \omega$ that $q_N^n = p_N^n$. Hence, again by (b), $\langle p_N^0, \ldots, p_N^{n-1}\rangle \in R_N^n$. And since this holds for each $n \geq 1$, N is sound. □

8.2 The domains of $\mathtt{S}$ and Λ, part 1

Now one of the main results on the functions $\mathtt{S}$ and Λ can be given. The following definition facilitates its statement:

Definition 8.8.

$\mathbf{pP\lambda}$	:=	The class of pPλ-structures.
$\mathbf{pP}s$	:=	The class of pPs-structures.
$\mathbf{p\lambda}$	:=	The class of pλ-structures.
$\mathbf{p}s$	:=	The class of ps-structures.

Lemma 8.9. $\mathtt{S}$ *is a bijection between* $\mathbf{pP\lambda}$ *and* $\mathbf{pP}s$, *and* Λ *is the inverse of* $\mathtt{S}$, *hence a bijection between* $\mathbf{pP}s$ *and* $\mathbf{pP\lambda}$.

Proof. By the remarks 4.2 and 4.10, it suffices to show that $\mathtt{S}(M)$ is defined for every pPλ-structure M defined, and that $\Lambda(N)$ is defined for every pPs-structure N. This is because by the Lemmata 4.11 and 4.13, $\mathtt{S}$ and Λ are mutually inverse.

So assume that M is a pPλ-structure, for which $\mathtt{S}(M)$ is undefined. Let M be a $<_0$-least such structure. By remark 4.2, item 3, it is clear that M cannot be a limit in $<_0$, and of course, $M \neq \langle \emptyset, \emptyset, \emptyset, \emptyset \rangle$. There are two successor cases:

Case 1: $\mathrm{ht}(M) = \mu + 1$.

Then let $M = \langle \mathrm{J}^E_{\mu+1}, \emptyset, D \rangle$. Let $\bar{M} = M||\mu = \langle \mathrm{J}^{E^M}_\mu, F, D_{M||\mu} \rangle$. By minimality of M, $\bar{N} = \mathtt{S}(\bar{M})$ exists. Let $\bar{N} = \langle \mathrm{J}^{E'}_\nu, ((F|\nu)^{\mathtt{h}})^c \rangle$. I have to show that $N := \langle \mathrm{J}^{\tilde{E}}_{\nu+1}, \emptyset \rangle$ is a pPs-structure, where $\tilde{E} = E' {}^\frown \langle \nu, ((F_\nu)^{\mathtt{h}})^c \rangle$. So properties 1.-3. of Definition 3.4 have to be checked. As $\bar{N}$ is a pPs-structure, 1. and 2. only have to be shown for $\gamma = \nu+1$. But since $N =$ "$N||\nu + 1$" is passive, this is trivial. So it remains to check 3. for $\gamma = \nu$. So I have to show that $R^*_{N||\nu} = R^*_{\bar{N}} \neq \emptyset$. For this, I will show inductively that for $1 \leq n < \omega$, $R^n_{\bar{N}} \neq \emptyset$, by constructing sequences $\langle p_1, p_2, \ldots \rangle$ and $\langle q_1, q_2, \ldots \rangle$, so that for every $n < \omega$, $\langle p_1, \ldots, p_n \rangle \in R^n_{\bar{M}}$, $\langle q_1, \ldots, q_n \rangle \in R^n_{\bar{N}}$, and $\mathbf{\Sigma}_1(\bar{M}^{\langle p_1, \ldots, p_n \rangle}) = \mathbf{\Sigma}_1(\bar{N}^{\langle q_1, \ldots, q_n \rangle})$.

$\boxed{n = 1}$ Since $R^1_M \neq \emptyset$, Corollary 4.5 yields a $\bar{p}_1 \in R^1_{\bar{M}}$, so that $\bar{p}_1 \in \bar{N}$. By Lemma 5.18 then, $\bar{p}_1 \cup \{\nu \dot{-} 1\} \in R^1_{\bar{N}}$ (I could show at this point that $\bar{p}_1 \in R^1_{\bar{N}}$, but this is not needed here – see Lemma 8.13). Set $q_1 = \bar{p}_1 \cup \{\nu \dot{-} 1\}$. Using Lemma 5.16 and Corollary 5.18, it is now easy to see that, setting $p_1 := \bar{p}_1 \cup \{\mu \dot{-} 1\}$, it follows that $\mathbf{\Sigma}_1(\bar{M}^{p_1}) = \mathbf{\Sigma}_1(\bar{N}^{q_1})$.

$\boxed{n \longrightarrow n+1}$ From the proof of Lemma 8.7, claims (a) and (b), it follows that we can now choose some $p_{n+1} \in R^1_{\bar{M}^{\langle p_1, \ldots, p_n \rangle}}$ that can be lengthened, and then set: $q_{n+1} = p_{n+1}$. It follows from that proof that then $\mathbf{\Sigma}_1(\bar{M}^{\langle p_1, \ldots, p_{n+1} \rangle}) = \mathbf{\Sigma}_1(\bar{N}^{\langle q_1, \ldots, q_{n+1} \rangle})$. From this, it obviously follows that $q_{n+1} \in R^1_{\bar{N}^{\langle q_1, \ldots, q_n \rangle}}$. But then $\langle q_1, \ldots, q_{n+1} \rangle \in R^{n+1}_{\bar{N}}$, as $\langle p^1, \ldots, p^{n+1} \rangle \in R^{n+1}_{\bar{M}}$, and so, $\omega\rho^1_{\bar{N}^{\langle q_1, \ldots, q_{n+1} \rangle}} = \omega\rho^1_{\bar{M}^{\langle p_1, \ldots, p_{n+1} \rangle}} = \omega\rho^{n+1}_{\bar{M}} = \omega\rho^{n+1}_{\bar{N}}$.

So case 1 cannot occur.

Case 2: M is active.

Then $\bar{M} := M^{\text{passive}} <_0 M$, so, by minimality of M, the structure $\bar{N} := \mathtt{S}(\bar{M})$ exists. Let $\bar{N} = \langle \mathrm{J}^{E'}_\nu, \emptyset \rangle$, $F := E^M_{\text{top}}$ and $s := s(F)$, $s^+ := s^+(F)$. Set: $N := \langle \mathrm{J}^{E'}_{s^+}, (F|s^+)^c \rangle$. I want to show that $N = \mathtt{S}(M)$.

According to the definition of $\mathtt{S}$ (Def. 4.1), it first has to be shown that $|\mathtt{S}(\bar{M})| = |\bar{M}|$. This follows from Lemma 4.15(c).

Secondly, I have to show that, setting $N' := \langle \mathrm{J}^{E'}_\nu, F \rangle$, it follows that $N' = \widehat{N}$. Let $\tau = \tau(F)$. So $N||\tau = \mathtt{S}(M||\tau)$. Let

$$\pi : \mathrm{J}^{E'}_\tau \longrightarrow_F \tilde{N}.$$

We know that $|N||\tau| = |\mathtt{S}(M||\tau)| = |M||\tau|$. Since π is a Σ_0-extender ultrapower embedding, it follows that

$$\pi : M||\tau \longrightarrow_F \bar{M}$$

and $|\tilde{N}| = |\bar{M}| = |\mathtt{S}(\bar{M})|$. So it remains to show that $\dot{E}^{\tilde{N}} = \dot{E}^{N'}$. To this end, let ψ_E be a Σ_1-formula as in Lemma 5.15. It follows that for $\langle \alpha, f \rangle \in \Gamma(N||\tau, \mathrm{crit}(F))$:

$$\begin{aligned} \dot{E}^{\tilde{N}}(\pi(f)(\vec{\alpha})) &\iff \prec\vec{\alpha}\succ \in F(\{\prec\vec{\beta}\succ \mid \dot{E}^{\mathtt{S}(M||\tau)}(f(\vec{\beta}))\} \\ &\iff \prec\vec{\alpha}\succ \in F(\{\vec{\beta} \mid M||\tau \models \psi_E[f(\vec{\beta})]\} \\ &\iff \bar{M} \models \psi_E(\pi(f)(\vec{\alpha})) \\ &\iff \dot{E}^{\mathtt{S}(\bar{M})}(\pi(f)(\vec{\alpha})). \end{aligned}$$

Note that $\bar{M}$ and $M||\tau$ are passive, from which it follows that $\mathtt{S}(\bar{M}) = \widehat{\mathtt{S}(\bar{M})}$ and $\mathtt{S}(M||\tau) = \widehat{\mathtt{S}(M||\tau)}$. That's why ψ_E can be used as above. Also, I applied Łoś's Theorem in moving from the second to the third line, to a Σ_1-formula, which is legitimate because π is Σ_ω-preserving. Since every element of $|\tilde{N}|$ is of the form $\pi(f)(\vec{\alpha})$ as above, we have thus shown that $\tilde{N} = \mathtt{S}(\bar{M})$. But as $\pi\restriction\kappa = F$, it follows immediately that $\widehat{N} = N'$, as claimed.

Finally, I have to show that N is a pPs-structure. The crucial property of Definition 3.4 here is the coherency condition. Obviously, it is enough to concentrate on the top extender. Since N^{passive} is a segment of $\widehat{N}^{\text{passive}} = \bar{N}$, it suffices to show that $E_{s^+}^{\text{Ult}(N,E^N_{\text{top}})} = \emptyset$. There are two cases:

If $s^+ < \nu$, then it is enough to show that $E'_{s^+} = \emptyset$. If not, then $\bar{N}||s^+ = \mathtt{S}(M||\bar{\mu})$ for some $\bar{\mu} < \text{ht}(M)$ with $s^+(\bar{\mu})^M = s^+ = s^+(M)$, a contradiction.

If $s^+ = \nu$, then let $\pi' : N \longrightarrow_F \tilde{N}'$. Then $\pi \subseteq \pi'$, $N^{\text{passive}} = \widehat{N}^{\text{passive}}$ and $\pi'(\tau) = \nu$, as standard arguments show. But τ is a cardinal in N, and so, τ indexes no extender in N. Hence, $E^{\tilde{N}'}_{s^+} = E^{\tilde{N}'}_{\pi'(\tau)} = \emptyset$.

Hence, $\mathtt{S}(M) = N$ is defined after all, and this case is impossible as well. The converse is shown analogously. □

Lemma 8.10. *$\mathtt{S}$ is a bijection between $\mathbf{p\lambda}$ and $\mathbf{ps}$, and Λ is the inverse of $\mathtt{S}$, hence a bijection between $\mathbf{ps}$ and $\mathbf{p\lambda}$.*

Proof. This follows from the previous Lemma 8.9, together with Lemma 4.16. □

8.3 Eliminating the additional parameters

Lemma 8.11. *Let $M = \langle \mathrm{J}^E_{\bar{\mu}+1}, \emptyset, D_M\rangle$ be a pPλ-structure for which $N = \mathtt{S}(M) = \langle \mathrm{J}^{E^N}_{\bar{\nu}+1}, \emptyset\rangle$ exists. Let $p \in [\mathrm{On}_{\mathtt{S}(M)}]^{<\omega}$ have the property that $\{\omega\bar{\mu}\}$ is $\Sigma_1(M)$ in p. Then $\omega\bar{\nu}$ is $\Sigma_1(N)$ in the same parameter p.*

Proof. Case 1: $\omega\bar{\mu} \in D_M$ – I.e., $M||\bar{\mu}$ is passive, or $M||\bar{\mu}$ is active, but $s^+(M||\bar{\mu}) = \bar{\mu}$.

In this case, $D_M = D_{M||\bar{\mu}} \cup \{\omega\bar{\mu}\}$. Let $\varphi(x, \vec{y})$ be a Σ_1 formula and $p = \{\eta_1, \ldots, \eta_n\}$, so that for every $\gamma \in |M|$,

$$\gamma = \omega\bar{\mu} \iff M \models \varphi[({}^x/_\gamma), ({}^{\vec{y}}/_{\vec{\eta}})].$$

By replacing every occurrence of "$\dot{D}(v)$" in φ with "$(\dot{D}(v) \vee v = z)$", for some new variable z, one gets a formula $\tilde{\varphi}(x, \vec{y}, z)$ with the property:

$$M \models \varphi[({}^x/_\gamma), ({}^{\vec{y}}/_{\vec{\eta}})] \iff \langle |M|, E^M, \emptyset, D_{M||\bar{\mu}}\rangle \models \tilde{\varphi}[({}^x/_\gamma), ({}^{\vec{y}}/_{\vec{\eta}}), ({}^z/_{\omega\bar{\mu}})].$$

Set: $\bar{E} := E\restriction\omega\bar{\mu}$, $F := E_{\omega\bar{\mu}}$ and $\bar{D} := D_{M||\bar{\mu}}$. Corollary 5.8 gives

$$\begin{array}{rcl} & M & \models \varphi[({}^x/_\gamma), ({}^{\vec{y}}/_{\vec{\eta}})] \\ \iff & \langle |M|, E^M, \emptyset, \bar{D}\rangle & \models \tilde{\varphi}[({}^x/_\gamma), ({}^{\vec{y}}/_{\vec{\eta}}), ({}^z/_{\omega\bar{\mu}})] \\ \iff & \langle |M|, \bar{E}, F, \bar{D}\rangle & \models \bar{T}_\lambda(\tilde{\varphi})[({}^x/_\gamma), ({}^{\vec{y}}/_{\vec{\eta}}), ({}^z/_{\omega\bar{\mu}}), ({}^v/_{|\mathrm{J}^E_{\bar{\mu}}|}), ({}^w/_{\omega\bar{\mu}})] \\ \iff & \exists u \in |M| & (u \text{ is transitive, and} \\ & \langle u, \bar{E}\cap u, F \cap u, \bar{D}\cap u\rangle & \models \bar{T}_\lambda(\tilde{\varphi})[({}^x/_\gamma), ({}^{\vec{y}}/_{\vec{\eta}}), ({}^z/_{\omega\bar{\mu}}), ({}^v/_{|\mathrm{J}^E_{\bar{\mu}}|}), ({}^w/_{\omega\bar{\mu}})]) \\ \iff & \exists m < \omega \exists u & (u = S^m_{\bar{E},F}(|M||\bar{\mu}| \cup \{|M||\bar{\mu}|\}) \wedge \\ & \langle u, \bar{E}, F, \bar{D}\rangle & \models \bar{T}_\lambda(\tilde{\varphi})[({}^x/_\gamma), ({}^{\vec{y}}/_{\vec{\eta}}), ({}^z/_{\omega\bar{\mu}}), ({}^v/_{|\mathrm{J}^E_{\bar{\mu}}|}), ({}^w/_{\omega\bar{\mu}})]). \end{array}$$

For $m < \omega$, let $\dot{f}_m$ be a code for a function rudimentary in $\dot{E}, \dot{F}$, so that for arbitrary e, f, a, $\mathtt{val}^{e,f}[\dot{f}_m](a) = S^m_{e,f}(a \cup \{a\})$[16] – it's obvious that such codes exist. Let $f_m = \mathtt{val}^{\bar{E},F}[\dot{f}_m]$, and fix $m < \omega$ large enough so that

$$\langle S^m_{\bar{E},F}(|M||\bar{\mu}| \cup \{|M||\bar{\mu}|\}), \bar{E}, F, \bar{D}\rangle \models \bar{T}_\lambda(\tilde{\varphi})[({}^x/_\gamma), ({}^{\vec{y}}/_{\vec{\eta}}), ({}^z/_{\omega\bar{\mu}}), ({}^v/_{|\mathrm{J}^E_{\bar{\mu}}|}), ({}^w/_{\omega\bar{\mu}})]),$$

if (and hence only if) $\gamma = \omega\bar{\mu}$. Note that $|\mathrm{J}^E_{\bar{\mu}+1}|$ is the closure of $|\mathrm{J}^E_{\bar{\mu}}| \cup \{|\mathrm{J}^E_{\bar{\mu}}|\}$ under functions rud in $\bar{E}, F$ and $\bar{D}$. Only functions rud in $\bar{E}$ and F are needed, as was shown in Lemma 5.5. But using $f_{D_{M||\bar{\mu}}}$ one does not get more, as $\bar{D}$, being definable in $\mathrm{J}^E_{\bar{\mu}}$, is an element of $|\mathrm{J}^E_{\bar{\mu}+1}|$. We have:

$$\begin{array}{lll} & M & \models \varphi[\gamma, \vec{\eta}] \\ \iff & \langle |M|, \bar{E}, F, \bar{D}\rangle & \models \bar{T}_\lambda(\tilde{\varphi})[({}^x/_\gamma), ({}^{\vec{y}}/_{\vec{\eta}}), ({}^z/_{\omega\bar{\mu}}), ({}^v/_{|\mathrm{J}^E_{\bar{\mu}}|}), ({}^w/_{\omega\bar{\mu}})] \\ \iff & \langle |M|, \bar{E}, F, \bar{D}\rangle & \models \bar{T}_\lambda(\tilde{\varphi})_{\langle f_m(|M||\bar{\mu}|), \bar{E}, F, D_{M||\bar{\mu}}\rangle}[({}^x/_\gamma), ({}^{\vec{y}}/_{\vec{\eta}}), ({}^z/_{\omega\bar{\mu}}), \\ & & \quad ({}^v/_{|\mathrm{J}^E_{\bar{\mu}}|}), ({}^w/_{\omega\bar{\mu}})] \\ \stackrel{\text{def}}{\iff} & \langle |M|, \bar{E}, F, \bar{D}\rangle & \models \psi[({}^{v_0}/_{f_m(|M||\bar{\mu}|)}), ({}^{v_1}/_{\bar{E}}), ({}^{v_2}/_F), ({}^{v_3}/_{\bar{D}}), \\ & & \quad ({}^x/_\gamma), ({}^{\vec{y}}/_{\vec{\eta}}), ({}^z/_{\omega\bar{\mu}}), ({}^v/_{|\mathrm{J}^E_{\bar{\mu}}|}), ({}^w/_{\omega\bar{\mu}})]. \end{array}$$

Here, $\varphi_{\langle w,x,y,z\rangle}$ denotes the relativization of φ to $\langle w, x, y, z\rangle$. φ is a formula in the language with additional symbols $\dot{D}, \dot{E}$ and $\dot{F}$, which are replaced in the relativized formula with x, y and z, respectively. Thus, a Σ_0 formula in the language without additional symbols is produced.

