THE SUBCOMPLETENESS OF MAGIDOR FORCING

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Abstract. It is shown that the Magidor forcing to collapse the cofinality of a measurable cardinal that carries a length ω_1 sequence of normal ultrafilters, increasing in the Mitchell order, to ω_1, is subcomplete.

1. Introduction

The paper [3] was dedicated to the question how "nice" a forcing can be if it changes the cofinality of a regular cardinal κ to an uncountable cardinal without collapsing κ as a cardinal. It was shown that such a forcing cannot be proper, and in particular, not countably closed, and that such a forcing necessarily does some "damage" to the universe, so the conclusion was that it can't be very nice. But in the present paper, I show that such a forcing can be nice in a different way: it can be subcomplete.[1] Namely, the Magidor forcing to change the cofinality of a measurable cardinal which carries a length ω_1 sequence of normal measures, increasing in the Mitchell order, to ω_1, is subcomplete.

The salient features of subcomplete forcings, introduced by Jensen in [4], are that they don't add reals, yet may add countable sequences, preserve stationary subsets of ω_1, and can be iterated with revised countable support. The class of subcomplete forcings includes Namba forcing (assuming (CH)), Příkrý forcing, and the forcing to shoot a club of order type ω_1 through a stationary subset of a regular cardinal greater than ω_1, consisting of ordinals of countable cofinality. The catalog of known subcomplete forcings is not very long, and proofs of subcompleteness are often not obvious. So another motivation for the work in this paper is the wish to extend this catalog by an interesting example, and to publicize the theory of subcomplete forcing and the methods for proving subcompleteness.

2. Preliminaries

We follow Jensen's exposition [5] of subcomplete forcing.

Definition 2.1. A transitive set N (usually a model of ZFC^-) is *full* if there is an ordinal γ such that $L_\gamma(N) \models \mathsf{ZFC}^-$ and N is regular in $L_\gamma(N)$, meaning that if $x \in N$, $f \in L_\gamma(N)$ and $f : x \longrightarrow N$, then $\mathrm{ran}(f) \in N$.

Definition 2.2. For a poset $\mathbb{P}$, $\delta(\mathbb{P})$ is the minimal cardinality of a dense subset of $\mathbb{P}$.

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[1]Of course, one could object that what this shows is that subcomplete forcing is not very nice.

Definition 2.3. Let $N = L^A_\tau = \langle L_\tau[A], \in, A \cap L_\tau[A]\rangle$ be a ZFC^- model, δ an ordinal and $X \cup \{\delta\} \subseteq N$. Then $C^N_\delta(X)$ is the smallest $Y \prec N$ such that $X \cup \delta \subseteq Y$.

Definition 2.4. A forcing $\mathbb{P}$ is *subcomplete* if for sufficiently large cardinals θ with $\mathbb{P} \in H_\theta$, any ZFC^- model $N = L^A_\tau$ with $\theta < \tau$ and $H_\theta \subseteq N$, any $\sigma : \bar{N} \longrightarrow_{\Sigma_\omega} N$ such that $\bar{N}$ is countable, transitive and full and such that $\mathbb{P}, \theta \in \mathrm{ran}(\sigma)$, any $\bar{G} \subseteq \bar{\mathbb{P}}$ which is $\bar{\mathbb{P}}$-generic over $\bar{N}$, and any $s \in \mathrm{ran}(\sigma)$, the following holds. Letting $\sigma(\bar{s}, \bar{\theta}, \bar{\mathbb{P}}) = s, \theta, \mathbb{P}$, there is a condition $p \in \mathbb{P}$ such that whenever $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over V with $p \in G$, there is in $\mathrm{V}[G]$ a σ' such that

1. $\sigma' : \bar{N} \longrightarrow_{\Sigma_\omega} N$,
2. $\sigma'(\bar{s}, \bar{\theta}, \bar{\mathbb{P}}) = s, \theta, \mathbb{P}$,
3. $(\sigma')``\bar{G} \subseteq G$,
4. $C^N_{\delta(\mathbb{P})}(\mathrm{ran}(\sigma')) = C^N_{\delta(\mathbb{P})}(\mathrm{ran}(\sigma))$.

In order to prove that a forcing is subcomplete, we adopt Jensen's approach to liftups, as presented in [5].