So the above formula holds in $\langle |M|, \bar{E}, F, \bar{D}\rangle$ iff $\gamma = \omega\bar{\mu}$. So one can continue as follows:

$$\begin{array}{lll} \iff & \langle |M|, \bar{E}, F, \bar{D}\rangle & \models \psi[({}^{v_0}/_{\mathtt{val}^{\bar{E},F,\bar{D}}[f_m](|M||\bar{\mu}|)}), ({}^{v_1}/_{\mathtt{val}^{\bar{E},F,\bar{D}}[f_{\dot{E}}](|\mathrm{J}^E_{\bar{\mu}}|)}), \\ & & \quad ({}^{v_2}/_{\mathtt{val}^{\bar{E},F,\bar{D}}[f_{\dot{F}}](|\mathrm{J}^E_{\bar{\mu}}|)}), ({}^{v_3}/_{\mathtt{val}^{\bar{E},F,\bar{D}}[f_{\dot{D}}](|\mathrm{J}^E_{\bar{\mu}}|)}), \\ & & \quad ({}^x/_{\mathtt{val}^{\bar{E},F,\bar{D}}[c_{\mathrm{On}}](|\mathrm{J}^E_{\bar{\mu}}|)}), ({}^{\vec{y}}/_{\vec{\eta}}), ({}^z/_{\omega\bar{\mu}}), ({}^v/_{|\mathrm{J}^E_{\bar{\mu}}|}), ({}^w/_{\omega\bar{\mu}})] \\ \iff & M||\bar{\mu} & \models T_{\dot{D},\dot{E},\dot{F}}(\psi, v_0, \dot{f}_m, \Phi, v_1, f_{\dot{E}}, \Phi, v_2, f_{\dot{F}}, \Phi, v_3, f_{\dot{D}}, \Phi, \\ & & \quad x, c_{\mathrm{On}}, \Phi, w, c_{\mathrm{On}}, \Phi)[({}^{\vec{y}}/_{\vec{\eta}})] \\ \stackrel{\text{def}}{\iff} & M||\bar{\mu} & \models \chi[({}^{\vec{y}}/_{\vec{\eta}})]. \end{array}$$

Here, I made use of the function $T_{\dot{D},\dot{E},\dot{F}}$ that was introduced in Lemma 5.4; see also Lemma 5.6 for the meaning of c_{On}.

(2) *$\bar{\mu}$ is the least ordinal δ such that $\omega\delta \in D_M$ and $(M||\delta) \models \chi[({}^{\vec{y}}/_{\vec{\eta}})]$. (So $\bar{\mu}$ is the only ordinal with that property, as $\omega\bar{\mu}$ is the largest limit ordinal of M).*

Proof of (2). It is clear that $\delta = \bar{\mu}$ has this property. So we are left to show its minimality. So suppose $\delta < \bar{\mu}$ is also such that $\omega\delta \in D_M$ and $(M||\delta) \models \chi[\vec{\eta}]$. By definition of χ, setting

[16] $S^m_{e,f}$ is the m-fold composition of the function $S_{e,f}$, i.e., the function S of [Jen72], relativized by e, f.

$\tilde{E} := E\restriction\omega\delta$, $\tilde{F} := E_{\omega\delta}$ and $\tilde{D} := D_{M||\delta}$, one gets that:

$$
\begin{array}{rll}
 & (M||\delta) & \models \chi[\vec{\eta}] \\
\iff & (M||\delta) & \models T_{\dot{D},\dot{E},\dot{F}}(\psi, v_0, \dot{f}_m, \Phi, v_1, f_{\dot{E}}, \Phi, v_2, f_{\dot{F}}, \Phi, v_3, f_{\dot{D}}, \Phi, \\
 & & \qquad x, c_{\mathrm{On}}, \Phi, w, c_{\mathrm{On}}, \Phi)[(^{\vec{y}}/_{\vec{\eta}})] \\
\iff & \langle |M||\delta+1|, \tilde{E}, \tilde{F}, \tilde{D}\rangle & \models \psi[(^{v_0}/_{\mathtt{val}^{\tilde{E},\tilde{F},\tilde{D}}[f_m](|M||\delta|)}), (^{v_1}/_{\mathtt{val}^{\tilde{E},\tilde{F},\tilde{D}}[f_{\dot{E}}](|\mathrm{J}^E_\delta|)}), \\
 & & (^{v_2}/_{\mathtt{val}^{\tilde{E},\tilde{F},\tilde{D}}[f_{\dot{F}}](|\mathrm{J}^E_\delta|)}), (^{v_3}/_{\mathtt{val}^{\tilde{E},\tilde{F},\tilde{D}}[f_{\dot{D}}](|\mathrm{J}^E_\delta|)}), \\
 & & (^{x}/_{\mathtt{val}^{\tilde{E},\tilde{F},\tilde{D}}[c_{\mathrm{On}}](|\mathrm{J}^E_\delta|)}), (^{\vec{y}}/_{\vec{\eta}}), (^{z}/_{\omega\delta}), (^{v}/_{|\mathrm{J}^E_\delta|}), (^{w}/_{\omega\delta})] \\
\iff & \langle |M||\delta+1|, \tilde{E}, \tilde{F}, \tilde{D}\rangle & \models \bar{T}_\lambda(\tilde{\varphi})_{\langle f_m(|M||\delta|), \tilde{E}, \tilde{F}, \tilde{D}\rangle}[(^{x}/_{\omega\delta}), (^{\vec{y}}/_{\vec{\eta}}), (^{z}/_{\omega\delta}), \\
 & & \qquad (^{v}/_{|\mathrm{J}^E_\delta|}), (^{w}/_{\omega\delta})].
\end{array}
$$

Hence

$$\langle |M||\delta+1|, \tilde{E}, \tilde{F}, \tilde{D}\rangle \models \bar{T}_\lambda(\tilde{\varphi})[(^{x}/_{\omega\delta}), (^{\vec{y}}/_{\vec{\eta}}), (^{z}/_{\omega\delta}), (^{v}/_{|\mathrm{J}^E_\delta|}), (^{w}/_{\omega\delta})],$$

and that means:

$$\langle |M||\delta+1|, E\restriction\omega(\delta+1), \tilde{D}\rangle \models \tilde{\varphi}[(^{x}/_{\omega\delta}), (^{\vec{y}}/_{\vec{\eta}}), (^{z}/_{\omega\delta})].$$

Remark: As $\omega\delta \in D_M$, it follows that $D_{M||\delta+1} = D_{M||\delta} \cup \{\omega\delta\} = \tilde{D} \cup \{\omega\delta\}$. For otherwise, δ would index an extender in M, so that $s^+(M||\delta) < \omega\delta$. But then it would follow that $\omega\delta \notin D_M$.

So the last line says:

$$(M||\delta+1) \models \varphi[(^{x}/_{\omega\delta}), (^{\vec{y}}/_{\vec{\eta}})].$$

But with $u = |M||\delta+1|$, it follows that $M||\delta+1 = M|u$, since $\omega\delta \in D_M$, hence $D_{M||\delta} = D_M \cap \omega\delta$, and $D_{M||\delta+1} = D_{M||\delta} \cup \{\omega\delta\}$, hence $D_{M||\delta+1} = D_M \cap \omega(\delta+1)$. So it follows that $M \models \varphi[\omega\delta, \vec{\eta}]$, too, since φ, being a Σ_1 formula, is persistent. By assumption, though, this means that $\delta = \bar{\mu}$, and hence that it was not the case that $\delta < \bar{\mu}$ after all. $\square_{(2)}$

In the following, I use the sequence $\langle f^N_\gamma \mid \gamma < \mathrm{ht}(N)\rangle$ of functions coming from Corollary 5.18. Let $N = \langle \mathrm{J}^{E^N}_{\bar{\nu}+1}, \emptyset\rangle$. We get:

(3) *$\bar{\nu}$ is the least ordinal δ satisfying:*

(a) $N||\delta$ is passive, or: $N||\delta$ is active and $\mathrm{On}_{\widehat{N||\delta}} = \mathrm{On}_{N||\delta}$.

(b) $(N||\delta) \models f^N_\delta(\chi)[\vec{\eta}, \delta \dot{-} 1]$.

So there is a Σ_1 formula $\tilde{\chi}$, so that for all γ,

$$\bar{\nu} = \gamma \iff N \models \tilde{\chi}[\gamma, \vec{\eta}].$$

Proof of (3). As to the minimality: $(N||\delta) \models f^N_\delta(\chi)[\vec{\eta}, \delta \dot{-} 1]$ just means that

$$\Lambda(N||\delta) \models \chi[\vec{\eta}].$$

Let $\Lambda(N||\delta) = M||\delta'$ for some $\delta' \leq \bar{\mu}$. In order to be able to use (2), I am going to show that $\omega\delta' \in D_M$:

As $\mathtt{S}(M||\delta')$ is a segment of $\mathtt{S}(M)$, Lemma 4.3 can be used:

$(*)$ *There is no $\mu \leq \mathrm{ht}(M)$, so that $s^+(M||\mu) \leq \omega\delta' < \mu$.*

Now suppose $\omega\delta' \notin D_M$. Let then $\omega\mu \in |M|$ be such that

(∗∗) $s^+(M||\mu) < \omega\delta' \le \mu$.

Then $\omega\delta' \neq \mu$, for otherwise, by (a), $M||\delta'$ would be active and $s^+(M||\delta') = \omega\delta'$, contradicting (∗∗). So $s^+(M||\mu) < \omega\delta' < \mu$, contradicting (∗). So $\omega\delta' \in D_M$, as claimed.

By (2), this means that $\delta' = \bar{\mu}$, which in turn means that $\delta = \bar{\nu}$.

In order to express this minimality characterization by a Σ_1 formula, I use for one thing the fact that $\langle f^N_\gamma \mid \gamma < \mathrm{ht}(N)\rangle$, viewed as a 3-ary relation, is $\Sigma_1(N)$ – see 5.18(d). As a second step, it is easy to verify that the statement $\mathrm{On}_{\widehat{N||\delta}} \subseteq \mathrm{On}_{N||\delta}$ can be expressed by a Σ_ω formula over $N||\delta$. For basically, this means (setting $F' = E^N_{\omega\delta}$):

$$N||\delta \models \forall f \in {}^{\mathrm{crit}(F')^n}\tau(F')\forall\vec{\alpha} < s(F')\exists\beta \ \prec\beta,\vec{\alpha}\succ \in F'(\{\prec\xi,\vec{\delta}\succ < \mathrm{crit}(F') \mid \xi = f(\vec{\delta})\}).$$

$\square_{(3)}$

Hence $\bar{\nu}$ is Σ_1 definable over N from $\vec{\eta}$, and hence so is $\omega\bar{\nu}$, which is what I wanted to show.

Case 2: $\omega\bar{\mu} \notin D_M$. This means that $M||\bar{\mu}$ is active and $s^+(M||\bar{\mu}) < \bar{\mu}$.

The main difference to the first case is that the relationship between D_M and $D_{M||\bar{\mu}}$ is different. Namely, it is easy to check that:

(∗) *Let $M||\gamma$ be active and $\omega\gamma \notin D_M$. Then $D_{M||\gamma+1} = D_{M||\gamma} \cap (s^+(M||\gamma)+1) = D^*_{M||\gamma}$.*

Let $\bar{E} = E{\restriction}\omega\bar{\mu}$, $F = E_{\omega\bar{\mu}}$ and $\bar{D} = D_M$.

In the following, I write $s^+(\gamma)$ for $s^+(M||\gamma)$. I argue as in case 1, but can avoid the de-tour via $\tilde{\varphi}$ in order to define χ, for now $\bar{D} = D_M$. Hence, I don't have to introduce the additional variable z. If I denote the resulting formula in the current case χ', the reflections from case 1, modulo the these changes, yield:

$$M \models \varphi[\omega\bar{\mu},\vec{\eta}] \iff \langle |M||\bar{\mu}|, \bar{E}, F, \bar{D}\rangle \models \chi'[({}^{\vec{y}}/_{\vec{\eta}})].$$

In order to express this over $M||\bar{\mu}$ (note that $\bar{D} = D_M$), every occurrence of "$\dot{D}(v)$" in χ' has to be replaced with "$\dot{D}(v) \wedge v \le z$", where this time, $s^+(\bar{\mu})$ must be substituted for the new variable z. Denote the resulting formula by $\tilde{\chi}'$. Hence:

$$M \models \varphi[\omega\bar{\mu},\vec{\eta}] \iff M||\bar{\mu} \models \tilde{\chi}'[({}^{\vec{y}}/_{\vec{\eta}}),({}^{z}/_{s^+(\bar{\mu})})].$$

So instead of (2), we get:

(2') *$\bar{\mu}$ is the least δ such that*

(a) $M||\delta$ is active and $\delta > s^+(\delta) \in D_M$.

(b) $M||\delta \models \tilde{\chi}'[\vec{\eta}, s^+(\delta)]$.

Proof. As in the proof of (2), it is obvious that $\delta = \bar{\mu}$ has these properties. Turning to the minimality, suppose, $\delta < \bar{\mu}$ had the same properties. Then, setting $\tilde{E} := E{\restriction}\omega\delta$, $\tilde{F} := E_{\omega\delta}$ and $\tilde{D} := D_{M||\delta+1}$, we'd get

$$\begin{array}{rcl} & (M||\delta) & \models \tilde{\chi}'[\vec{\eta}, s^+(\delta)] \\ \iff & \langle |M||\delta|, \tilde{E}, \tilde{F}, \tilde{D}\rangle & \models \chi'[\vec{\eta}] \\ \iff & M||\delta+1 & \models \varphi[\omega\delta,\vec{\eta}], \end{array}$$

as before – here I use that by (∗), $D_{M||\delta+1} = D_{M||\delta} \cap (s^+(\delta)+1)$.

Again, letting $u := |M||\delta+1|$, it follows that $M||\delta+1 = M|u$:

By assumption, $s^+(\delta) \in D_M$. So Lemma 3.19 says that $\omega(\delta+1) \in D_M$; it even says that this is the next element of D_M after $s^+(\delta)$. So by coherency,

$$D_M \cap \omega(\delta+1) = D_{M||\delta+1},$$

and that's what was needed.

Since φ is a Σ_1 formula, it follows by persistence that $M \models \varphi[\omega\delta, \vec{\eta}]$, i.e., we get the contradiction $\delta = \bar{\mu} > \delta$. $\square_{(2')}$

Correspondingly, we get:

(3') *$\bar{\nu}$ is the least ordinal δ such that*

(a) $N||\delta$ is active and $\mathrm{On}_{\widehat{N||\delta}} \neq \mathrm{On}_{N||\delta}$.

(b) $(N||\delta) \models f_\delta(\tilde{\chi}')[\vec{\eta}, \underbrace{\delta \dot{-} 1}_{0}]$.

So there is a Σ_1 formula χ^*, so that for all γ,

$$\bar{\nu} = \gamma \iff N \models \chi^*[\gamma, \vec{\eta}].$$

$\square$

Lemma 8.12. *Let $N = \langle \mathrm{J}^E_{\bar{\nu}+1}, \emptyset \rangle$ be a pPs-structure, for which $M = \Lambda(N) = \langle \mathrm{J}^{E^M}_{\bar{\mu}+1}, \emptyset, D \rangle$ exists. Let $p \in [\mathrm{On}_N]^{<\omega}$ be such that $\{\omega\bar{\nu}\}$ is $\Sigma_1(N)$ in p. Then $\{\omega\bar{\mu}\}$ is $\Sigma_1(M)$ in the same parameter p.*

Proof. Let φ be a Σ_1 formula and $p = \{\eta_1, \ldots, \eta_n\}$, so that for every $\gamma \in |N|$,

$$\gamma = \omega\bar{\nu} \iff N \models \varphi[\gamma, \vec{\eta}].$$

We have:

$$\begin{aligned} & N \models \varphi[\gamma, \vec{\eta}] \\ \iff & \exists u \in |N| \quad (u \text{ is transitive and } \langle u, E \cap u, \emptyset \rangle \models \varphi[\gamma, \vec{\eta}]) \\ \iff & \exists m < \omega \exists u \quad (u = S^m_{E\restriction\omega\bar{\nu}, E_{\omega\bar{\mu}}}(|N||\bar{\nu}| \cup \{|N||\bar{\nu}|\}) \wedge \langle u, E \cap u, \emptyset \rangle \models \varphi[\gamma, \vec{\eta}]). \end{aligned}$$

Now, for every $m < \omega$, fix a code $\dot{f}_m$ for a function that's rud in $\dot{E}, \dot{F}$, so that for all a and arbitrary sets e, f, $S^m_{e,f}(a) = \mathtt{val}^{e,f}[\dot{f}_m](a)$.