Definition 2.5. Let $\sigma : \bar{\mathfrak{A}} \longrightarrow_{\Sigma_0} \mathfrak{A}$, where $\bar{\mathfrak{A}}$ and $\mathfrak{A}$ are models of the language of set theory. Then σ is a *cofinal* embedding from $\bar{\mathfrak{A}}$ to $\mathfrak{A}$ if for every $x \in \mathfrak{A}$ there is an $\bar{x} \in \bar{\mathfrak{A}}$ such that $x \in^{\mathfrak{A}} \sigma(\bar{x})$. We write "$\sigma : \bar{\mathfrak{A}} \longrightarrow_{\Sigma_0} \mathfrak{A}$ cofinally" to express that σ is a cofinal embedding. If $\bar{\mathfrak{A}}$ is a ZFC^--model and $\bar{\tau} \in \bar{\mathfrak{A}}$ is a $\bar{\mathfrak{A}}$-cardinal, then σ is a $\bar{\tau}$-*cofinal* from $\bar{\mathfrak{A}}$ to $\mathfrak{A}$ if for every $x \in \mathfrak{A}$, there is a $\bar{x} \in \bar{\mathfrak{A}}$ such that in $\bar{\mathfrak{A}}$, the cardinality of $\bar{x}$ is less than $\bar{\tau}$ and $x \in^{\mathfrak{A}} \sigma(\bar{x})$. We write "$\sigma : \bar{\mathfrak{A}} \longrightarrow_{\Sigma_0} \mathfrak{A}$ $\bar{\tau}$-cofinally" to express that σ is a $\bar{\tau}$-cofinal embedding.

Observation 2.6. *Let $\sigma : \bar{N} \longrightarrow_{\Sigma_0} N$ $\bar{\tau}$-cofinally, where $\bar{\tau} \in \bar{N}$ is an uncountable cardinal in $\bar{N}$ and $\bar{N} \models \mathsf{ZFC}^-$. Let $\bar{\kappa} \geq \bar{\tau}$ such that in $\bar{N}$, the cofinality of $\bar{\kappa}$ is either equal to ω or greater than or equal to $\bar{\tau}$. Then σ is continuous at $\bar{\kappa}$, i.e.,*

$$\sigma(\bar{\kappa}) = \sup \sigma``\bar{\kappa}$$

Proof. Set $\kappa = \sigma(\bar{\kappa})$. Note that $\sup \sigma``\bar{\kappa} \leq \kappa$, so we only have to prove that $\kappa \leq \sup \sigma``\bar{\kappa}$.

If the cofinality of $\bar{\kappa}$ in $\bar{N}$ is ω, then, letting $f : \omega \longrightarrow \bar{\kappa}$ be cofinal, with $f \in \bar{N}$, it is easy to see that the set $\{\sigma(f(n)) \mid n < \omega\}$ is cofinal in κ.

Now suppose that the cofinality of $\bar{\kappa}$ in $\bar{N}$ is at least $\bar{\tau}$, and let $\alpha < \kappa$ be arbitrary. By $\bar{\tau}$-cofinality, there is a set $a \in \bar{N}$ such that in $\bar{N}$ the cardinality of a is less than $\bar{\tau}$, and such that $\alpha \in \sigma(a)$. By replacing a with $a \cap \bar{\kappa}$, we may assume that $a \subseteq \bar{\kappa}$. Since the cofinality of $\bar{\kappa}$ in $\bar{N}$ is at least $\bar{\tau}$, it follows that a is bounded in $\bar{\kappa}$, say by $\gamma < \bar{\kappa}$. But then, $\alpha \in \sigma(a) \subseteq \sigma(\gamma) < \sup \sigma``\bar{\kappa}$. □

The following fact about cofinal embeddings is well-known.

Fact 2.7. *If $\sigma : \bar{N} \longrightarrow_{\Sigma_0} N$ cofinally, where $\bar{N} \models \mathsf{ZFC}^-$ and $\bar{N}$, N are transitive, then σ is fully elementary, and hence, $N \models \mathsf{ZFC}^-$.*

Definition 2.8. A model $\mathfrak{A}$ of the language of set theory is *solid* if its well-founded part is transitive.

Definition 2.9. Let $\bar{\mathfrak{A}}$ be a solid model of ZFC^-, and let $\bar{\tau}$ be an uncountable cardinal in the well-founded part of $\bar{\mathfrak{A}}$. Let $\bar{H} = (H_{\bar{\tau}})^{\bar{\mathfrak{A}}}$, and let $\bar{\pi} : \bar{H} \longrightarrow_{\Sigma_0} H$ cofinally, where H is transitive. Then we write $\pi : \bar{\mathfrak{A}} \longrightarrow_{\bar{\pi}} \mathfrak{A}$, or say that $\langle \mathfrak{A}, \pi\rangle$ is a *liftup* of $\bar{\mathfrak{A}}, \bar{\pi}$, to express that $\bar{\pi} \subseteq \pi$, H is contained in the well-founded part of $\mathfrak{A}$, $\mathfrak{A}$ is solid, and π is Σ_0-preserving and $\bar{\tau}$-cofinal.

A liftup always exists ([5, p. 118, Lemma 3.3]), and it is unique, up to isomorphism ([5, p. 117, Lemma 3.1]). In particular, if there is a transitive liftup, then that is the unique transitive liftup. Transitive liftups exist in the scenario described in the following *interpolation lemma.*

Lemma 2.10 ([5, Lemma 5.1])**.** *Let $\bar{N}$ and N be transitive models of* ZFC^-*, and let $\sigma : \bar{N} \longrightarrow_{\Sigma_\omega} N$. Let $\bar{\tau}$ be a cardinal in $\bar{N}$. Let $\bar{H} = H_{\bar{\tau}}^{\bar{N}}$ and $\tilde{H} = \bigcup\{\sigma(u) \mid u \in \bar{H}\}$.*[2] *Then*

1. *The transitive liftup $\tilde{\sigma} : \bar{N} \longrightarrow_{\sigma\restriction\bar{H}} \tilde{N}$ exists.*
2. *There is a canonical elementary embedding $k : \tilde{N} \longrightarrow N$ such that $k \circ \tilde{\sigma} = \sigma$ and $k\restriction\tilde{H} = \mathrm{id}$.*
3. *k is the unique $\tilde{k} : \tilde{N} \longrightarrow_{\Sigma_0} N$ with $\tilde{k}\circ\tilde{\sigma} = \sigma$ and $\tilde{k}\restriction\tilde{\tau} = \mathrm{id}$, where $\tilde{\tau} = \mathrm{On}\cap\tilde{H}$.*

Corollary 2.11. *In the notation of the previous lemma, if σ is a cofinal embedding, then so is the canonical embedding k.*

We will also use some Barwise theory, as presented in [5, p. 102 ff]. For a more detailed treatment, see [1]. Recall that a structure $\langle M, A_1, \ldots, A_n\rangle$ is admissible if it is transitive and satisfies KP, using the predicates $A_1, \ldots, A_n$. For admissible M, Barwise developed an infinitary logic where the infinitary formulas are (coded by) elements of M. Thus, infinitary conjunctions and disjunctions are allowed, as long as they are in M. Let A be a $\Sigma_1(M)$ set of such infinitary formulas. The logic comes with a proof theory and a model theory. The main features are:

1. The M-finiteness lemma: if a formula φ is provable from A, then there is a $u \in M$ such that $u \subseteq A$ and φ is provable from u.
2. The correctness theorem: if there is a model $\mathfrak{A}$ with $\mathfrak{A} \models A$, then A is consistent.
3. The Barwise completeness theorem: if M is countable and A is consistent, then there is a model $\mathfrak{A}$ with $\mathfrak{A} \models A$.

Definition 2.12. Let M be admissible. If A consists of infinitary formulas in M, then A is a *theory on M*. A is an *$\in$-theory* on M if the language it is formulated in contains the symbol $\in$, a constant symbol $\underline{x}$, for every $x \in M$, and if the theory contains the extensionality axiom, as well as the *basic axiom*

$$\forall y \quad (y \in \underline{x} \iff \bigvee_{z\in x} y = \underline{z})$$

for every $x \in M$. It is a ZFC^--theory on M if it is an $\in$-theory on M that contains the ZFC^- axioms (viewed as a set of finitary formulas, which are also in M).

If A is an $\in$-theory on M and $\mathcal{A}$ is a solid model for A, then automatically, $\underline{x}^{\mathfrak{A}} = x$, which is why I won't specify the interpretation of these constants by such a model.

Definition 2.13. A transitive model N of ZFC^- is *almost full* if there is a solid model $\mathfrak{A}$ of ZFC^-, such that N is contained in the well-founded part of $\mathfrak{A}$ and N is *regular* in $\mathfrak{A}$, i.e., if $x \in N$, $f \in \mathfrak{A}$, and $f : x \longrightarrow N$, then $\mathrm{ran}(f) \in N$.

Definition 2.14. If N is a transitive set, then we write $\alpha(N)$ for the least $\alpha > \omega$ such that $L_\alpha(N) \models \mathsf{KP}$.

[2] There's a typo here in the original lemma.

We will use the following lemma several times in the main proof.

Lemma 2.15 ([5, p. 123, Lemma 4.5]). *Let $\bar{N}$ and N be a transitive ZFC^--models. Let $\bar{N}$ be almost full and $\sigma : \bar{N} \longrightarrow_{\Sigma_0} N$ be cofinal. Then N is almost full. Further, let $\bar{\mathcal{L}}$ be a theory in an infinitary language on $L_{\alpha(\bar{N})}(\bar{N})$ that has a Σ_1-definition in $L_{\alpha(\bar{N})}(\bar{N})$ in the parameters $\bar{N}$ and $p_1, \ldots, p_n \in \bar{N}$. Let $\mathcal{L}$ be the infinitary theory on $L_{\alpha(N)}(N)$ defined over $L_{\alpha(N)}(N)$ by the same Σ_1-formula, using the parameters N, $\sigma(p_1), \ldots, \sigma(p_n)$. If $\bar{\mathcal{L}}$ is consistent, then so is $\mathcal{L}$.*