Choose $m \in \omega$ large enough so that

$$\langle S^m_{E\restriction\omega\bar{\nu}, E_{\omega\bar{\nu}}}(|N||\bar{\nu}| \cup \{|N||\bar{\nu}|\}), E \cap S^m_{E\restriction\omega\bar{\nu}, E_{\omega\bar{\nu}}}(|N||\bar{\nu}| \cup \{|N||\bar{\nu}|\}), \emptyset \rangle \models \varphi[\omega\bar{\nu}, \vec{\eta}]).$$

Then, setting $\bar{E} := E\restriction\omega\bar{\nu}$, $F := E_{\omega\bar{\nu}}$ and $\gamma = \omega\bar{\nu}$,

$$\begin{array}{rcl}
 & N & \models \varphi[\gamma, \vec{\eta}] \\
\iff & \langle |N|, \bar{E}, F\rangle & \models \bar{T}_s(\varphi)[\gamma, \vec{\eta}, |\mathrm{J}^E_{\bar{\nu}}|, \omega\bar{\nu}] \\
\iff & \langle |N|, \bar{E}, F\rangle & \models (\langle f_m(|N||\bar{\nu}|), \bar{E}, F\rangle \models \bar{T}_s(\varphi)[\gamma, \vec{\eta}, |\mathrm{J}^E_{\bar{\nu}}|, \omega\bar{\nu}]) \\
\iff & \langle |N|, \bar{E}, F\rangle & \models (\underbrace{\bar{T}_s(\varphi)_{\langle f_m(|N||\bar{\nu}|), \bar{E}, F\rangle}[\gamma, \vec{\eta}, |\mathrm{J}^E_{\bar{\nu}}|, \omega\bar{\nu}]}_{\psi[f_m(|N||\bar{\nu}|), \bar{E}, F, \omega\bar{\nu}, \vec{\eta}, |\mathrm{J}^E_{\bar{\nu}}|]}) \\
\iff & \langle |N|, \bar{E}, F\rangle & \models \psi[({}^{v_0}/{}_{\mathtt{val}^{\bar{E},F}[\dot{f}_m](|\mathrm{J}^E_{\bar{\nu}}|)}), ({}^{v_1}/{}_{\mathtt{val}^{\bar{E},F}[\dot{f}_E](|\mathrm{J}^E_{\bar{\nu}}|)}), ({}^{v_2}/{}_{\mathtt{val}^{\bar{E},F}[\dot{f}_F](|\mathrm{J}^E_{\bar{\nu}}|)}), \\
 & & \quad ({}^{v_3}/{}_{\mathtt{val}^{\bar{E},F}[c_{\mathrm{On}}](|\mathrm{J}^E_{\bar{\nu}}|)}), ({}^{\vec{w}}/{}_{\vec{\eta}}), ({}^{v_4}/{}_{\mathtt{val}^{\bar{E},F}[\pi^1_0](|\mathrm{J}^E_{\bar{\nu}}|)})] \\
\iff & (N||\bar{\mu}) & \models T_{\dot{E},\dot{F}}(\psi, v_0, \dot{f}_m, \Phi, v_1, \dot{f}_E, \Phi, v_2, \dot{f}_F, \Phi, v_3, c_{\mathrm{On}}, \Phi, \\
 & & \quad v_4, \pi^1_0, \Phi)[({}^{\vec{w}}/{}_{\vec{\eta}})] \\
\stackrel{\mathrm{def}}{\iff} & (N||\bar{\nu}) & \models \chi[({}^{\vec{w}}/{}_{\vec{\eta}})].
\end{array}$$

$\bar{T}_s$ is the function from Corollary 5.10, and $T_{\dot{E},\dot{F}}$ is the one from Lemma 5.4. The latter Lemma is also used in going from the fifth to the sixth line.

(2) *$\bar{\nu}$ is the least ordinal δ such that $(N||\delta) \models \chi[\vec{\eta}]$.*

Proof of (2). It's clear that $\delta = \bar{\mu}$ has this property. It remains to show its minimality. So suppose $\delta < \bar{\mu}$ had the same property. By passing through the above chain of equivalences backwards, setting $\tilde{E} := E\restriction\delta$ and $\tilde{F} := E_{\omega\delta}$, one arrives at:

$$\begin{array}{rcl}
 & (N||\delta) & \models \chi[\vec{\eta}] \\
\iff & (N||\delta) & \models T_{\dot{E},\dot{F}}(\psi, v_0, \dot{f}_m, \Phi, v_1, \dot{f}_E, \Phi, v_2, \dot{f}_F, \Phi, v_3, c_{\mathrm{On}}, \Phi, \\
 & & \quad v_4, \pi^1_0, \Phi)[({}^{\vec{w}}/{}_{\vec{\eta}})] \\
\iff & \langle |N||\delta+1|, \tilde{E}, \tilde{F}\rangle & \models \psi[({}^{v_0}/{}_{\mathtt{val}^{\tilde{E},\tilde{F}}[\dot{f}_m](|\mathrm{J}^E_{\delta}|)}), ({}^{v_1}/{}_{\mathtt{val}^{\tilde{E},\tilde{F}}[\dot{f}_E](|\mathrm{J}^E_{\delta}|)}), \\
 & & \quad ({}^{v_2}/{}_{\mathtt{val}^{\tilde{E},\tilde{F}}[\dot{f}_F](|\mathrm{J}^E_{\delta}|)}), ({}^{v_3}/{}_{\mathtt{val}^{\tilde{E},\tilde{F}}[c_{\mathrm{On}}](|\mathrm{J}^E_{\delta}|)}), \\
 & & \quad ({}^{\vec{w}}/{}_{\vec{\eta}}), ({}^{v_4}/{}_{\mathtt{val}^{\tilde{E},\tilde{F}}[\pi^1_0](|\mathrm{J}^E_{\delta}|)})] \\
\iff & \langle |N||\delta+1|, \tilde{E}, \tilde{F}\rangle & \models \bar{T}_s(\varphi)_{\langle f_m(|N||\bar{\nu}|), \bar{E}, F\rangle}[\omega\delta, \vec{\eta}, |\mathrm{J}^E_{\delta}|, \omega\delta].
\end{array}$$

Hence, $N||\delta+1 \models \varphi[\omega\delta, \vec{\eta}]$. But with $u = |N||\delta+1|$, $N||\delta+1 = N|u$, and hence it follows that $N \models \varphi[\omega\delta, \vec{\eta}]$, as φ is a Σ_1 formula. By assumption, this means that $\delta = \bar{\nu}$, contradicting the assumption that $\delta < \bar{\nu}$. $\square_{(2)}$

(3) *$\bar{\mu}$ is the least ordinal δ with:*

(a) $\tilde{\mathcal{C}}_0(M||\delta) \models g(\chi_{\tilde{\mathcal{C}}_0(M||\delta)})[\vec{\eta}, \delta \dot{-} 1]$. (For the definition of g, see Lemma 5.16.)

(b) $\omega\delta \in D_M$, or: $M||\delta$ is active and $s^+(M||\delta) \in D_M$.

This can be formulated as $M \models \tilde{\chi}[\gamma, \vec{\eta}]$, where $\tilde{\chi}$ is a Σ_1 formula.

Proof of (3). This follows from (2). Condition (b) says that $\mathtt{S}(M||\delta)$ is a segment of N. $\square_{(3)}$

Hence $\bar{\mu}$ (and thus also $\omega\bar{\mu}$) is Σ_1-definable over M from the parameters $\vec{\eta}$, as was to be shown. $\square$

8.4 Very good parameters in M and N

Lemma 8.13. *Let M be a pPλ structure for which $N = \mathtt{S}(M)$ exists. Let $p \in R^1_M$ be such that $p \in [\mathrm{On}_N]^{<\omega}$. Then also $p \in R^1_N$. The analogous statement remains true if one replaces M, N with $\tilde{\mathcal{C}}_0(M)$, $\tilde{\mathcal{C}}_0(N)$, respectively. If M is a pλ structure, then the analogous statement remains true if one replaces M, N with $\mathcal{C}_0(M)$, $\mathcal{C}_0(N)$, respectively.*

Proof. I prove the Lemma for M and N, indicating the changes necessary to prove the variants. As $p \in R^1_M$, there is a function f which is definable over M by a Σ_1 formula φ, so that

$$y = f(x) \longleftrightarrow M \models \varphi[y, x, p],$$

and so that $\mathrm{ran}(f \restriction \omega\rho^1_M) = \mathrm{On}_M$.

By Corollary 5.18, there is a Σ_1 formula $\varphi' = f_N(\varphi)$ (here I use the notation from Corollary 5.19), defining $f \cap \mathrm{On}^2_N$ over N (maybe using $\mathrm{ht}(N) \dot{-} 1$ as an additional parameter). So, for $\xi, \zeta < \mathrm{On}_N$,

$$\zeta = f(\xi) \iff N \models \varphi'[\zeta, \xi, p, q],$$

where $q = \emptyset$ if the height of N is a limit, and $q = \mathrm{ht}(N) - 1$ otherwise; in order to prove the claim for the (pseudo) Σ_0-codes, one has to use $\varphi' = f^{\mathrm{p}}_N(\varphi)$ here; see 5.20. Hence, $\mathrm{On}_N \subseteq h^1_N(\omega\rho^1_N \cup \{p^1_M\} \cup \{q\})$. But $\mathrm{ht}(M) \dot{-} 1 \in h^1_M(\omega\rho^1_M \cup p)$, as $p \in R_M$. Now, by Lemma 8.11, $q = \mathrm{ht}(N) \dot{-} 1 \in h^1_N(\omega\rho^1_M \cup \{p\})$. This means:

$$\mathrm{On}_N \subseteq h^1_N(\omega\rho^1_M \cup p).$$

But already since Lemma 6.4 we know that $\rho^1_M = \rho^1_N$ (and the corresponding is true of the pseudo Σ_0-codes of these structures). Hence, it has been shown that $\mathrm{On}_N \subseteq h^1_N(\omega\rho^1_N \cup p)$, i.e., $p \in R^1_N$. □

The converse of the part of the previous Lemma 8.13 concerning the pseudo Σ_0-codes is shown entirely analogously:

Lemma 8.14. *Let N be a pPs structure, for which $M = \Lambda(N)$ exists. Let $p \in R^1_{\tilde{\mathcal{C}}_0(N)}$. Then $p \in R^1_{\tilde{\mathcal{C}}_0(M)}$.*

So, together with Lemma 8.13 this yields:

$$R^1_{\tilde{\mathcal{C}}_0(M)} \cap [\mathrm{On}_{\tilde{\mathcal{C}}_0(N)}]^{<\omega} = R^1_{\tilde{\mathcal{C}}_0(N)} \cap [\mathrm{On}_{\tilde{\mathcal{C}}_0(N)}]^{<\omega}.$$

If N is a ps structure, then the analogous statement remains true if one replaces $\tilde{\mathcal{C}}_0(M)$ with $\mathcal{C}_0(M)$ and $\tilde{\mathcal{C}}_0(N)$ with $\mathcal{C}_0(N)$.

Proof. Like the proof of Lemma 8.13; instead of f^{p}_N one has to use the function g from Lemma 5.16. □

8.5 Soundness and Solidity from N to M

Here and in the following sections, M always denotes a pPs-structure, and N is supposed to be $\mathtt{S}(M)$. Before beginning the proof that 1-solidity carries over from $\tilde{\mathcal{C}}_0(N)$ to $\tilde{\mathcal{C}}_0(M)$, I clarify some terminology I use – I follow [Zem02, p. 43].

Definition 8.15. For an acceptable J-structure M, if $\nu \in \mathrm{On}_M$ and $p \in [\mathrm{On}_M]^{<\omega}$, $W^{\nu,p}$ denotes the transitive collapse of $M|X$, where X is the $\Sigma^{(n)}_1$-hull of $\nu \cup (p \setminus (\nu + 1))$ in M; $n \in \omega$ here is chosen so that $\omega\rho^{n+1}_M \leq \nu < \omega\rho^n_M$. This structure is called the witness w.r.t. ν, p in M. The inverse of the collapsing map is called the *witness map*.

M is *solid above* $\alpha \in \mathrm{On}_M$, if for every $\nu \in p_M \setminus \alpha$, the witness W_M^{ν,p_M} is an element of $|M|$. M is *solid* if M is *solid* above 0. M is *n-solid* (for some $n \in \omega$), if M is *solid* above $\omega\rho_M^n$.

Lemma 8.16. *Let N be a pPs structure, so that $\tilde{\mathcal{C}}_0(N)$ is* 1-solid*. Let $M = \Lambda(N)$. Then for every $\gamma \in p^0_{\tilde{\mathcal{C}}_0(N)}$, the corresponding witness on the M-side, $W_{\tilde{\mathcal{C}}_0(M)}^{\gamma,p^0_{\tilde{\mathcal{C}}_0(N)}}$, is an element of $|M|$.*

Proof. First, assume N is not of type III. At the end of the proof, I will sketch how to argue in the type III case. So $|N| = |\tilde{\mathcal{C}}_0(N)|$. Let $N = \langle \mathrm{J}_\nu^{E^N}, E^N_{\omega\nu}\rangle$ and $M = \langle \mathrm{J}_\mu^{E^M}, E^M_{\omega\mu}\rangle$. Further, let γ be an element of $p^0_{\tilde{\mathcal{C}}_0(N)}$. I have to show that $W_{\tilde{\mathcal{C}}_0(M)}^{\gamma,p^0_{\tilde{\mathcal{C}}_0(N)}} \in |M|$. Set:

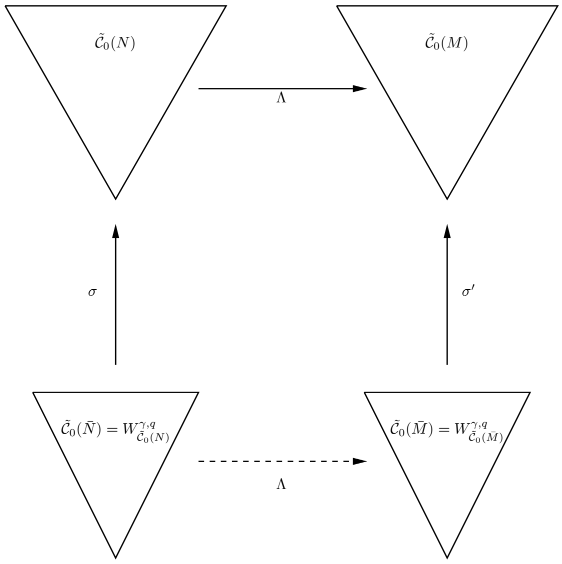
$$q := p^0_{\tilde{\mathcal{C}}_0(N)} \setminus (\gamma+1) \;\text{ and }\; X := h_{\tilde{\mathcal{C}}_0(N)}(\gamma \cup q).$$

Let $\sigma : \bar{N} \overset{\sim}{\longleftrightarrow} N|X$ be the transitive collapse of $N|X$, let $\bar{q} := \sigma^{-1}(q)$, and let $\bar{N} = \langle \mathrm{J}_{\bar{\nu}}^{E^{\bar{N}}}, \bar{F}\rangle$. Obviously, $\bar{N}$ is a pPs-structure of the same type as N: If N is passive, then so is $\bar{N}$, and it's clear that $\bar{N}$ is a pPs-structure. Now let N be active. As N is not of type III, $s := s(N)$ is available as a constant in $\tilde{\mathcal{C}}_0(N)$, and consequently is in the range of σ. Let $\bar{s} = \sigma^{-1}(s)$. If N is of type I, then so is $\bar{N}$, for if there were a generator of $\bar{F}$ other than $\mathrm{crit}(\bar{F})$, then its image under σ would be a generator of F different from $\mathrm{crit}(F)$, so that N wouldn't be type I. Moreover, letting $\bar{\tau} = \tau(\bar{F})$, $\sigma(\bar{\tau}) = \tau(F) = s$ (as σ takes cardinals in the sense of $\bar{N}$ to cardinals in the sense of N), hence $\bar{\tau} = \bar{s} = s(\bar{F})$. If N is of type II, then an analogous argument shows that $\sigma(\bar{s}) = s$ (and thus, that $\bar{N}$ is of type II). I use here that being a generator is Π_1: Let $s = \gamma + 1$, $\bar{\gamma} = \sigma^{-1}(\gamma)$. Then $\bar{\gamma}$ is a generator of $\bar{F}$, and there can be no larger generator $\bar{\delta}$ of $\bar{F}$, for the image of such a larger generator would have to be a generator of F which is greater than s. Hence $\bar{s} = s(\bar{F})$ in this case as well.

The property of being continuable carries over from N to $\bar{N}$. Of course, this is only of interest if the structures are active. In this case, the embedding σ extends canonically to an embedding from $\mathbb{D}(\bar{N}||\bar{\tau}, \bar{F})$ into $\widehat{N}$, proving its well-foundedness.

We have $\tilde{\mathcal{C}}_0(\bar{N}) = W^{\gamma,q}_{\tilde{\mathcal{C}}_0(N)} \in |N|$, because $\tilde{\mathcal{C}}_0(N)$ is 1-solid. By Lemma 8.9, $\bar{M} := \Lambda(\bar{N}) := \langle \mathrm{J}_{\bar{\mu}}^{E^{\bar{M}}}, E^{\bar{M}}_{\omega\bar{\mu}}, D_{\bar{M}}\rangle$ exists. Obviously, $|\bar{N}| = h^1_{\tilde{\mathcal{C}}_0(\bar{N})}(\gamma \cup \bar{q})$.

Claim: $\tilde{\mathcal{C}}_0(\bar{M}) = W^{\gamma,q}_{\tilde{\mathcal{C}}_0(M)}$.



Proof of Claim.

Case 1: $\nu = \nu' + 1$.

Then μ is of the form $\mu' + 1$ as well, and $\bar{N}$ is passive.

Case 1.1: $\bar{\nu} = \bar{\nu}' + 1$.