3. The proof

Let us fix the following for the remainder of the paper. κ is a measurable cardinal which carries a sequence $\vec{U} = \langle U_i \mid i < \omega_1 \rangle$ of normal measures on κ, increasing in the Mitchell-order. For $i < j < \omega_1$, f_i^j is a function with domain κ such that $U_i = [f_i^j]_{U_j}$. I write $\vec{f}$ for this sequence of functions, $\vec{f} = \langle f_i^j \mid i < j < \omega_1 \rangle$. $\mathbb{M} = \mathbb{M}(\vec{U}, \vec{f})$ is then the forcing introduced by Magidor in [6]. If $\alpha < \omega_1$, then I use the notation $\mathbb{M}{\restriction}\alpha$ for the Magidor forcing $\mathbb{M}(\vec{U}{\restriction}\alpha, \langle f_i^j \mid i < j < \alpha \rangle)$.

The generic filter added by $\mathbb{M}$ corresponds to a normal function $c : \omega_1 \longrightarrow \kappa$ called a Magidor sequence. That is, the generic filter is definable from the sequence c, and vice versa. So instead of working with $\mathbb{M}$-generic filters, I will work with the sequences they correspond to.

The following sets play a crucial role in the very definition of $\mathbb{M}$. For $0 < \gamma < \omega_1$, let

$$
\begin{aligned}
A_\gamma &= \{\delta < \kappa \mid \forall \mu < \nu < \gamma \quad f_\mu^\gamma(\delta) \lhd f_\nu^\gamma(\delta) \text{ are normal ultrafilters on } \delta\} \\
B_\gamma &= \{\delta \in A_\gamma \mid \forall \mu < \nu < \gamma \quad [f_\mu^\nu{\restriction}\delta]_{f_\nu^\gamma(\delta)} = f_\mu^\gamma(\delta)\}
\end{aligned}
$$

Then $A_\gamma, B_\gamma \in U_\gamma$, and letting

$$B_0 = \{\delta < \kappa \mid \delta \text{ is inaccessible}\}$$

it follows that $B_0 \in U_0$ also. These facts were shown by Magidor in [6].

I will need the following characterization of Magidor genericity.

Theorem 3.1 ([2, Theorem 4.4]). *Let V be an inner model of W, and let $\mathbb{M} = \mathbb{M}(\langle \tilde{U}_\gamma | \gamma < \alpha \rangle, \langle \tilde{f}_\mu^\nu \mid \mu < \nu < \alpha \rangle)$ be a Magidor forcing in V. Then c in W is $\mathbb{M}$-generic over V iff c is a strictly increasing sequence in $\prod_{\gamma<\alpha} \tilde{B}_\gamma$ (where these sets are defined as above, with respect to the sequences $\vec{\tilde{U}}$ and $\vec{\tilde{f}}$), such that*

1. *For every function $X \in \mathrm{V} \cap \prod_{\gamma<\alpha} \tilde{U}_\gamma$, there is a $\zeta < \alpha$ such that for all $\xi < \alpha$ with $\xi > \zeta$, $c(\xi) \in X(\xi)$.*
2. *For every limit $\beta < \alpha$, and for every function $X \in \mathrm{V} \cap \prod_{\gamma<\beta} \tilde{f}_\gamma^\beta(c(\beta))$, there is a $\zeta < \beta$ such that for all $\xi < \beta$ with $\xi > \zeta$, $c(\xi) \in X(\xi)$.*

Now all the machinery needed for the proof of the main theorem is assembled.

Theorem 3.2. *$\mathbb{M}$ is subcomplete.*

Proof. Let $\theta > 2^{2^\kappa}$ be a regular cardinal, $\tau > \theta$ regular, $N = L_\tau[A]$ a ZFC^- model with $H_\theta \subseteq N$, and let $\sigma : \bar{N} \longrightarrow_{\Sigma_\omega} N$ be an elementary embedding with $\theta, \mathbb{M} \in \operatorname{ran}(\sigma)$ and $\bar{N}$ countable, transitive and full. Let's fix $s \in \operatorname{ran}(\sigma)$, and let $\delta = \delta(\mathbb{M})$, the smallest cardinality of a dense subset of $\mathbb{M}$. Let $\bar{s} = \sigma^{-1}(s)$,

$\bar{\mathbb{M}} = \sigma^{-1}(\mathbb{M})$, $\bar{\theta} = \sigma^{-1}(\theta)$ and $\bar{\delta} = \sigma^{-1}(\delta)$. Let $\Omega = \omega_1^{\bar{N}}$, and let $\bar{c} : \Omega \longrightarrow \bar{\kappa}$ be a Magidor sequence (for $\bar{\mathbb{M}}$) over $\bar{N}$. Note that Ω is the critical point of σ.

To prove that $\mathbb{M}$ is subcomplete, we have to show that there is a condition in $\mathbb{M}$ such that whenever G is $\mathbb{M}$-generic and G contains that condition, then, letting d be the Magidor sequence corresponding to G, in $\mathrm{V}[d]$, there is an elementary embedding $\sigma' : \bar{N} \longrightarrow N$ such that $\sigma'(\bar{s}) = s$, $\sigma'``\bar{c} \subseteq d$ and $C^N_\delta(\mathrm{ran}(\sigma)) = C^N_\delta(\mathrm{ran}(\sigma'))$.

It is easy to see that $\delta \geq \kappa^+$. The point is that if U is any normal ultrafilter on κ and $\vec{A} = \langle A_i \mid i < \kappa\rangle$ is a sequence of members of U, then there is an $A \in U$ such that no A_i is contained in A. Because otherwise, U would be the set of subsets A of κ such that there is an $i < \kappa$ with $A_i \subseteq A$. But if M is the ultrapower of V by U, then $\vec{A} \in M$, and so, it would follow that $U \in M$, being definable from $\vec{A}$. This contradicts the well-foundedness of the Mitchell order. Now if $D = \{\langle s_i, A_i\rangle \mid i < \kappa\} \subseteq \mathbb{M}$ were dense in $\mathbb{M}$, then we could, by this fact, for every $j < \omega_1$ choose a $B_j \in U_j$ such that for every $i < \kappa$, if A_i is defined at j and $A_i(j) \in U_j$, then $A_i(j)$ is not contained in B_j. Then $\langle \emptyset, \vec{B}\rangle$ is a condition in $\mathbb{M}$ that does not have a strengthening in D, because for every $i < \kappa$, we can take j large enough that $\mathrm{dom}(s_i) \subseteq j$, in which case $A_i(j) \in U_j$, and hence, by definition, $A_i(j)$ is not contained in B_j, so that it is not the case that $\langle s_i, A_i\rangle \leq \langle \emptyset, \vec{B}\rangle$. The reader is referred to [6] for the definition of the ordering on $\mathbb{M}$.

Let $k_0 : N_0 \longrightarrow N$ be the inverse of the Mostowski collapse of $C^N_\delta(\mathrm{ran}(\sigma))$. Let $\sigma_0 = k_0^{-1} \circ \sigma$. So $\sigma_0 : \bar{N} \longrightarrow N_0$. Let $s^{N_0} = k_0^{-1}(s)$, $\mathbb{M}^{N_0} = k_0^{-1}(\mathbb{M})$, $\theta^{N_0} = k_0^{-1}(\theta)$ and $\delta^{N_0} = k_0^{-1}(\delta)$ (which is δ). As pointed out in [5, p. 129], it follows that $\sigma_0 : \bar{N} \longrightarrow_{\sigma\restriction\bar{H}} N_0$, where $\bar{H} = (H_{\bar{\delta}^+})^{\bar{N}}$. In particular, the cofinality of $\mathrm{On} \cap N_0$ is ω, since σ_0 is a cofinal embedding.

Let $\mathcal{L}_0$ be the ZFC^--theory on $L_{\alpha(N_0)}(N_0)$ in the language with extra constants $\dot{\sigma}$ and $\dot{c}$, consisting of the following additional axioms:[3]

1. $\dot{\sigma} : \underline{\bar{N}} \longrightarrow \underline{N_0}$ is a $\underline{\bar{c}(\omega)}$-cofinal, elementary embedding.
2. $\dot{\sigma}(\underline{\bar{s}}, \underline{\bar{\mathbb{M}}}, \underline{\bar{\theta}}, \underline{\bar{\delta}}) = \underline{s}^{N_0}, \underline{\mathbb{M}}^{N_0}, \underline{\theta}^{N_0}, \underline{\delta}^{N_0}$.
3. $\dot{\sigma}``\underline{\bar{c}} = \dot{c}$.
4. $\dot{c}$ is generic over $\underline{N_0}$ for $\underline{\mathbb{M}}^{N_0}\restriction\underline{\Omega}$.

The basic axioms are intended to insure that in any solid model of $\mathcal{L}_0$, $\underline{x}$ will be interpreted as x, for every $x \in L_{\alpha(N_0)}(N_0)$. So when dealing with a solid model of a theory containing these basic axioms, I omit the specification of how that model interprets the constants of the form $\underline{x}$.