Then $\bar{\mu} = \bar{\mu}' + 1$ as well, and N and $\bar{N}$ are in the same case of Lemma 5.19 (in the current case, $N = \tilde{\mathcal{C}}_0(N)$ and $\bar{N} = \tilde{\mathcal{C}}_0(\bar{N})$ in the sense that the additional constants in both structures are interpreted by 0. So I am not going to distinguish between these structures and their codes for the rest of the treatment of this case). I will make use of this in the following.

(1) *Let $\varphi(\vec{x}, y)$ be a Σ_1 formula, and let $\vec{\alpha} < \gamma$. Then:*

$$M \models \varphi[\vec{\alpha}, q] \iff \bar{M} \models \varphi[\vec{\alpha}, \bar{q}].$$

Proof of (1). We have that $\sigma(\vec{\alpha}, \bar{\nu}') = \vec{\alpha}, \nu'$, and as σ is Σ_1-preserving, Corollary 5.19 tells us that $f_N = f_{\bar{N}}$ (we use the notation of that Corollary), because in the current case, both structures have successor height, and which of the possible cases listed in the Corollary applies to N and $\bar{N}$ is uniformly $\Sigma_1(N)$ in the parameter $N||\nu'$, and $\Sigma_1(\bar{N})$ in the parameter $\bar{N}||\bar{\nu}'$, respectively. But since $\sigma(\bar{N}||\bar{\nu}') = N||\nu'$, this is decided in $\bar{N}$ and in N in the same way. So we can conclude:

$$\begin{aligned} M \models \varphi[\vec{\alpha}, q] &\iff N \models f_N(\varphi)[\vec{\alpha}, q, \nu'] \\ &\iff \bar{N} \models f_N(\varphi)[\vec{\alpha}, \bar{q}, \bar{\nu}'] \\ &\iff \bar{N} \models f_{\bar{N}}(\varphi)[\vec{\alpha}, \bar{q}, \bar{\nu}'] \\ &\iff \bar{M} \models \varphi[\vec{\alpha}, \bar{q}]. \end{aligned}$$

$\Box_{(1)}$

(2) $|\bar{M}| = h^1_{\tilde{\mathcal{C}}_0(\bar{M})}(\gamma \cup \bar{q})$.

Proof of (2).

(2.1) $\bar{\mu}' \in h^1_{\tilde{\mathcal{C}}_0(\bar{M})}(\gamma \cup \bar{q})$.

Proof of (2.1). We know that $|\bar{N}| = h^1_{\tilde{\mathcal{C}}_0(\bar{N})}(\gamma \cup \bar{q})$. So $\bar{\nu}'$ is $\Sigma_1(\bar{N})$ in $\bar{q} \cup \{\vec{\alpha}\}$ for a finite set $\{\vec{\alpha}\} \subseteq \gamma$. Now Lemma 8.12 can be applied, showing $\bar{\mu}'$ is $\Sigma_1(\bar{M})$ in $\bar{q} \cup \{\vec{\alpha}\}$, and this yields the claim. $\square_{(2.1)}$

Since Corollary 3.26, it's clear that $h^1_{\bar{M}}(\mathrm{otp}(D_{\bar{M}})) = |\bar{M}|$. Since $\bar{N}$ is passive, we also already know that $\mathrm{ht}(\bar{N}) = \mathrm{otp}(D_{\bar{M}})$ (see Lemma 4.5). So, for (2), it suffices to prove that $\mathrm{ht}(\bar{N}) \subseteq h^1_{\bar{M}}(\gamma \cup \bar{q})$. This can be seen as follows: $\mathrm{On}_{\bar{N}} \subseteq h^1_{\tilde{\mathcal{C}}_0(\bar{N})}(\gamma \cup \bar{q})$, so it follows from Lemma 5.16 that $\mathrm{On}_{\bar{N}} \subseteq h^1_{\tilde{\mathcal{C}}_0(\bar{M})}(\gamma \cup \bar{q} \cup \{\bar{\mu}'\})$. But by (2.1), $\bar{\mu}' \in h^1_{\tilde{\mathcal{C}}_0(\bar{M})}(\gamma \cup \bar{q})$, which obviously gives the claim. $\square_{(2)}$

Now define $\sigma' : \bar{M} \longrightarrow M$ by $\sigma'(h^1_{\bar{M}}(i, \langle \vec{\beta}, \bar{q} \rangle)) := h^1_M(i, \langle \vec{\beta}, q \rangle)$ (for $i < \omega$ and $\vec{\beta} < \gamma$).

(3) $\sigma' : \bar{M} \longrightarrow_{\Sigma_1} M$, $\sigma' \restriction \gamma = \mathrm{id} \restriction \gamma$ *and* $X' := \mathrm{ran}(\sigma') = h^1_M(\gamma \cup q)$.

Proof of (3). Obvious. $\square_{(3)}$

But this shows that $\bar{M} = W^{\gamma;q}_M$, hence the claim is proven in case 1.1. In the other cases, I shall try to repeat this argument as closely as possible. The difficulty is that $\bar{N}$ and N may not be of the same type in the sense of 5.19.

Case 1.2: $\bar{\nu}$ is a limit ordinal.

Then let λ be the largest limit ordinal below ν, and let $n \in \omega$ be such that $\nu = \lambda + n$ (hence $n > 0$). Obviously, $\lambda + m \notin X$ for $m < n$, or else $\lambda + (n-1) = \nu' \in X$, making $\bar{\nu}$ a successor ordinal. Set: $\tilde{N} := (N||\lambda)^{\mathrm{passive}}$ and $\tilde{M} := \Lambda(\tilde{N})$ – again, the Λ-image of $\tilde{N}$ exists by Lemma 8.9. Further, let λ' be the largest limit ordinal below μ. We have:

($*$) $\tilde{\mathcal{C}}_0(N)|X \prec_{\Sigma_1} \tilde{\mathcal{C}}_0(\tilde{N})$.

This is obvious, as $X \subseteq |\tilde{N}|$. Note that in the current case, N is passive, and so, the constants in $\tilde{\mathcal{C}}_0(N)$ are interpreted by 0. The same applies to $\tilde{\mathcal{C}}_0(\tilde{N})$, being passive as well.

So, $X = h^1_{\tilde{\mathcal{C}}_0(\tilde{N})}(\gamma \cup q)$, because: For every $a \in X$, the statement "$a = h(i, \langle \vec{\beta}, q \rangle)$" holds in $\tilde{\mathcal{C}}_0(N)$ of some $\vec{\beta} < \gamma$ and $i < \omega$. But this is a Σ_1 statement in parameters from X, so because $\tilde{\mathcal{C}}_0(N)|X \prec_{\Sigma_1} \tilde{\mathcal{C}}_0(N)$, it holds in $\tilde{\mathcal{C}}_0(N)|X$ as well, and due to ($*$), also in $\tilde{\mathcal{C}}_0(\tilde{N})$. For the converse, one can argue in the same way: If the statement "$a = h(i, \langle \vec{\beta}, q \rangle)$" is true in $\tilde{\mathcal{C}}_0(\tilde{N})$, for some $\vec{\beta} < \gamma$ and $i < \omega$, then, in particular, the statement "there is some x, so that $x = h(i, \langle \vec{\beta}, q \rangle)$" holds in $\tilde{\mathcal{C}}_0(\tilde{N})$. The same holds then in $\tilde{\mathcal{C}}_0(N)|X$, as well. The witness for the truth of the statement has to be a, so that $a \in X$.

Hence $\tilde{\mathcal{C}}_0(\bar{N}) = W^{\gamma,q}_{\tilde{\mathcal{C}}_0(\tilde{N})}$. We get the equivalents of (1)-(3) from case 1.1, where N and M have to be replaced by $\tilde{N}$ and $\tilde{M}$, respectively. The point is that now the heights of $\tilde{N}$ and $\bar{N}$ are limit ordinals, so that $\tilde{N}$ and $\bar{N}$ are of the same type, in the sense of the distinction made in Corollary 5.19, namely of type (3) – note that $\bar{N}$ is passive, since N is. So it follows by almost the same proofs:

(1) Let $\varphi(\vec{x}, y)$ be a Σ_1 formula, and let $\vec{\alpha} < \gamma$. Then:

$$\tilde{M} \models \varphi[\vec{\alpha}, q] \iff \bar{M} \models \varphi[\vec{\alpha}, \bar{q}].$$

(2) $\bar{M} = h^1_{\bar{M}}(\gamma \cup \bar{q})$.

(3) $\sigma' : \bar{M} \longrightarrow_{\Sigma_1} \tilde{M}$, $\sigma' \restriction \gamma = \mathrm{id} \restriction \gamma$ and $X' := \mathrm{ran}(\sigma') = h^1_{\tilde{M}}(\gamma \cup q)$,

where $\sigma' : \bar{M} \longrightarrow \tilde{M}$ is defined by $\sigma'(h^1_{\bar{M}}(i, \langle \vec{\beta}, \bar{q} \rangle)) := h^1_{\tilde{M}}(i, \langle \vec{\beta}, q \rangle)$ (for $i < \omega$ and $\vec{\beta} < \gamma$).

To prove (2), the equivalent of (2.1) in case 1.1. is obsolete, since the additional parameter doesn't show up in the case of limit height.

(4) $X' = h^1_{\tilde{\mathcal{C}}_0(M)}(\gamma \cup q)$.

Proof of (4). Let $Y := h^1_{\tilde{\mathcal{C}}_0(M)}(\gamma \cup q)$.

(4.1) $\lambda', \lambda'+1, \ldots, \lambda'+(n-1) \notin Y$.

Proof of (4.1). We know that $\mu' \notin Y$, as otherwise it would follow by Lemma 8.12 that $\nu' \in X$, which is not the case. But if there were some $m < n$ such that $\lambda' + m \in Y$, then it would follow that $\mu' \in Y$ as well, a contradiction. $\square_{(4.1)}$

(4.2) $Y \subseteq |\tilde{M}|$.

Proof of (4.2). From the definition of Λ, it is easy to see that $\lambda' = \mathrm{ht}(\Lambda(\widehat{N||\lambda}^{\text{passive}}))$. If $N||\lambda$ is passive, then the claim follows immediately from (4.1), for then $\lambda' = \mathrm{ht}(\tilde{M})$. If $N||\lambda$ is active, then an additional argument is needed. Then $M||\lambda' = \Lambda(N||\lambda)$ is active and $\lambda = s^+(M||\lambda')$. Moreover, $\mathrm{ht}(\tilde{M}) = \mathrm{ht}(\tilde{N}) = \lambda$.

If $\lambda' = s^+(M||\lambda')$, then $\lambda = \lambda'$, and it follows by (4.1) that $Y \cap \mathrm{On} \subseteq \lambda' = \lambda$, which entails the claim.

So let $\lambda' > s^+(M||\lambda')$. As $\lambda' \notin Y$, it follows that $\lambda \notin Y$: Otherwise, $\lambda' \in Y$ as well, since λ' =the unique ξ such that $s^+(M||\xi) = \lambda$, and this is a Σ_1 definition of λ' from λ.

But then $Y \cap \mathrm{On} \subseteq \lambda$: Suppose $\beta \in Y \setminus \lambda$. By (4.1), $Y \cap \mathrm{On} \subseteq \lambda'$, and I have just shown that $\lambda \notin Y$. Hence, $\lambda < \beta < \lambda'$. But $(\lambda, \lambda'] \cap D_M = \emptyset$ and $\lambda \in D_M$. So λ =the unique δ such that

$$M \models \dot{D}(\delta) \wedge (\forall \bar{\beta} < \beta(\delta < \bar{\beta} \longrightarrow \neg \dot{D}(\bar{\beta}))),$$

since $\lambda = \max(D_M \cap \beta)$. So $\lambda \in Y$, which is, as we have already seen, not the case.

So $Y \cap \mathrm{On} \subseteq \lambda = \mathrm{On}_{\tilde{M}}$, and the claim follows. $\square_{(4.2)}$

(4.3) $\tilde{\mathcal{C}}_0(M)|Y \prec_{\Sigma_1} \tilde{\mathcal{C}}_0(M)$.

Proof of (4.3). This follows from the definition of Y. $\square_{(4.3)}$

(4.4) $\tilde{\mathcal{C}}_0(\tilde{M}) = \tilde{\mathcal{C}}_0(M)|(|\tilde{M}|)$.

Proof of (4.4). $\tilde{M}$ and M both are passive, so $E^{\tilde{M}}_{\mathrm{top}} = E^M_{\mathrm{top}} = \emptyset$, and the constants in the languages of $\tilde{\mathcal{C}}_0(\tilde{M})$ and $\tilde{\mathcal{C}}_0(M)$ are interpreted in both structures by $\emptyset$.

Finally, $D_{\tilde{M}} = D_M \cap |\tilde{M}|$:

If $N||\lambda$ is passive, then $\tilde{N} = N||\lambda$ is a segment of $N = \mathtt{S}(M)$. Then $\tilde{M} = \Lambda(\tilde{N})$ is a segment of M, and all these structures are passive. Hence $\mathrm{ht}(\tilde{M}) = \lambda' \in D_M$, and the claim follows from the coherency of enhancements.

If $N||\lambda$ is active, then $\Lambda(N||\lambda) = M||\lambda'$ and $\lambda = s^+(M||\lambda') \in D_M$. Moreover, $\tilde{M} = (M||\lambda)^{\mathrm{passive}}$ (I have to write $(M||\lambda)^{\mathrm{passive}}$ here, for it could be that $\lambda = \lambda'$). Hence, $D_{\tilde{M}} = D_{M||\lambda} = D_M \cap \mathrm{On}_{\tilde{M}}$. $\square_{(4.4)}$

(4.5) $\tilde{\mathcal{C}}_0(\tilde{M})|Y \prec_{\Sigma_1} \tilde{\mathcal{C}}_0(M)$.

Proof of (4.5). This follows immediately from (4.2)-(4.4), since $\tilde{\mathcal{C}}_0(\tilde{M})|Y = \tilde{\mathcal{C}}_0(M)|Y$. $\square_{(4.5)}$

(4.6) $\tilde{\mathcal{C}}_0(\tilde{M})|Y \prec_{\Sigma_1} \tilde{\mathcal{C}}_0(\tilde{M})$.

Proof of (4.6). I have to show that for every Σ_1 formula $\varphi(\vec{x})$ and every tuple $\vec{a} \in Y$,

$$\tilde{\mathcal{C}}_0(\tilde{M})|Y \models \varphi[\vec{a}] \iff \tilde{\mathcal{C}}_0(\tilde{M}) \models \varphi[\vec{a}].$$

The direction from left to right:

Let $\varphi \equiv \exists y \quad \psi$, where $\psi(y, \vec{x})$ is a Σ_0 formula. Suppose $\tilde{\mathcal{C}}_0(\tilde{M})|Y \models \varphi[\vec{a}]$. Then let $b \in Y$ be such that $\tilde{\mathcal{C}}_0(\tilde{M})|Y \models \psi[b, \vec{a}]$. By (4.5), it follows that $\tilde{\mathcal{C}}_0(M) \models \psi[b, \vec{a}]$. But ψ is Σ_0. $\tilde{\mathcal{C}}_0(M)$ and $\tilde{\mathcal{C}}_0(\tilde{M}) = \tilde{\mathcal{C}}_0(M)|(|\tilde{M}|)$ are transitive, hence we have $\tilde{\mathcal{C}}_0(\tilde{M}) \models \psi[b, \vec{a}]$ (clearly, $b \in Y \subseteq |\tilde{M}|$ by (4.2)).

For the converse, we have:

$$\begin{aligned} \tilde{\mathcal{C}}_0(\tilde{M}) \models \varphi[\vec{a}] \quad &\Rightarrow \quad \tilde{\mathcal{C}}_0(M) \models \varphi[\vec{a}] \quad \text{(persistence and (4.4))} \\ &\Rightarrow \quad \tilde{\mathcal{C}}_0(\tilde{M})|Y \models \varphi[\vec{a}] \text{ by (4.5).} \end{aligned}$$

$\Box_{(4.6)}$

(4.7) $X' \subseteq Y$.

Proof of (4.7). X' is minimal with $\gamma \cup q \subseteq X'$ and the property that $\tilde{\mathcal{C}}_0(\tilde{M})|X' \prec_{\Sigma_1} \tilde{\mathcal{C}}_0(\tilde{M})$. Y has both of these properties, by (4.6) and by definition of Y. $\Box_{(4.7)}$

(4.8) $Y \subseteq X'$.

Proof of (4.8). Let $b \in Y$. Then there are $\vec{\alpha} \in \gamma \cup q$ and a Σ_1 formula $\varphi(\vec{x})$, so that b is the unique c such that $\tilde{\mathcal{C}}_0(M) \models \varphi[c, \vec{\alpha}]$. By (4.5), it follows that $(\tilde{\mathcal{C}}_0(\tilde{M})|Y) \models \varphi[b, \vec{\alpha}]$. So by (4.6), one can deduce that $\tilde{\mathcal{C}}_0(\tilde{M}) \models \varphi[b, \vec{\alpha}]$. But b is also uniquely determined by that property, for if there were some $b' \neq b$ with $\tilde{\mathcal{C}}_0(\tilde{M}) \models \varphi[b', \vec{\alpha}]$, this would imply the contradiction $\tilde{\mathcal{C}}_0(M) \models \varphi[b', \vec{\alpha}]$ (by (4.4) and persistence). Hence $b \in X'$, which was to be shown. $\Box_{(4.8)}$

(4.7) and (4.8) together prove the claim. $\Box_{(4)}$

So, from (4), (4.5) and (1)-(3) it follows that:

(1') Let $\varphi(\vec{x}, y)$ be a Σ_1 formula, and let $\vec{\alpha} < \gamma$. Then we have:

$$M \models \varphi[\vec{\alpha}, q] \iff \bar{M} \models \varphi[\vec{\alpha}, \bar{q}].$$

(2') $\bar{M} = h^1_{\bar{M}}(\gamma \cup \bar{q})$.