(1) $\mathcal{L}_0$ *is consistent.*

Proof of (1). Let $\sigma_1 : \bar{N} \longrightarrow N_1$ be the liftup of $\bar{N}$ by $\sigma\restriction(H_{\bar{c}(\omega)})^{\bar{N}}$, which is well-founded and hence can be taken to be transitive, by Lemma 2.10, and let $c_1 = \sigma_1``\bar{c}$. Moreover, let $s^{N_1} = \sigma_1(\bar{s})$, $\mathbb{M}^{N_1} = \sigma_1(\bar{\mathbb{M}})$, $\theta^{N_1} = \sigma_1(\bar{\theta})$ and $\delta^{N_1} = \sigma_1(\bar{\delta})$.

By design, $\langle H_\theta, \sigma_1, c_1\rangle$ is a model of the ZFC^--theory $\mathcal{L}_1$ on $L_{\alpha(N_1)}(N_1)$ in the language with extra constants $\dot{\sigma}$, $\dot{c}$ and the following additional axioms:

1. $\dot{\sigma} : \underline{\bar{N}} \longrightarrow \underline{N_1}$ is a $\underline{\bar{c}(\omega)}$-cofinal, elementary embedding.
2. $\dot{\sigma}(\underline{\bar{s}}, \underline{\bar{\mathbb{M}}}, \underline{\bar{\theta}}, \underline{\bar{\delta}}) = \underline{s}^{N_1}, \underline{\mathbb{M}}^{N_1}, \underline{\theta}^{N_1}, \underline{\delta}^{N_1}$.

[3]Being a ZFC^--theory on $L_{\alpha(N_0)}(N_0)$, the language of $\mathcal{L}_0$ contains the constants $\underline{x}$ ($x \in L_{\alpha(N_0)}(N_0)$), the basic axioms and the ZFC^- axioms, see Definition 2.12. That's why here, and in similar situations to come, only the extra constants and additional axioms of the theory are explicitly listed.

3. $\dot{\sigma}``\underline{\bar{c}} = \dot{c}$.
4. $\dot{c}$ is generic over $\underline{N_1}$ for $\underline{\mathbb{M}}^{N_1}{\restriction}\underline{\Omega}$.

The first three points are obviously satisfied in $\langle H_\theta, \sigma_1, c_1\rangle$ (of course, $\underline{x}$ is interpreted as x). For the last point, we have to show that c_1 is $\sigma_1(\mathbb{M}){\restriction}\Omega$-generic over N_1. To this end, we verify the characterization of Magidor genericity given in Theorem 3.1.

Let's write $\bar{\vec{U}} = \sigma^{-1}(\vec{U})$, $\bar{\vec{f}} = \sigma^{-1}(\vec{f})$, and $\vec{U}^{N_1} = \sigma_1(\bar{\vec{U}})$, $\vec{f}^{N_1} = \sigma_1(\bar{\vec{f}})$. Note that since $\bar{c}(\gamma) \in (B_\gamma)^{\bar{N}}$, where $(B_\gamma)^{\bar{N}}$ is defined like B_γ with respect to the sequences $\bar{\vec{U}}$ and $\bar{\vec{f}}$ in $\bar{N}$, it's clear that $\sigma_1(\bar{c}(\gamma)) = c_1(\gamma) \in (B_\gamma)^{N_1}$ in the obvious sense.

So the first point to verify is that given a sequence $\vec{X} = \langle X_i \mid i < \Omega\rangle \in N_1 \cap \prod_{i<\Omega} U_i^{N_1}$, there is a $\zeta < \Omega$ such that for all $\xi \in (\zeta, \Omega)$, $c_1(\xi) \in X_\xi$. Of course, we want to use the fact that $\bar{c}$ satisfies the corresponding condition over $\bar{N}$. Since σ_1 is $\bar{c}(\omega)$-cofinal, there is a set $w \in \bar{N}$ of $\bar{N}$-cardinality less than $\bar{c}(\omega)$ such that $\vec{X} \in \sigma_1(w)$. We may assume that w consists of members of $\prod_{i<\Omega} \bar{U}_i$, and then, we may define $\langle Y_i \mid i < \Omega\rangle \in \prod_{i<\Omega} \bar{U}_i$ in $\bar{N}$ by setting

$$Y_i = \bigcap\{z_i \mid z = \langle z_j \mid j < \Omega\rangle \in w\}$$

(we used that U_i is $\bar{\kappa}$-closed in $\bar{N}$ and that the $\bar{N}$-cardinality of w is less than $\bar{\kappa}$). By genericity, there is a $\zeta < \Omega$ such that for all $\xi \in (\zeta, \Omega)$, $\bar{c}(\xi) \in Y_\xi$. But this means that, fixing ξ temporarily, it's true in $\bar{N}$ that for every $z \in w$, $\bar{c}(\xi) \in z_\xi$. By elementarity of σ_1, it follows that for every $z \in \sigma_1(w)$, $c_1(\xi) \in z_\xi$. In particular, $c_1(\xi) \in X_\xi$, and this holds for all $\xi \in (\zeta, \Omega)$.