(3') $\sigma' : \bar{M} \longrightarrow_{\Sigma_1} M$, $\sigma' \restriction \gamma = \mathrm{id} \restriction \gamma$ and $\mathrm{ran}(\sigma') = h^1_M(\gamma \cup q)$.

But from (1')-(3') it follows that $\tilde{\mathcal{C}}_0(\tilde{M}) = W^{\gamma, q}_{\tilde{\mathcal{C}}_0(M)}$, and that σ' is the associated *witness map.*

Case 2: ν is a limit ordinal.

Then $\bar{\nu}$ is a limit as well. Moreover, N is active iff $\bar{N}$ is. So either both N and $\bar{N}$ are of type (3) or both are of type (4) in the sense of the distinction made in the statement of Corollary 5.19. So one can argue in the current case as we did in case 1.1, with the simplification that the parameters ν' and $\bar{\nu}'$ don't show up.

This concludes the proof of the claim. $\Box_{Claim}$

Let me now show the analogous claim in the case that N is active of type III:

Then "$\tilde{\mathcal{C}}_0(N) = \widehat{N}$" and "$\tilde{\mathcal{C}}_0(M) = M$" in the sense, that the additional constants available in the Pseudo-Σ_0-Codes are easily definable in the reduced structures. Now form $\tilde{N} := W^{\gamma, q}_{\widehat{N}}$. It is obvious that there is a pPs structure $\bar{N}$ so that $\tilde{N} = \widehat{\bar{N}}$ (whether or not this structure is of type III, does not matter for the rest of the argument). Let $\bar{M} = \Lambda(\bar{N})$. As in case 1.1., it follows that $\bar{M} = W_M^{\gamma;q}$; instead of f_N and f_N^{p}, one can now use the function $\widehat{f}_N$ from Corollary 5.19.

So, $W^{\gamma,q}_{\tilde{\mathcal{C}}_0(M)} = \tilde{\mathcal{C}}_0(\Lambda(\bar{N}))$, and $\bar{N} \in |N|$. But $|N| \subseteq |M|$, i.e., $\bar{N} \in |M|$.

In order to finish the proof of the Lemma, it remains to show that $\Lambda(\bar{N}) \in |M|$, as well. This can be seen as follows: $|\bar{N}| = h^1_{\bar{N}}(\gamma \cup \bar{q})$, and so, $\bar{N}$ can be reconstructed from $\bar{A} := A^{\bar{q}}_{\bar{N}} \cap (\gamma \times \omega)$. Let $\kappa = \mathrm{crit}(\sigma)$. Then $\kappa \geq \gamma$, and $\sigma(\kappa)$ is a cardinal in N. Obviously, $\sigma(\kappa) > \kappa \geq \gamma$. $\bar{A}$ can be coded as a subset A' of κ, which is an element of N. As N is acceptable, $A' \in |N||\sigma(\kappa)|$. But $N' := (N||\sigma(\kappa))^{\mathrm{passive}}$ is a ZFC^- model, so $\bar{N} \in |N'|$, and $|\Lambda(N')| = |N'|$. Hence, $\bar{N} \in |\Lambda(N')|$. And $M' := \Lambda(N')$ is a ZFC^- model as well, which implies that $\Lambda\restriction|M'|$ can be defined in M', so that $\Lambda(\bar{N}) \in |M'|$. But obviously, $|M'| \subseteq |M|$, so, putting all of this together, $\Lambda(\bar{N}) \in |M|$. □

Lemma 8.17. *Let $\tilde{\mathcal{C}}_0(N)$ be a pPs-structure that's 1-solid and sound. Let $M := \Lambda(N)$. Then $\tilde{\mathcal{C}}_0(M)$ is 1-solid and sound, too.*

Proof. I want to use Lemma 8.7 in order to prove that $\tilde{\mathcal{C}}_0(M)$ is sound (there, M now plays the role of $\tilde{\mathcal{C}}_0(N)$ and N that of $\tilde{\mathcal{C}}_0(M)$). So the points (i)-(v) need verification. Of these, only (ii) and (v) are not obvious.

For (v): I have to show that $q^0_{\tilde{\mathcal{C}}_0(N)} = q^0_{\tilde{\mathcal{C}}_0(M)} \in R^1_{\tilde{\mathcal{C}}_0(M)}$. Since $\tilde{\mathcal{C}}_0(N)$ is sound,

$$q^0_{\tilde{\mathcal{C}}_0(N)} = p^0_{\tilde{\mathcal{C}}_0(N)} \in R^1_{\tilde{\mathcal{C}}_0(N)} \subseteq R^1_{\tilde{\mathcal{C}}_0(M)},$$

by Corollary 8.4 and Lemma 8.14. By Lemma 8.16,

$$\forall\gamma \in q^0_{\tilde{\mathcal{C}}_0(N)} \quad W^{\gamma, q^0_{\tilde{\mathcal{C}}_0(N)}}_{\tilde{\mathcal{C}}_0(M)} \in |M|.$$

But this means, by [Zem02, Cor. 1.12.4]:

$$q^0_{\tilde{\mathcal{C}}_0(N)} = p_{\tilde{\mathcal{C}}_0(M),1} = q^0_{\tilde{\mathcal{C}}_0(M)}.$$

This shows (v).

For (ii): Let's write q^0 for $q^0_{\tilde{\mathcal{C}}_0(M)} = q^0_{\tilde{\mathcal{C}}_0(N)}$ in the following. I have to show that for all $q \in H^1 := H^1_{\tilde{\mathcal{C}}_0(M)} = H^1_{\tilde{\mathcal{C}}_0(N)}$,

$$\{a \mid a \text{ is } \Sigma_1(\tilde{\mathcal{C}}_0(M)^{q^0}) \text{ in } q\} = \{a \mid a \text{ is } \Sigma_1(\tilde{\mathcal{C}}_0(N)^{q^0}) \text{ in } q\}.$$

For the left-to-right direction, let A be $\Sigma_1(\tilde{\mathcal{C}}_0(M)^{1,q^0})$ in the parameter $q \in H^1$. Let φ be a Σ_1-formula defining A over $\tilde{\mathcal{C}}_0(M)^{1,q^0}$. So, for $a \in H^1$,

$$\begin{aligned} a \in A \quad &\iff \quad \tilde{\mathcal{C}}_0(M)^{1,q^0} \models \varphi[a, q] \\ &\iff \quad \langle H^1, A^{q^0}_{\tilde{\mathcal{C}}_0(M)}\rangle \models \varphi[a, q]. \end{aligned}$$

Now I would like to define the first standard-code of $\tilde{\mathcal{C}}_0(M)$ over $\tilde{\mathcal{C}}_0(N)$, which is possible in principle, because it is $\mathbf{\Sigma}_1$. But there is the subtlety that Lemma 5.17 only applies to sets of (n-tuples of) ordinals. So I have to code the standard-code by such a set first. To this end, let $f : \mathrm{On}_M \leftrightarrow |M|$ the canonical $\Sigma_1(M)$-bijection. Set:

$$\tilde{A} = \{\langle i, \gamma\rangle \mid \tilde{\mathcal{C}}_0(M) \models \varphi_i[f(\gamma), q^0]\}.$$

Then $\tilde{A}$ is a set of pairs of ordinals which is $\Sigma_1(\tilde{\mathcal{C}}_0(M))$ in q^0.

By Lemma 5.17, $\tilde{A} \cap |N|$ is $\Sigma_1(\tilde{\mathcal{C}}_0(N))$ in $q^0 \cup \{\mathrm{ht}(N) \dot{-} 1\}$. Consider now the case that the height of M (and hence also that of N) is a successor.

Let $\mu = \bar{\mu} + 1 = \mathrm{ht}(M)$ and $\nu = \bar{\nu} + 1 = \mathrm{ht}(N)$. As $\tilde{\mathcal{C}}_0(N)$ is sound, $\bar{\nu} = h^1_{\tilde{\mathcal{C}}_0(N)}(j, q^0 \cup \{\vec{\alpha}\})$ for some $\vec{\alpha} \in (\omega p^1)^{<\omega}$. Let $\vec{\alpha}$ be lexicographically minimal so that a j with that property exists, and let j be minimal with respect to $\vec{\alpha}$.

Let i be the Gödel-number of the Σ_1-formula defining $\tilde{A}$, after substituting $h^1_{\tilde{\mathcal{C}}_0(N)}(j, q^0 \cup \{\vec{\alpha}\})$. So for all $a \in |N|$,

$$a \in \tilde{A} \iff \tilde{\mathcal{C}}_0(N) \models \varphi_i[\langle a, \vec{\alpha}\rangle, q^0].$$

Claim: $\{\vec{\alpha}\}$ is $\Sigma_1(\tilde{\mathcal{C}}_0(N)^{q^0})$-definable without parameters.

Proof of Claim. The point is that $\{\omega\bar{\nu}\}$ is $\Pi_1(N)$ (without parameters), for it is the largest limit ordinal of N. Write $\pi(x)$ for this Π_1-formula. Now $\{\bar{\nu}\}$ can be defined from $\omega\bar{\nu}$ without additional parameters by a Σ_1-formula. Hence $\vec{\alpha}$ is $<_{\mathtt{lex}}$-minimal with the property that $\omega\bar{\nu} = h^1_{\tilde{\mathcal{C}}_0(N)}(j, q^0_N \cup \{\vec{\alpha}\})$. So $\vec{\alpha}$ is the $<_N$-minimal finite sequence of ordinals $\vec{\beta}$ such that

$$N \models \pi[h_N(j, \langle \vec{\beta}, q^0_N\rangle)].$$

Since $h_{\tilde{\mathcal{C}}_0(N)}$ is a good Σ_1-function, it can be substituted in Σ_1-formulae. So one can argue:

$$\begin{aligned}
\tilde{\mathcal{C}}_0(N) \models \pi[h_{\tilde{\mathcal{C}}_0(N)}(j, \langle \vec{\beta}, q^0\rangle)] \quad &\iff \quad \neg(\tilde{\mathcal{C}}_0(N) \models \underbrace{\neg\pi}_{\sigma}[h_{\tilde{\mathcal{C}}_0(N)}(j, \langle \vec{\beta}, q^0\rangle)]) \\
&\iff \quad \neg(\tilde{\mathcal{C}}_0(N) \models \sigma[h_{\tilde{\mathcal{C}}_0(N)}(j, \langle \vec{\beta}, q^0\rangle)]) \\
&\iff \quad \neg(\tilde{\mathcal{C}}_0(N) \models \tilde{\sigma}[\vec{\beta}, q^0]) \\
&\iff \quad \neg A^{q^0}_N(k, \vec{\beta}),
\end{aligned}$$

where k is the Gödel-number of the Σ_1-formula $\tilde{\sigma}$. So $\vec{\beta} = \vec{\alpha}$ iff

$$\tilde{\mathcal{C}}_0(N)^{q^0} \models \neg\dot{A}(k, \langle\vec{\beta}\rangle) \wedge \forall \vec{\gamma} <_N \vec{\beta} \quad \dot{A}(k, \langle\vec{\gamma}\rangle),$$

which is even a Σ_0-formula. $\square_{\text{Claim}}$

Write ψ for the Σ_1-formula defining $\vec{\alpha}$. Then for $a \in H^1$,

$$\begin{aligned}
a \in \tilde{A} \quad &\iff \quad A^{q^0}_{\tilde{\mathcal{C}}_0(N)}(i, \langle a, \vec{\alpha}\rangle) \\
&\iff \quad \tilde{\mathcal{C}}_0(N)^{1,q^0} \models \exists \vec{\beta} \quad (\psi(\beta) \wedge \dot{A}(i, \langle a, \vec{\beta}\rangle).
\end{aligned}$$

We have for all $a \in H^1$:

$$\begin{aligned}
a \in A \quad &\iff \quad \tilde{\mathcal{C}}_0(M)^{1,q^0} \models \varphi[a, q] \\
&\iff \quad \langle H^1, \tilde{A}\rangle \models \varphi'[a, q],
\end{aligned}$$

where φ' arises from substituting every occurrence of $\dot{A}(i, x)$ in φ with $\dot{A}(i, f^{-1}(x))$, and then substituting the Σ_1-definition of f for f. The result is a Σ_1-formula. Further, we have for all $a \in H^1$:

$$\begin{aligned}
a \in A \quad &\iff \quad \langle H^1, \tilde{A}\rangle \models \varphi'[a, q] \\
&\iff \quad \langle H^1, A^{q^0}_{\tilde{\mathcal{C}}_0(N)}\rangle \models \exists \vec{\beta} \quad (\psi(\beta) \wedge \varphi^*[a, q]),
\end{aligned}$$

where φ^* results from substituting every occurrence of $\dot{A}(l,a)$ in φ' with $\dot{A}(i,\langle\langle l,a\rangle,\vec{\beta}\rangle)$. Then φ^* is again Σ_1, and so the proof is complete.

If the heights of M and N are limit ordinals, the complication with the additional parameter $\mathrm{ht}(N)\dot{-}1$ doesn't arise, which simplifies the proof.

For the right-to-left direction, one can argue similarly. But this time, one has to express $A_N^{q^0}$ over M. Let $\tilde{A} = \{\langle i,a\rangle \mid \tilde{\mathcal{C}}_0(N) \models \varphi_i[a,q^0]\}$ (the step of coding by a set of ordinals is not necessary now), which is obviously a $\Sigma_1(N)$ relation. Now the function g from Lemma 5.16 can be used in order to see that $\tilde{A}$ is $\Sigma_1(M)$ in $q^0 \cup \mathrm{ht}(M)\dot{-}1$. If the heights of M and N are successor ordinals, it must be shown that $\mathrm{ht}(M)-1 = h^1_{\tilde{\mathcal{C}}_0(M)}(j, q^0 \cup \{\vec{\alpha}\})$, for suitable $\vec{\alpha} < \omega\rho^1_{\tilde{\mathcal{C}}_0(M)}$. For this, the soundness of $\tilde{\mathcal{C}}_0(N)$ can be applied, yielding that $\mathrm{ht}(N) = h^1_{\tilde{\mathcal{C}}_0(N)}(j', q^0 \cup \{\vec{\alpha}\})$, and thus allows us to use Lemma 8.12. The rest of the argument is as before. □

The previous Lemmas remain true, mutatis mutandis, if they are stated for the full Σ_0-Codes $\mathcal{C}_0(M)$ and $\mathcal{C}_0(N)$ instead of $\tilde{\mathcal{C}}_0(M)$ and $\tilde{\mathcal{C}}_0(N)$. I need some facts on the downward preservation of s-structures, though.

Lemma 8.18. *There is a Π_1 formula $\psi(x,y)$, such that for every active pPs-structure N and every ordinal ξ, the following holds: If $E^{\widehat{N}}_{\mathrm{top}}|\xi \in |\widehat{N}|$, then $\langle a,f\rangle$ is the $\prec_N$-minimal*[17] *member of $\Gamma(N,\kappa(N))$ with $a \in [s(N)]^{<\omega}$ and $\pi^N_{s(N)}(f)(a) = E^{\widehat{N}}_{\mathrm{top}}|\xi$, if and only if*

$$\tilde{\mathcal{C}}_0(N) \models \psi[\langle a,f\rangle,\xi].$$

Proof. For an active pPs-structure N, let $\Gamma_N := \Gamma(N||\tau(N),\kappa(N))$. The statement "$x = E^{\widehat{N}}_{\mathrm{top}}|\xi$" is uniformly $\Pi_1(\widehat{N})$, denote this formula by $\psi_{\mathtt{F}}(v,w)$. So $\widehat{N} \models \psi_{\mathtt{F}}[x,\xi] \iff x = E^{\widehat{N}}_{\mathrm{top}}|\xi$. In the following, I shall use the function $\mathbf{d}$ from Lemma 5.11. Let $\langle\alpha,f\rangle \in \Gamma_N$. Then

$$\begin{aligned} \pi^N_{s(N)}(f)(a) = E^{\widehat{N}}_{\mathrm{top}}|\xi &\iff \neg(\widehat{N} \models \neg\psi_{\mathtt{F}}[\pi^N_{s(N)}(f)(a), \pi^N_{s(N)}(\mathrm{id})(\xi)]) \\ &\iff \neg(\tilde{\mathcal{C}}_0(N) \models \mathbf{d}(\neg\psi_{\mathtt{F}})[\langle f,a\rangle,\langle \mathrm{id},\xi\rangle]) \\ &\iff \tilde{\mathcal{C}}_0(N) \models \neg\mathbf{d}(\neg\psi_{\mathtt{F}})[\langle f,a\rangle,\langle \mathrm{id},\xi\rangle] \\ &\iff \tilde{\mathcal{C}}_0(N) \models \tilde{\psi}[\langle f,a\rangle,\xi] \end{aligned}$$

for a Π_1 formula $\tilde{\psi}$. This formula is independent of N. So we get:

(1) *For every active pPs-structure N' of type II, every $\langle a,f\rangle \in \Gamma_N$ and every ξ,*

$$\pi^{N'}_{s(N')}(f)(a) = E^{\widehat{N'}}_{\mathrm{top}}|\xi \iff \tilde{\mathcal{C}}_0(N') \models \tilde{\psi}[\langle a,f\rangle,\xi].$$

For $\langle a,f\rangle,\langle b,g\rangle \in \Gamma_N$, let

$$a_{\langle a,f\rangle,\langle b,g\rangle} := \{c \in [\kappa]^n \mid f_{a,a\cup b}(c) = g_{b,b\cup c}(c)\},$$

where $f_{a,a\cup b}$ and $g_{b,b\cup c}$ result from adding appropriate "dummy" variables to f and g, see [Ste00, P. 4f]. Let n be the number of Elements of $a \cup b$. We have:

(2) *Let $\langle a,f\rangle \in \Gamma_N$, and let θ be minimal such that $f \in |\mathrm{J}^{E^N}_\theta|$ (hence $\theta < \tau(N)$). Let $\langle b,g\rangle \prec_N \langle a,f\rangle$. Then $a_{\langle a,f\rangle,\langle b,g\rangle} \in |\mathrm{J}^{E^N}_{\theta+1}|$.*

[17] For the definition of $\prec_N$, see 3.37.

Proof of (2). Since $\langle b,g\rangle \prec_N \langle a,f\rangle$, $g \leq_N f$, and hence also $g \in |\mathrm{J}_\theta^{E^N}|$. For the definition of, $a_{\langle a,f\rangle,\langle b,g\rangle}$ no exact knowledge about a and b is needed; it suffices to know "how a and b lie in $a \cup b$". There are only finitely many possibilities. Hence $a_{\langle a,f\rangle,\langle b,g\rangle}$ is definable from f and g, and hence a member of $\mathrm{J}_{\theta+1}^{E^N}$. $\square_{(2)}$

(3) *The relation* $\{\langle\gamma,\xi\rangle \in |N| \mid \xi < \tau(N) \wedge \gamma \geq \gamma_\xi\}$ *is uniformly* $\Sigma_0(N)$ *in the parameter* $\{1\}$, *for active pPs-structures* N. *For the definition of* γ_ξ *see the proof of 5.12.*

Proof of (3). I have to go back to Definition 3.3, where $(E_{\mathrm{top}}^N)^c$ is introduced. We have:

$$\gamma \geq \gamma_\xi \iff E_{\mathrm{top}}^N(\gamma,\xi,1,\{1\});$$

obviously, $E_{\mathrm{top}}^N(1,\{1\})$, since $1=\{0\}\in[\kappa]^1$, $\{1\}=\{\{0\}\}\subseteq[\kappa]^1$, and $1\in\pi_s^N(\{1\})=\{1\}$. Hence, $E_{\mathrm{top}}^N(\gamma,\xi,1,\{1\})$ means that $F\cap([s(N)]^{<\omega}\times|\mathrm{J}_\xi^{E^N}|)\in|\mathrm{J}_\gamma^{E^N}|$, i.e., $\gamma\geq\gamma_\xi$. $\square_{(3)}$

So we have: $\langle a,f\rangle$ is $\prec_N$-minimal with $a\in[s(N)]^{<\omega}$ and $\pi_{s(N)}^N(f)(a)=E_{\mathrm{top}}^{\widehat{N}}|\xi$ iff

$$\begin{aligned}\tilde{\mathcal{C}}_0(N) \models a \in [\dot{s}]^{<\omega} \wedge \tilde{\psi}[\langle a,f\rangle,\xi] \wedge \forall\langle b,g\rangle\forall\theta\forall\gamma\forall c \\ \big((\langle a,f\rangle \in |\mathrm{J}_\theta^{E^N}| \wedge \forall\tilde{\theta}<\theta \quad \langle a,f\rangle \notin |\mathrm{J}_{\tilde{\theta}}^{E^N}| \wedge \text{"}\gamma\geq\gamma_{\theta+1}\text{"} \wedge \langle b,g\rangle \prec_N \langle a,f\rangle \wedge \\ \wedge\, c = a_{\langle a,f\rangle,\langle b,g\rangle}) \longrightarrow \neg\dot{F}(\gamma,\theta+1,a\cup b,c)\big).\end{aligned}$$

Note here that $\neg\dot{F}(\gamma,\theta+1,a\cup b,c)$ means that $\pi_{s(N)}^N(f)(a)\neq\pi_{s(N)}^N(g)(b)$. This is because $c=a_{\langle a,f\rangle,\langle b,g\rangle}\in|\mathrm{J}_{\theta+1}^{E^N}|$, by (2), and $\gamma\geq\gamma_{\theta+1}$. $\square$

Lemma 8.19. *Let N be a ps-structure of type II and $\sigma:\bar{N}\longrightarrow_{\Sigma_1} N$ an embedding such that $\dot{q}^{\mathcal{C}_0(N)}$ and $\dot{s}^{\mathcal{C}_0(N)}\in\mathrm{ran}(\sigma)$. Then $\bar{N}$ is also a ps-structure of type II, $\sigma(\dot{s}^{\mathcal{C}_0(\bar{N})})=\dot{s}^{\mathcal{C}_0(N)}$ and $\sigma(\dot{q}^{\mathcal{C}_0(\bar{N})})=\dot{q}^{\mathcal{C}_0(N)}$.*

Proof. The map σ can be extended to a Σ_1-preserving embedding σ' from $\widehat{\bar{N}}$ to $\widehat{N}$ in a canonical way. The proof of Lemma 8.27 can be applied to $\sigma':\widehat{\bar{N}}\longrightarrow_{\Sigma_1}\widehat{N}$ and shows that $\sigma'(\dot{q}^{\mathcal{C}_0(\widehat{\bar{N}})})=\dot{q}^{\mathcal{C}_0(\widehat{N})}$, and that $\widehat{\bar{N}}$ satisfies the s'-ISC. It follows from the preceding Lemma 8.18 that $\sigma(\dot{q}^{\mathcal{C}_0(\bar{N})})=\dot{q}^{\mathcal{C}_0(N)}$. Moreover, it's obvious that $\bar{N}$ satisfies the s'-ISC and is of type II. $\square$

Lemma 8.20. *Let M be an active $p\lambda$ structure of type III. Then $\omega\rho_M^1=s(M)$.*

Proof. Obviously, $\omega\rho_M^1\leq s(M)$ since there is a $\Sigma_1(M)$-surjection from $s(M)$ onto $|M|$. Assume that $\omega\rho_M^1<s(M)$. Then let A be a set which is $\Sigma_1(M)$-definable in the parameter p such that $A\cap\omega\rho_M^1\notin M$. Let $F:=E_{\mathrm{top}}^M$. Then for $\delta\in\mathrm{gen}_F$ and $\delta<\delta'\leq s$, $\mathrm{crit}(\sigma_{\delta,\delta'}^M)=\delta$, as is easily checked. Moreover,

(∗) $|M|=\bigcup_{\delta\in\mathrm{gen}_F}\mathrm{ran}(\sigma_{\delta,s}^M)$.

Proof of (∗). By definition of s,

$$\pi_s:\mathrm{J}_\tau^{E^M}\longrightarrow_{F|s}\mathrm{J}_\nu^{E^M},$$

where $\tau=\tau(M)$ and $\nu=\mathrm{ht}(M)$. Let $x\in\mathrm{J}_\nu^{E^M}$ and $\kappa=\mathrm{crit}(F)$. Then there are n, $\vec{\alpha}\in s^n$ and a function $f:\kappa^n\to\mathrm{J}_\tau^{E^M}$ with $f\in\mathrm{J}_\tau^{E^M}$, so that $x=\pi_s^M(f)(\vec{\alpha})$. Let $\max(\vec{\alpha})<\delta\in\mathrm{gen}_F$ (this is possible since gen_F has no maximum). Then

$$\sigma_{\delta,s}^M(\pi_\delta^M(f)(\vec{\alpha}))=\pi_s^M(f)(\vec{\alpha})=x\in\mathrm{ran}(\sigma_{\delta,s}^M).$$

$\square_{(*)}$

Now let μ be a cutpoint of F with the following properties:

1. $p \in \operatorname{ran}(\sigma^M_{\mu,s})$.
2. $\mu \geq \omega\rho^1_M$.
3. $[M]_\mu$ satisfies the s'-MISC.

Such a μ is easily found, using Lemma 8.26. Then $\sigma^M_{\mu,s} : [M]_\mu \longrightarrow_{\Sigma_0} M$ cofinally, hence Σ_1−preserving. Let $\bar{A}$ be $\mathbf{\Sigma}_1([M]_\mu)$ in $\bar{p}$ by the same formula that defines A in a $\mathbf{\Sigma}_1(M)$ way using the parameter p, where $\sigma^M_{\mu,s}(\bar{p}) = p$. Since $\omega\rho^1_M \leq \mu \leq \operatorname{crit}(\sigma^M_{\mu,s})$, and since $\sigma^M_{\mu,s}$ is Σ_1−preserving, it follows that

$$\bar{A} \cap \omega\rho^1_M = A \cap \omega\rho^1_M.$$

But $[M]_\mu$ satisfies the s'-MISC and M is a pλ structure, hence $[M]_\mu \in M$. Hence, everything that's definable over $[M]_\mu$ belongs to M, in particular $\bar{A} \cap \omega\rho^1_M = A \cap \omega\rho^1_M$, a contradiction. □

Corollary 8.21. *Let N be an active ps structure of type III. Let $s = s(E^N_{\text{top}})$. Then $s = \omega\rho^1_{\tilde{\mathcal{C}}_0(N)}$, and $p_{\tilde{\mathcal{C}}_0(N),1} = \langle\emptyset\rangle$.*

Proof. This is shown like Lemma 8.20; it's obvious that one can define a Σ_1-surjection from s onto $|\widehat{N}|$ in $\widehat{N}$ using the top extender. □

Lemma 8.22. *Let N be a ps-structure such that $\mathcal{C}_0(N)$ is 1-solid. Let $M = \Lambda(N)$. Then for every $\gamma \in p^0_{\mathcal{C}_0(N)}$, the corresponding witness on the M-side, $W^{\gamma,p^0_{\mathcal{C}_0(N)}}_{\mathcal{C}_0(M)}$, is an element of $|M|$.*

Proof. If N is of type III, then $q^0_{\mathcal{C}_0(N)} = \emptyset$; see Lemma 8.21. So in that case, nothing is to be shown – the situation in the more general case that N is a pPs-structure was different.

The only point that deserves extra attention is the verification that $W^{\gamma,q^0_{\mathcal{C}_0(N)}}$ is a ps-structure. This follows from Lemma 8.19. The rest of the proof remains more ore less the same. The main efforts were in the case that N is passive, anyway, and in that case, the Pseudo-Σ_0-Codes are essentially the same as the full Σ_0-Codes. □

Lemma 8.23. *Let $\mathcal{C}_0(N)$ be a ps-structure that's 1-solid and sound. Let $M = \Lambda(N)$. Then $\mathcal{C}_0(M)$ is 1-solid and sound.*

Proof. Like before. □

8.6 Soundness and Solidity from M to N

Let's turn around what was done in the previous section now.

Lemma 8.24. *Let M be a pPλ-structure, so that $\tilde{\mathcal{C}}_0(M)$ is 1-solid. Let $N = \mathtt{S}(M)$. Then for every $\gamma \in p^0_{\tilde{\mathcal{C}}_0(M)}$, the corresponding witness on the N-side, $W^{\gamma,p^0_{\tilde{\mathcal{C}}_0(M)}}_{\tilde{\mathcal{C}}_0(N)}$, is an element of $|N|$.*

Proof. Let $M = \langle \mathrm{J}^{E^M}_\mu, E^M_{\omega\mu}\rangle$, $N = \langle \mathrm{J}^{E^N}_\nu, E^N_{\omega\nu}\rangle$, and suppose first that M is not active of type III.

Let $\gamma \in p^0_{\tilde{\mathcal{C}}_0(M)}$, and set:

$$q := p^0_{\tilde{\mathcal{C}}_0(M)} \setminus (\gamma+1) \text{ and } X := h_{\tilde{\mathcal{C}}_0(M)}(\gamma \cup q).$$

Let

$$\sigma : \bar{M} \stackrel{\sim}{\longleftrightarrow} M|X$$

invert the collapse of $M|X$, and set $\bar{q} := \sigma^{-1}(q)$. Finally, let

$$\bar{M} = \langle \mathrm{J}^{E^{\bar{M}}}_{\bar{\mu}}, \bar{F}, \bar{D} \rangle.$$

Obviously, $\bar{M}$ is a pPλ-structure of the same type as M; that $\bar{D} = D_{\bar{M}}$ follows from the uniform Π_1-definability of enhancements (Lemma 3.12). We have that $\tilde{\mathcal{C}}_0(\bar{M}) = W^{\gamma, p^0_{\tilde{\mathcal{C}}_0(M)}}_{\tilde{\mathcal{C}}_0(M)} \in |M|$, as $\tilde{\mathcal{C}}_0(M)$ is 1-solid. Let $\bar{N} := \mathtt{S}(\bar{M}) = \langle \mathrm{J}^{E^{\bar{N}}}_{\bar{\nu}}, E^{\bar{M}}_{\omega\bar{\mu}} \rangle$. Obviously, $|\bar{M}| = h^1_{\tilde{\mathcal{C}}_0(\bar{M})}(\gamma \cup \bar{q})$.

Claim: $\tilde{\mathcal{C}}_0(\bar{N}) = W^{\gamma, p^0_{\tilde{\mathcal{C}}_0(M)}}_{\tilde{\mathcal{C}}_0(N)}$.
Proof of the claim.
Case 1: $\mu = \mu' + 1$.
Then ν is of the form $\nu' + 1$ as well. So in particular, $\bar{M}$ is passive.
Case 1.1: $\bar{\mu} = \bar{\mu}' + 1$.
Then $\bar{\nu} = \bar{\nu}' + 1$.

(1) *Let $\varphi(\vec{x}, y)$ be a Σ_1-formula, and let $\vec{\alpha} < \gamma$. Then we have:*

$$\tilde{\mathcal{C}}_0(N) \models \varphi[\vec{\alpha}, q] \iff \tilde{\mathcal{C}}_0(\bar{N}) \models \varphi[\vec{\alpha}, \bar{q}].$$

Proof of (1). First, it has to be argued that $\bar{q} \in |\bar{N}|$ and $q \in |N|$. We have that $q \in |N|$, because $q \subseteq p^0_{\tilde{\mathcal{C}}_0(M)} \subseteq \mathrm{ht}(N)$, by Lemma 6.2. This means that for every element ζ of q, the formula $\varphi_{\mathrm{V}}[\zeta, \mu']$ holds in M. The corresponding is true in $\bar{M}$, where ζ of course has a pre-image under σ. So the claim follows from the uniformity of φ_{V} – see Lemma 5.15. One shows analogously that $\gamma \subseteq |\bar{N}|$.

We have that $\sigma(\vec{\alpha}, \bar{\mu}') = \vec{\alpha}, \mu'$. In the following, I shall make use of the function g from 5.16.

$$\begin{aligned} \tilde{\mathcal{C}}_0(N) \models \varphi[\vec{\alpha}, q] &\iff \tilde{\mathcal{C}}_0(M) \models g(\varphi)[\vec{\alpha}, q, \nu'] \\ &\iff \tilde{\mathcal{C}}_0(\bar{M}) \models g(\varphi)[\vec{\alpha}, \bar{q}, \bar{\nu}'] \\ &\iff \tilde{\mathcal{C}}_0(\bar{N}) \models \varphi[\vec{\alpha}, \bar{q}]. \end{aligned}$$

Here, I used that $\sigma : \tilde{\mathcal{C}}_0(\bar{M}) \longrightarrow_{\Sigma_1} \tilde{\mathcal{C}}_0(M)$, and that $g(\varphi)$ is a Σ_1-formula. $\Box_{(1)}$

(2) $|\bar{N}| = h^1_{\tilde{\mathcal{C}}_0(\bar{N})}(\gamma \cup \bar{q})$.

Proof of (2).

(2.1) $\bar{\nu}' \in h^1_{\tilde{\mathcal{C}}_0(\bar{N})}(\gamma \cup \bar{q})$.

Proof of (2.1). We know that $|\bar{M}| = h^1_{\tilde{\mathcal{C}}_0(\bar{M})}(\gamma \cup \bar{q})$. Hence, $\bar{\mu}'$ is $\Sigma_1(\tilde{\mathcal{C}}_0(\bar{M}))$ in $\bar{q} \cup \{\vec{\alpha}\}$ for some finite set $\{\vec{\alpha}\} \subseteq \gamma$. Now we can apply Lemma 8.11 to get that $\bar{\nu}'$ is $\Sigma_1(\tilde{\mathcal{C}}_0(\bar{N}))$ in $\bar{q} \cup \{\vec{\alpha}\}$, and this gives the claim. $\Box_{(2.1)}$

Since $\mathrm{On}_{\bar{N}} \subseteq h^1_{\tilde{\mathcal{C}}_0(\bar{M})}(\gamma \cup \bar{q})$, it follows from Lemma 5.20 that $\mathrm{On}_{\bar{N}} \subseteq h^1_{\tilde{\mathcal{C}}_0(\bar{N})}(\gamma \cup \bar{q} \cup \{\bar{\nu}'\})$. But by (2.1), $\bar{\nu}' \in h^1_{\tilde{\mathcal{C}}_0(\bar{N})}(\gamma \cup \bar{q})$, which gives the claim. $\Box_{(2)}$

Now define $\sigma' : \tilde{\mathcal{C}}_0(\bar{N}) \longrightarrow \tilde{\mathcal{C}}_0(N)$ by $\sigma'(h^1_{\tilde{\mathcal{C}}_0(\bar{N})}(i, \langle \vec{\beta}, \bar{q} \rangle)) := h^1_{\tilde{\mathcal{C}}_0(N)}(i, \langle \vec{\beta}, q \rangle)$ (for $i < \omega$ and $\vec{\beta} < \gamma$).

(3) $\sigma' : \tilde{\mathcal{C}}_0(\bar{N}) \longrightarrow_{\Sigma_1} \tilde{\mathcal{C}}_0(N)$, *$\sigma' \restriction \gamma = \mathrm{id} \restriction \gamma$ and $X' := \mathrm{ran}(\sigma') = h^1_{\tilde{\mathcal{C}}_0(N)}(\gamma \cup q)$.*

Proof of (3). Obvious. $\square_{(3)}$

But this shows that $\tilde{\mathcal{C}}_0(\bar{N}) = W^{\gamma,q}_{\tilde{\mathcal{C}}_0(N)}$, and so we're done in case 1.1.