The second point that needs to be verified in order to see that c_1 is $\mathbb{M}^{N_1}{\restriction}\Omega$-generic over N_1 is that for any limit ordinal $\beta < \Omega$ and every $\vec{X} = \langle X_i \mid i < \beta\rangle \in \prod_{i<\beta}(f_i^\beta)^{N_1}(c_1(\beta))$, there is a $\zeta < \beta$ such that for all $\xi \in (\zeta, \beta)$, $c_1(\xi) \in X_\xi$. As before, let $w \in \bar{N}$ have $\bar{N}$-cardinality less than $\bar{c}(\omega)$, consisting of members of $\prod_{i<\beta} \bar{f}_i^\beta(\bar{c}(\beta))$, such that $\vec{X} \in \sigma_1(w)$. Define $Y_i = \bigcap\{z_i \mid z = \langle z_j \mid j < \beta\rangle \in w\}$, for $i < \beta$. Since $\bar{f}_i^\beta(\bar{c}(\beta))$ is $\bar{c}(\beta)$-closed in $\bar{N}$, and since in $\bar{N}$, $\overline{\overline{w}} < \bar{c}(\omega) \le \bar{c}(\beta)$, it follows that $Y_i \in \bar{f}_i^\beta(\bar{c}(\beta))$. By genericity, let $\zeta < \beta$ be such that for all $\xi \in (\zeta, \beta)$, $\bar{c}(\xi) \in Y_\xi$. So for every $z \in w$ and $\xi \in (\zeta, \beta)$, $\bar{c}(\xi) \in z_\xi$. Applying σ_1, this means that for every $z \in \sigma_1(w)$ (and in particular, for $\vec{X}$), $c_1(\xi) \in z_\xi$.

So since $\mathcal{L}_1$ has a model, it is consistent. And since σ_0 is the liftup of $\bar{N}$ by $\sigma{\restriction}(H_{\bar{\delta}^+})^{\bar{N}}$, σ_1 is also the liftup of $\bar{N}$ by $\sigma_0{\restriction}(H_{\bar{c}(\omega)})^{\bar{N}}$, as σ_0 and σ coincide on $(H_{\bar{c}(\omega)})^{\bar{N}}$. Hence, there is a canonical embedding $k_1 : N_1 \longrightarrow N_0$ (defined by $k_1(\sigma_1(f)(\gamma)) = \sigma_0(f)(\gamma)$, for $\gamma < \bar{c}(\omega)$), so that $k_1 \circ \sigma_1 = \sigma_0$, and this embedding is cofinal, by Corollary 2.11. But then the consistency of $\mathcal{L}_1$ implies that of $\mathcal{L}_0$ by Lemma 2.15, since $\mathcal{L}_0$ is like $\mathcal{L}_1$, with the parameters moved by k_1, and with N_1 replaced by N_0. Note here that N_1 is almost full by the same lemma, since $\sigma_1 : \bar{N} \longrightarrow N_1$ is cofinal and $\bar{N}$ is (almost) full. $\Box_{(1)}$

Now let F be generic for $\mathrm{Col}(\omega, 2^\theta)$. By Barwise compactness, in $\mathrm{V}[F]$, there is a model $\mathcal{A}$ for the theory $\mathcal{L}_0$, since this theory is countable in $\mathrm{V}[F]$. Let $\sigma' = \dot{\sigma}^{\mathcal{A}}$ and $c' = \dot{c}^{\mathcal{A}}$. So

$$\sigma' : \bar{N} \longrightarrow N_0 \quad \bar{c}(\omega)\text{-cofinally, and } c' = (\sigma')``\bar{c} \text{ is } (\mathbb{M}^{N_0}{\restriction}\Omega)\text{-generic over } N_0$$

and

$$\sigma'(\bar{s}, \bar{\mathbb{M}}, \bar{\theta}, \bar{\delta}) = s^{N_0}, \mathbb{M}^{N_0}, \theta^{N_0}, \delta^{N_0}$$

So this embedding is already pretty close to what we want to find in an $\mathbb{M}$-generic extension of V, the most obvious flaw being that it should map $\bar{c}$ into an $\mathbb{M}^{N_0}$-generic, not an $\mathbb{M}^{N_0}\restriction\Omega$-generic. We will fix that by introducing a discontinuity point at κ. Let

$$j : \mathrm{V} \longrightarrow_{U_\Omega} M$$

be the ultrapower and embedding by U_Ω, M being transitive. Let $M_0 = j(N_0)$, and set

$$j(s^{N_0}, \mathbb{M}^{N_0}, \theta^{N_0}, \delta^{N_0}, \vec{U}^{N_0}, \vec{f}^{N_0}) = s^{M_0}, \mathbb{M}^{M_0}, \theta^{M_0}, \delta^{M_0}, \vec{U}^{M_0}, \vec{f}^{M_0}$$