Again, the problem that can occur in the other cases is that M could have successor height, while $\bar{M}$ has limit height.

Case 1.2: $\bar{\mu}$ is a limit.

Then let λ be the largest limit below μ, and let $n = \mu - \lambda$. Then $\lambda + m \notin X$, for $m < n$. Set:

$$\tilde{\lambda} := \begin{cases} \lambda & \text{if } M||\lambda \text{ is passive,} \\ s^{+M}(\lambda) & \text{otherwise.} \end{cases}$$

Set: $\tilde{M} := (M||\tilde{\lambda})^{\text{passive}}$ and $\tilde{N} := \Lambda(\tilde{M})$. Note that $M||\tilde{\lambda}$ is active in case $\lambda = s^+(M||\lambda)$. Let λ' be the largest limit below ν. Obviously then $\lambda' = \text{ht}(\tilde{N})$. Let $\tilde{X} := h^1_{\tilde{\mathcal{C}}_0(\tilde{M})}(\gamma \cup q)$.

(1) $X = \tilde{X}$.

Proof of (1).

(1.1) $\lambda, \lambda+1, \ldots, \lambda+(n-1) \notin X$.

Proof of (1.1). We know that $\mu' \notin X$. But if there was some $m < n$ such that $\lambda + m \in X$, then $\mu' \in X$ as well, a contradiction. $\square_{(1.1)}$

(1.2) $X \subseteq |\tilde{M}|$.

Proof of (1.2). If $\lambda = \tilde{\lambda}$, then this follows immediately from (1.1), since $\tilde{\lambda} = \text{ht}(\tilde{M})$. Otherwise, $M||\lambda$ is active, and one can argue as follows. Since $\lambda \notin X$, $\tilde{\lambda} \notin X$: If we had that $\tilde{\lambda} \in X$, then $\lambda \in X$ as well, for $\lambda =$the unique ξ such that $s^+(M||\xi) = \tilde{\lambda}$, and this is a Σ_1-definition of λ from $\tilde{\lambda}$. But from this, we can conclude that $X \cap \text{On} \subseteq \tilde{\lambda}$: Assume $\beta \in X \setminus \tilde{\lambda}$. By (1.1), $X \cap \text{On} \subseteq \lambda$, and we have already shown that $\tilde{\lambda} \notin X$. Hence, $\tilde{\lambda} < \beta < \lambda$. We have $(\tilde{\lambda}, \lambda] \cap D_M = \emptyset$ and $\tilde{\lambda} \in D_M$. Hence $\tilde{\lambda} =$the unique δ with

$$M \models \dot{D}(\delta) \wedge (\forall \bar{\beta} < \beta(\delta < \bar{\beta} \longrightarrow \neg \dot{D}(\bar{\beta}))),$$

since $\tilde{\lambda} = \max(D_M \cap \beta)$. Hence $\tilde{\lambda} \in X$, which we already know is impossible. So $X \cap \text{On} \subseteq \tilde{\lambda} = \text{On}_{\tilde{M}}$, and the claim follows. $\square_{(1.2)}$

(1.3) $\tilde{\mathcal{C}}_0(M)|X \prec_{\Sigma_1} \tilde{\mathcal{C}}_0(M)$.

Proof of (1.3). This follows from the definition of X. $\square_{(1.3)}$

(1.4) $\tilde{\mathcal{C}}_0(\tilde{M}) = \tilde{\mathcal{C}}_0(M)|(|\tilde{M}|)$.

Proof of (1.4). $\tilde{M}$ and M both are passive, hence $E^{\tilde{M}}_{\text{top}} = E^M_{\text{top}} = \emptyset$, and the additional constants appearing in $\tilde{\mathcal{C}}_0(\tilde{M})$ and $\tilde{\mathcal{C}}_0(M)$ are interpreted as $\emptyset$.

Finally, $D_{\tilde{M}} = D_M \cap \text{On}_{\tilde{M}}$:

We have that $\omega\tilde{\lambda} \in D_M$, for either $M||\lambda$ is passive, $\omega\lambda \in D_M$ and $\tilde{\lambda} = \lambda$, or $M||\lambda$ is active and $\tilde{\lambda} = s^+(M||\lambda) \in D_M$ (this is also true if $\lambda = s^+(M||\lambda)$). But $\tilde{\lambda} = \text{ht}(\tilde{M})$, so that $D_{\tilde{M}} = D_{M||\tilde{\lambda}^{\text{passive}}} = D_{M||\tilde{\lambda}} = D_M \cap \text{On}_{\tilde{M}}$, by the coherency of enhancements. $\square_{(1.4)}$

(1.5) $\tilde{\mathcal{C}}_0(\tilde{M})|X \prec_{\Sigma_1} \tilde{\mathcal{C}}_0(M)$.

Proof of (1.5). This follows from (1.2)-(1.4): $\tilde{\mathcal{C}}_0(\tilde{M})|X = \tilde{\mathcal{C}}_0(M)|X$. $\square_{(1.5)}$

(1.6) $\tilde{\mathcal{C}}_0(\tilde{M})|X \prec_{\Sigma_1} \tilde{\mathcal{C}}_0(\tilde{M})$.

Proof of (1.6). It has to be shown that for every Σ_1-formula $\varphi(\vec{x})$ and every tuple $\vec{a} \in X$,

$$\tilde{\mathcal{C}}_0(\tilde{M})|X \models \varphi[\vec{a}] \iff \tilde{\mathcal{C}}_0(\tilde{M}) \models \varphi[\vec{a}].$$

From left to right: Let $\varphi \equiv \exists y \quad \psi$, where $\psi(y, \vec{x})$ is Σ_0. Suppose $\tilde{\mathcal{C}}_0(\tilde{M})|X \models \varphi[\vec{a}]$. Let then $b \in X$ be so that $\tilde{\mathcal{C}}_0(\tilde{M})|X \models \psi[b, \vec{a}]$. By (1.5) then $\tilde{\mathcal{C}}_0(M) \models \psi[b, \vec{a}]$. But ψ is Σ_0 and $b \in |\tilde{M}|$ by (1.2), and $\tilde{\mathcal{C}}_0(M)$ and $\tilde{\mathcal{C}}_0(\tilde{M}) = \tilde{\mathcal{C}}_0(M)|(|\tilde{M}|)$ are transitive models, hence we have $\tilde{\mathcal{C}}_0(\tilde{M}) \models \psi[b, \vec{a}]$. For the opposite direction, we have:

$$\begin{aligned} \tilde{\mathcal{C}}_0(\tilde{M}) \models \varphi[\vec{a}] \quad &\Rightarrow \quad \tilde{\mathcal{C}}_0(M) \models \varphi[\vec{a}] \quad \text{(Persistency and (1.4))} \\ &\Rightarrow \quad \tilde{\mathcal{C}}_0(\tilde{M})|X \models \varphi[\vec{a}] \text{ by (1.5).} \end{aligned}$$

$\Box_{(1.6)}$

(1.7) $\tilde{X} \subseteq X$.

Proof of (1.7). $\tilde{X}$ is minimal with $\gamma \cup q \subseteq \tilde{X}$ and the property that $\tilde{\mathcal{C}}_0(\tilde{M})|\tilde{X} \prec_{\Sigma_1} \tilde{\mathcal{C}}_0(\tilde{M})$. Both are true of X, by (1.6) and the very definition of X. $\Box_{(1.7)}$

(1.8) $X \subseteq \tilde{X}$.

Proof of (1.8). Let $b \in X$. Then there are $\vec{a} \in \gamma \cup q$ and a Σ_1-formula $\varphi(y, \vec{x})$, so that b is the unique c with the property that $\tilde{\mathcal{C}}_0(M) \models \varphi[c, \vec{a}]$. By (1.5) and (1.2) it follows that $(\tilde{\mathcal{C}}_0(\tilde{M})|X) \models \varphi[b, \vec{a}]$. By (1.6), this implies that $\tilde{\mathcal{C}}_0(\tilde{M}) \models \varphi[b, \vec{a}]$. But b is uniquely determined by this again, because another $b' \neq b$ with $\tilde{\mathcal{C}}_0(\tilde{M}) \models \varphi[b', \vec{a}]$ would give the contradiction $\tilde{\mathcal{C}}_0(M) \models \varphi[b', \vec{a}]$ (by (1.4) and persistency). Hence $b \in \tilde{X}$. $\Box_{(1.8)}$

(1.7) and (1.8) show the claim. $\Box_{(1)}$

Hence $\tilde{\mathcal{C}}_0(\bar{M}) = W^{\gamma, q}_{\tilde{\mathcal{C}}_0(\tilde{M})}$.

We get the equivalents of (1)-(3) of case 1.1, where M and N have to be replaced with $\tilde{M}$ and $\tilde{N}$, respectively. For now, the heights of $\tilde{N}$ and $\bar{N}$ are limits. Of course $\bar{N}$ is passive, as N is. So we get:

(a) Let $\varphi(\vec{x}, y)$ be a Σ_1-formula, and let $\vec{\alpha} < \gamma$. Then we have:

$$\tilde{\mathcal{C}}_0(\tilde{N}) \models \varphi[\vec{\alpha}, q] \iff \tilde{\mathcal{C}}_0(\bar{N}) \models \varphi[\vec{\alpha}, \bar{q}].$$

(b) $|\bar{N}| = h^1_{\tilde{\mathcal{C}}_0(\bar{N})}(\gamma \cup \bar{q})$.

(c) $\sigma' : \tilde{\mathcal{C}}_0(\bar{N}) \longrightarrow_{\Sigma_1} \tilde{\mathcal{C}}_0(\tilde{N})$, $\sigma' \restriction \gamma = \mathrm{id} \restriction \gamma$ and $X' := \mathrm{ran}(\sigma') = h^1_{\tilde{N}}(\gamma \cup q)$,

where $\sigma' : \tilde{\mathcal{C}}_0(\bar{N}) \longrightarrow \tilde{\mathcal{C}}_0(\tilde{N})$ is defined by $\sigma'(h^1_{\bar{N}}(i, \langle \vec{\beta}, \bar{q} \rangle)) := h^1_{\tilde{N}}(i, \langle \vec{\beta}, q \rangle)$ (for $i < \omega$ and $\vec{\beta} < \gamma$).

By definition, $X' = h^1_{\tilde{\mathcal{C}}_0(\tilde{N})}(\gamma \cup q)$.

(2) $X' = h^1_{\tilde{\mathcal{C}}_0(N)}(\gamma \cup q)$.

Proof of (2). Let $Y := h^1_{\tilde{\mathcal{C}}_0(N)}(\gamma \cup q)$.

(2.1) $\lambda', \lambda' + 1, \ldots, \lambda' + (n - 1) \notin Y$.

Proof of (2.1). Note that $\nu' \notin Y$, or else, by Lemma 8.12, it would follow that $\mu' \in X$, which is not the case. The rest is clear. $\Box_{(2.1)}$

(2.2) $Y \subseteq |\tilde{N}|$.

Proof of (2.2). $\lambda' = \mathrm{ht}(\tilde{N})$, so the claim follows from (2.1). $\square_{(2.2)}$

(2.3) $\tilde{\mathcal{C}}_0(N)|Y \prec_{\Sigma_1} \tilde{\mathcal{C}}_0(N)$.

Proof of (2.3). By definition of Y. $\square_{(2.3)}$

(2.4) $\tilde{\mathcal{C}}_0(\tilde{N}) = \tilde{\mathcal{C}}_0(N)|(|\tilde{N}|)$.

Proof of (2.4). $\tilde{N}$ and N both are passive, hence $E^{\tilde{N}}_{\mathrm{top}} = E^N_{\mathrm{top}} = \emptyset$, and the additional constants of $\tilde{\mathcal{C}}_0(\tilde{N})$ and $\tilde{\mathcal{C}}_0(N)$ are interpreted in both structures as $\emptyset$. $\square_{(2.4)}$

(2.5) $\tilde{\mathcal{C}}_0(\tilde{N})|Y \prec_{\Sigma_1} \tilde{\mathcal{C}}_0(N)$.

Proof of (2.5). By (2.3) and (2.4). $\square_{(2.5)}$

(2.6) $\tilde{\mathcal{C}}_0(\tilde{N})|Y \prec_{\Sigma_1} \tilde{\mathcal{C}}_0(\tilde{N})$.

Proof of (2.6). It has to be shown that for every Σ_1-formula $\varphi(\vec{x})$ and every tuple $\vec{a} \in Y$,

$$\tilde{\mathcal{C}}_0(\tilde{N})|Y \models \varphi[\vec{a}] \iff \tilde{\mathcal{C}}_0(\tilde{N}) \models \varphi[\vec{a}].$$

The direction from left to right: Let $\varphi \equiv \exists y \quad \psi$, where $\psi(y,\vec{x})$ is a Σ_0-formula. Suppose $\tilde{\mathcal{C}}_0(\tilde{N})|Y \models \varphi[\vec{a}]$. Let then $b \in Y$ be so that $\tilde{\mathcal{C}}_0(\tilde{N})|Y \models \psi[b,\vec{a}]$. By (2.5), $\tilde{\mathcal{C}}_0(N) \models \psi[b,\vec{a}]$. Since ψ is Σ_0, $\tilde{\mathcal{C}}_0(\tilde{N}) \models \psi[b,\vec{a}]$. For the opposite direction,

$$\begin{aligned} \tilde{\mathcal{C}}_0(\tilde{N}) \models \varphi[\vec{a}] \quad &\Rightarrow \quad \tilde{\mathcal{C}}_0(N) \models \varphi[\vec{a}] \quad \text{(persistency and (2.4))} \\ &\Rightarrow \quad \tilde{\mathcal{C}}_0(\tilde{N})|Y \models \varphi[\vec{a}] \text{ by (2.5).} \end{aligned}$$

$\square_{(2.6)}$

(2.7) $X' \subseteq Y$.

Proof of (2.7). X' is minimal with $\gamma \cup q \subseteq X'$ and the property that $\tilde{\mathcal{C}}_0(\tilde{N})|X' \prec_{\Sigma_1} \tilde{\mathcal{C}}_0(\tilde{N})$. Both properties are shared by Y. $\square_{(2.7)}$

(2.8) $Y \subseteq X'$.

Proof of (2.8). Let $b \in Y$. Then there are $\vec{a} \in \gamma \cup q$ and a Σ_1-formula $\varphi(\vec{x})$, so that b is the unique c with the property that $\tilde{\mathcal{C}}_0(N) \models \varphi[c,\vec{a}]$. By (2.5), $(\tilde{\mathcal{C}}_0(\tilde{N})|Y) \models \varphi[b,\vec{a}]$ and by (2.6), it follows that $\tilde{\mathcal{C}}_0(\tilde{N}) \models \varphi[b,\vec{a}]$. This again determines b, for if there were some $b' \neq b$ with $\tilde{\mathcal{C}}_0(\tilde{N}) \models \varphi[b',\vec{a}]$, we would get the contradiction $\tilde{\mathcal{C}}_0(N) \models \varphi[b',\vec{a}]$ (by (2.4) and persistency). So $b \in X'$, as wished. $\square_{(2.8)}$

(2.7) and (2.8) prove the claim. $\square_{(2)}$

So we get:

(1') Let $\varphi(\vec{x},y)$ be a Σ_1-formula, and let $\vec{\alpha} < \gamma$. Then

$$\tilde{\mathcal{C}}_0(N) \models \varphi[\vec{\alpha},q] \iff \tilde{\mathcal{C}}_0(\bar{N}) \models \varphi[\vec{\alpha},\bar{q}].$$

(2') $|\bar{N}| = h^1_{\bar{N}}(\gamma \cup \bar{q})$.

(3') $\sigma' : \tilde{\mathcal{C}}_0(\bar{N}) \longrightarrow_{\Sigma_1} \tilde{\mathcal{C}}_0(N)$, $\sigma' \restriction \gamma = \mathrm{id} \restriction \gamma$ and $\mathrm{ran}(\sigma') = h^1_{\tilde{\mathcal{C}}_0(N)}(\gamma \cup q)$.

But from (1')-(3'), it follows that $\tilde{\mathcal{C}_0}(\bar{N}) = W^{\gamma, q^0_{\tilde{\mathcal{C}}_0(M)}}_{\tilde{\mathcal{C}_0}(N)}$, and that σ' is the corresponding witness map.

Case 2: μ is a limit.

Then $\bar{\mu}$ is also a limit. So one can argue in this case like in case 1.1., where the parameters ν' and $\bar{\nu}'$ don't occur. This finishes the proof of the claim. $\square_{Claim}$

The case that M is active of type III can be treated with the methods of the proof of the opposite direction of the current lemma.

So $W^{\gamma, q^0_{\tilde{\mathcal{C}}_0(M)}}_{\tilde{\mathcal{C}_0}(N)} = \mathtt{S}(\bar{M})$ and $\bar{M} \in |M|$. The proof that this implies that $\bar{N} \in |N|$, is as before, in the end of the proof of Lemma 8.16. $\square$

Lemma 8.25. *Let $\tilde{\mathcal{C}_0}(M)$ be a pPλ-structure that's 1-solid and sound. Let $N = \mathtt{S}(M)$. Then $\tilde{\mathcal{C}_0}(N)$ is 1-solid and sound.*

Proof. The proof of 8.17 works mutatis mutandis. $\square$

The same results hold true of the full Σ_0-codes $\mathcal{C}_0(M)$ and $\mathcal{C}_0(N)$. In preparation for this, a close look at the s'-initial segment conditions is needed.

Lemma 8.26. *Let M be an active pλ structure. Let $\tau(M) \leq \xi < s(M)$ be a cutpoint such that $\xi \notin C_M$.* [18] *Then $\xi = \bar{\xi} + 1$ for a cutpoint $\bar{\xi}$ of $F = E^M_{\text{top}}$. (So $\bar{\xi}$ is a limit of generators of F). Moreover, $(\bar{\xi}^+)^{[M]_{\bar{\xi}}} = (\bar{\xi}^+)^{[M]_\xi}$ - the proof shows that $\bar{\xi}$ is the only cutpoint less than ξ with this property.*

Proof. Assume the contrary. Let ξ be the least counterexample. Then $\xi \neq \tau := \tau(M)$, since by definition $\tau \in C_M$.

(1) *ξ is not a limit of* gen_F.

Proof of (1). Assume ξ were a limit of generators of F. Since $\xi \notin C_M$, $[M]_\xi$ does not satisfy the s'-MISC, because ξ is a cutpoint of F. So pick a cutpoint $\zeta \in [\tau, \xi)$ such that $(\zeta^+)^{[M]_\zeta} = (\zeta^+)^{[M]_\xi}$. Let $\theta = \min(\text{gen}_F \cap (\zeta, \xi))$ (this θ exists, since by assumption, ξ is a limit of gen_F. By minimality of ξ, $[M]_{\theta+1}$ satisfies the s'-MISC, as θ is not a limit of generators and $\theta + 1$ is a cutpoint of F. So we have:

$$(\zeta^+)^{[M]_\zeta} < (\zeta^+)^{[M]_{\theta+1}} \leq (\zeta^+)^{[M]_\xi},$$

contradicting the choice of ζ. $\square_{(1)}$ So let $\xi = \bar{\xi} + 1$. Then $\bar{\xi} \in \text{gen}_F$. Obviously, $\bar{\xi} \neq \tau$.

(2) *$\bar{\xi}$ is a limit of* gen_F.

Proof of (2). Again, pick a cutpoint $\zeta < \xi$ so that $(\zeta^+)^{[M]_\zeta} = (\zeta^+)^{[M]_\xi}$. Assume that $\bar{\xi}$ is not a limit of generators. Then $\bar{\xi}$ is not a cutpoint, as $\bar{\xi}$ obviously is not the successor of a generator. Hence $\zeta < \bar{\xi}$.

(2.1) $(\zeta^+)^{[M]_\zeta} = (\zeta^+)^{[M]_{\bar{\xi}}}$.

Proof of (2.1).

$$(\zeta^+)^{[M]_{\bar{\xi}}} \leq (\zeta^+)^{[M]_\xi} = (\zeta^+)^{[M]_\zeta} \leq (\zeta^+)^{[M]_{\bar{\xi}}}.$$

$\square_{(2.1)}$

(2.2) $(\zeta^+)^{[M]_\zeta} = (\zeta^+)^M$.

[18] For the definition of C_M, see Definition 3.30.

Proof of (2.2). Otherwise, $(\zeta^+)^{[M]_\zeta} < (\zeta^+)^M$, hence

$$(\zeta^+)^M = (\sigma_{\bar{\xi}}(\zeta)^+)^M = \sigma_{\bar{\xi}}((\zeta^+)^{[M]_{\bar{\xi}}}) > (\zeta^+)^{[M]_{\bar{\xi}}} = (\zeta^+)^{[M]_\zeta}$$

by (2.1). Hence $(\zeta^+)^{[M]_{\bar{\xi}}} = \bar{\xi} = (\zeta^+)^{[M]_\zeta}$, as $\bar{\xi} = \mathrm{crit}(\sigma_{\bar{\xi}})$ is a cardinal in $[M]_{\bar{\xi}}$. But this entails that

$$(\zeta^+)^{[M]_\xi} = (\sigma_{\bar{\xi},\xi}(\zeta)^+)^{[M]_\xi} = \sigma_{\bar{\xi},\xi}((\zeta^+)^{[M]_{\bar{\xi}}}) = \sigma_{\bar{\xi},\xi}(\bar{\xi}) > \bar{\xi} = (\zeta^+)^{[M]_\zeta},$$

contradicting the choice of ζ. $\square_{(2.2)}$

(2.3) $[M]_\zeta$ *does not satisfy the s'-MISC.*

Proof of (2.3). Otherwise $[M]_\zeta \in M$, and hence $(\zeta^+)^M > (\zeta^+)^{[M]_\zeta}$, contradicting (2.2). $\square_{(2.3)}$

(2.4) $\zeta = \bar{\zeta} + 1$, *where $\bar{\zeta}$ is a limit of generators. Moreover,*

$$(\zeta^+)^{[M]_\zeta} = (\bar{\zeta}^+)^{[M]_\zeta} = (\bar{\zeta}^+)^{[M]_{\bar{\zeta}}} = (\zeta^+)^{[M]_{\bar{\zeta}}}.$$

Proof of (2.4). By minimality of ξ, and since ζ is a cutpoint the conclusion of the lemma can be applied to ζ, by (2.3). $\square_{(2.4)}$

So we have:

$$(\zeta^+)^{[M]_\xi} = (\zeta^+)^{[M]_\zeta} = (\zeta^+)^{[M]_{\bar{\zeta}}}.$$

But $[M]_{\bar{\zeta}} \in M$, since $\bar{\zeta} \in C_M$ by minimality of ξ as a counterexample to the lemma, hence $(\zeta^+)^M > (\zeta^+)^{[M]_{\bar{\zeta}}}$, since in M there is a surjection even from $\bar{\zeta}$ onto $|[M]_{\bar{\zeta}}|$. Hence $(\zeta^+)^M > (\zeta^+)^{[M]_\zeta}$ by (2.4), contradicting (2.2). $\square_{(2)}$

In order to finish the proof, pick $\zeta < \xi$ in such a way that ζ is a cutpoint and $(\zeta^+)^{[M]_\zeta} = (\zeta^+)^{[M]_\xi}$. I show:

(3) $\zeta = \bar{\xi}$.

Proof of (3). Assuming the contrary, it follows that $\zeta < \bar{\xi}$. Let $\theta = \min(\mathrm{gen}_F \setminus (\zeta + 1))$. Then $\theta + 1 \in C_M$, so, $[M]_{\theta+1}$ satisfies the s'-MISC. Hence we have:

$$(\zeta^+)^{[M]_\zeta} < (\zeta^+)^{[M]_{\theta+1}} \leq (\zeta^+)^{[M]_\xi},$$

contradicting the choice of ζ. $\square_{(3),Lemma}$

Lemma 8.27. *Let M be a λ structure of type II and $\sigma : \bar{M} \longrightarrow_{\Sigma_1} M$ be an embedding with $q_M, s(M) \in \mathrm{ran}(\sigma)$. Then $\bar{M}$ is also a λ structure of type II, $\sigma(s(\bar{M})) = s(M)$ and $\sigma(q_{\bar{M}}) = q_M$. The corresponding is true if M is a $p\lambda$ structure of type II.*

Proof. Let $M = \langle \mathrm{J}^E_\nu, F\rangle$ and $\bar{M} = \langle \mathrm{J}^{\bar{E}}_{\bar{\nu}}, \bar{F}\rangle$. Let $s(M) = \xi + 1$, $\kappa = \mathrm{crit}(F)$, $\tau = \tau(F)$, and correspondingly, $\bar{\kappa} = \mathrm{crit}(\bar{F})$, $\bar{\tau} = \tau(\bar{F})$. Firstly, it is easy to see that $\sigma(s(\bar{M})) = s(M)$. This is because being a generator is Π_1, and because $s(M) \in \mathrm{ran}(\sigma)$.

Let $\bar{s} := \sigma^{-1}(s)$ and $\bar{\xi} := \sigma^{-1}(\xi)$, hence $\bar{s} = \bar{\xi} + 1$.

Case 1: $\xi = \max C_M$

Since $s = \xi + 1$, ξ is a generator of F, and since $\xi \in C_M$, ξ is a cutpoint of F. Hence, ξ is a limit of generators of F.

Since M satisfies the s'-ISC, $[M]_\xi \in M$. The statement "$x = F|\xi$" is $\Pi_1(M)$ and true of q_M in M. So if $\bar{q} = \sigma^{-1}(q_M)$, then $\bar{q} = \bar{F}|\bar{\xi}$. Moreover, $\sigma^{-1}([M]_\xi) = [\bar{M}]_{\bar{\xi}}$, as da $[M]_\xi$ is coded by q_M the same way $[\bar{M}]_{\bar{\xi}}$ is coded by $\bar{q}$. I show now that $\bar{\xi} = \max C_{\bar{M}}$.

Since $\sigma\restriction[\bar{M}]_{\bar{\xi}} : [\bar{M}]_{\bar{\xi}} \longrightarrow_{\Sigma_\omega} [M]_\xi$, the property of ξ of being a limit of generators of $E_{\text{top}}^{[M]_\xi}$ is preserved downwards, as this can be formulated in $[M]_\xi$. So $\bar{\xi}$ is a limit of generators of $E_{\text{top}}^{[\bar{M}]_{\bar{\xi}}}$, and hence of $\bar{F}$. Hence, $\bar{\xi}$ is a cutpoint of $\bar{F}$. It remains to show that $[\bar{M}]_{\bar{\xi}}$ satisfies the s'-MISC. But the statement that $[M]_\xi$ satisfies the s'-MISC is also $\Sigma_\omega([M]_\xi)$, and thus carries over to $[\bar{M}]_{\bar{\xi}}$. So $\bar{q} = q_{\bar{M}}$. This proves the lemma in case 1.

Case 2: $\xi \notin C_M$.

Then ξ is not a limit of generators of F, by Lemma 8.26. Set: $\zeta = \sup(\text{gen}_F \cap \xi)$. Hence, $\zeta < \xi$.

Case 2.1: ζ is a limit of generators, and $\zeta \notin \text{gen}_F$.

Then $\zeta \in C_M$ by Lemma 8.26. Hence, $\zeta = \max C_M$, $[M]_\zeta \in M$ and $q_M = F|\zeta \in \text{ran}(\sigma)$, so $\zeta \in \text{ran}(\sigma)$. Let $\bar{\zeta} = \sigma^{-1}(\zeta)$. It follows that $\bar{\zeta}$ is a limit of generators of $\bar{F}$, as ζ is a limit of generators of $E_{\text{top}}^{[M]_\zeta}$ and $\sigma\restriction[\bar{M}]_{\bar{\zeta}} \longrightarrow_{\Sigma_\omega} [M]_\zeta$. Hence, $\bar{\zeta}$ is a cutpoint of $\bar{F}$, and $[\bar{M}]_{\bar{\zeta}}$ satisfies the s'-MISC, again by elementarity. Hence $\bar{\zeta} \in C_{\bar{M}}$. It suffices to show that $\bar{\zeta} = \max C_{\bar{M}}$, and for this, it suffices to see that $[\bar{\zeta}, \bar{\xi}) \cap \text{gen}_{\bar{F}} = \emptyset$, since then in $\bar{M}$, there is no cutpoint that's greater than $\bar{\zeta}$ and less than $s(M)$. But that follows immediately from the fact that $[\zeta, \xi) \cap \text{gen}_F = \emptyset$, as σ maps generators of $\bar{F}$ to generators of F. The other requirements of the s'-ISC for $\bar{M}$ are easily verified.

Case 2.2: $\zeta \in \text{gen}_F$.

Case 2.2.1: $\zeta + 1 \in C_M$.

One can argue here similarly as in case 2.1. Since $\zeta + 1 = \max C_M$, it follows that $q_M = F|\zeta + 1 \in \text{ran}(\sigma)$. So $\zeta + 1 \in \text{ran}(\sigma)$, and hence $\zeta \in \text{ran}(\sigma)$. Let $\bar{\zeta} = \sigma^{-1}(\zeta)$. It follows that $[\bar{M}]_{\bar{\zeta}+1} \in \bar{M}$, that $\bar{\zeta} + 1$ is a cutpoint of $\bar{F}$, and that $[\bar{M}]_{\bar{\zeta}+1}$ satisfies the s'-MISC, hence that $\bar{\zeta} + 1 \in C_{\bar{M}}$. Finally, $(\bar{\zeta}, \bar{\xi}) \cap \text{gen}_{\bar{F}} = \emptyset$.

Case 2.2.2: $\zeta + 1 \notin C_M$.

Since $\zeta \in \text{gen}_F$, $\zeta + 1$ is a cutpoint of F, and hence, by Lemma 8.26, ζ is a limit of generators of F. Further, $\zeta = \max C_M$, by the same lemma. Since $q_M = F|\zeta \in \text{ran}(\sigma)$, it follows again that $\zeta \in \text{ran}(\sigma)$. So set $\bar{\zeta} = \sigma^{-1}(\zeta)$, as before. It follows that $\bar{q} := \sigma^{-1}(q_M) = \bar{F}|\bar{\zeta}$ and $\sigma([\bar{M}]_{\bar{\zeta}}) = [M]_\zeta$. Moreover, $[\bar{M}]_{\bar{\zeta}}$ satisfies the s'-MISC, so that $\bar{\zeta} \in C_{\bar{M}}$. Finally, $(\bar{\zeta}, \bar{\xi}) \cap \text{gen}_{\bar{F}} = \emptyset$, since a generator of $\bar{F}$ lying in this interval would be mapped to a generator of F lying in the interval (ζ, ξ), which cannot be, by definition of ζ. Hence $\bar{\zeta} = \max C_{\bar{M}}$ and $\bar{q} = q_{\bar{M}}$. The other requirements of the s'-ISC are again easily verified. □

Lemma 8.28. *Let M be a pλ-structure, so that $\mathcal{C}_0(M)$ is 1-solid. Let $N = \Lambda(M)$. Then for every $\gamma \in p^0_{\mathcal{C}_0(M)}$, the corresponding witness on the N-side, $W^{\gamma, p^0_{\mathcal{C}_0(M)}}_{\mathcal{C}_0(N)}$, is a member of $|N|$.*

Proof. The only additional point here is that $W^{\gamma, q^0_{\mathcal{C}_0(M)}}$ is a pλ-structure, as follows from Lemma 8.27. □

Lemma 8.29. *Let $\mathcal{C}_0(M)$ be a pλ-structure that's sound and 1-solid. Let $N = \mathtt{S}(M)$. Then $\mathcal{C}_0(N)$ is sound and 1-solid.*

Proof. As before. □

8.7 The domains of $\mathtt{S}$ and Λ, part 2

Definition 8.30. In order to state the following results in a compact way, let's introduce the following notation:

$\mathbf{P\lambda}$	:=	The class of Pλ-structures.
$\mathbf{P}\boldsymbol{s}$	:=	The class of Ps-structures.
$\mathbf{\Lambda}$	:=	The class of λ-structures.
$\mathfrak{S}$	:=	The class of s-structures.

Theorem 8.31. $\mathtt{S}\restriction\mathbf{P\lambda}$ *is a bijection between* $\mathbf{P\lambda}$ *and* $\mathbf{P}\boldsymbol{s}$*.* $\Lambda\restriction\mathbf{P}\boldsymbol{s}$ *is the inverse of* $\mathtt{S}\restriction\mathbf{P\lambda}$*, hence a bijection between* $\mathbf{P}\boldsymbol{s}$ *and* $\mathbf{P\lambda}$*.*

Proof. I show by $<_0$-induction on Pλ-structures that $\mathtt{S}(M)$ is a Ps-structure. By Lemma 8.9, it's known already that $\mathtt{S}(M)$ is a pPs-structure, so that it merely has to be verified that for $\mu < \mathrm{ht}(\mathtt{S}(M))$, the structure $\tilde{\mathcal{C}}_0(\mathtt{S}(M)||\mu)$ is sound and 1-solid. But this structure is always of the form $\mathtt{S}(M||\gamma)$, for some $\gamma < \mathrm{ht}(M)$. As M is a Pλ-structure, $\tilde{\mathcal{C}}_0(M||\gamma)$ is sound and 1-solid. Now it follows from Lemma 8.25 that $\tilde{\mathcal{C}}_0(\mathtt{S}(M||\gamma)) = \tilde{\mathcal{C}}_0(\mathtt{S}(M)||\mu)$ has the desired properties.

For the opposite direction, I argue by induction on $<_1$. In the case that N has successor height $\nu+1$, it suffices to know that $\Lambda(N) =$ "$\Lambda(N||\nu)+1$", for inductively, $\Lambda(N||\nu)$ is a Pλ-structure. So it remains to verify soundness and 1-solidity of $\Lambda(N||\nu)$. But that follows from Lemma 8.17. The other successor case is that N is active. But in that case, there is nothing to prove, as $\Lambda(N) =$ "$\langle\Lambda(\widehat{N}^{\mathrm{passive}}), E^{\widehat{N}}_{\mathrm{top}}, D_{\Lambda(N)}\rangle$", and $\Lambda(\widehat{N}^{\mathrm{passive}})$ already has the desired pre-soundness/solidity-properties. The limit case is trivial. □

Theorem 8.32. $\mathtt{S}\restriction\mathbf{\Lambda}$ *is a bijection between* $\mathbf{\Lambda}$ *and* $\mathfrak{S}$*.* $\Lambda\restriction\mathfrak{S}$ *is the inverse of* $\mathtt{S}\restriction\mathbf{\Lambda}$*, and hence a bijection between* $\mathfrak{S}$ *and* $\mathbf{\Lambda}$*.*

Proof. In order to prove pre-soundness/solidity, I argue as in the proof of Theorem 8.31, with the difference that now the Lemmas 8.29 and 8.23 are used. It follows from Lemma 4.16 that the s'-ISC carries over. □

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