Let $\mathcal{L}_2$ be the ZFC^--theory on $L_{\alpha(M_0)}(M_0)$ with extra constants $\dot{\sigma}$, $\dot{c}$ and the following additional axioms, with the differences to $\mathcal{L}_0$ highlighted:

1. $\dot{\sigma} : \underline{\bar{N}} \longrightarrow \underline{M_0}$ is a **cofinal**, elementary embedding.
2. $\dot{\sigma}(\underline{\bar{s}}, \underline{\bar{\mathbb{M}}}, \underline{\bar{\theta}}, \underline{\bar{\delta}}) = \underline{s}^{M_0}, \underline{\mathbb{M}}^{M_0}, \underline{\theta}^{M_0}, \underline{\delta}^{M_0}$.
3. $\boldsymbol{\dot{\sigma}``\underline{\bar{c}} \subseteq \dot{c}}$**.**
4. $\dot{c}$ **is generic over** $\underline{M_0}$ **for** $\underline{\mathbb{M}}^{M_0}$**.**

(2) *$\mathcal{L}_2$ is consistent.*

Proof of (2). In $\mathrm{V}[F]$, set $\sigma_2 = j \circ \sigma' : \bar{N} \longrightarrow M_0$. Since the critical point of j is κ, $c' = j``c'$.

We will need the following notation, see [2, Def. 4.1]. If $\mathbb{M}$ is a Magidor forcing of length λ, $\beta < \lambda$ and $\delta \in B^{\mathbb{M}}_\beta$, then we let $\mathbb{M}_{\langle\beta,\delta\rangle} = \{\langle t, T\rangle \mid \langle t, T\rangle \in \mathbb{M} \text{ and } t(\beta) = \delta\}$. Further, we write $\mathbb{M}^-_{\langle\beta,\delta\rangle} = \{\langle t\restriction\beta, T\restriction\beta\rangle \mid \langle t, T\rangle \in \mathbb{M}_{\langle\beta,\delta\rangle}\}$ and $\mathbb{M}^+_{\langle\beta,\delta\rangle} = \{\langle t\restriction(\beta,\lambda), T\restriction(\beta,\lambda)\rangle \mid \langle t, T\rangle \in \mathbb{M}_{\langle\beta,\delta\rangle}\}$. If all of these sets are equipped with the natural partial ordering coming from $\mathbb{M}$, it follows that $\mathbb{M}_{\langle\beta,\delta\rangle}$ is isomorphic to $\mathbb{M}^-_{\langle\beta,\delta\rangle} \times \mathbb{M}^+_{\langle\beta,\delta\rangle}$. And clearly, $\mathbb{M}_{\langle\beta,\delta\rangle}$ is like forcing with $\mathbb{M}$ below the condition $\langle\{\langle\beta,\delta\rangle\}, T\rangle$, where T is the weakest possible second coordinate for that first coordinate.

The first thing I want to verify is that it makes sense to form, in M_0, the poset $(\mathbb{M}^{M_0})^-_{\langle\Omega,\kappa\rangle}$. The requirement is merely that $\kappa \in (B_\Omega)^{M_0}$, where $(B_\Omega)^{M_0}$ is defined in M_0 from $\vec{U}^{M_0}$ and $\vec{f}^{M_0}$ as B_γ was defined in V. So $(B_\Omega)^{M_0} = j(B^{N_0}_\Omega)$. Since $\delta \geq \kappa^+$, it is clear that $\vec{U}^{N_0}_\Omega = U_\Omega \cap N_0$. So since $B^{N_0}_\Omega \in \vec{U}^{N_0}_\Omega$, it follows that $B^{N_0}_\Omega \in U_\Omega$. So $\kappa \in j(B^{N_0}_\Omega) = B^{M_0}_\Omega$, as wished.

The second point is that c' is $(\mathbb{M}^{M_0})^-_{\langle\Omega,\kappa\rangle}$-generic over M_0. Noting that by [2, Lemma 4.2], $(\mathbb{M}^{M_0})^-_{\langle\Omega,\kappa\rangle}$ is a Magidor forcing itself, we again have to verify the two conditions of Theorem 3.1 characterizing Magidor genericity. Since $\mathrm{V}^{N_0}_\kappa = \mathrm{V}^{M_0}_\kappa$, and c' is $\mathbb{M}^{N_0}\restriction\Omega$-generic over N_0, only the first condition needs to be checked. So let $\vec{X} = \langle X_i \mid i < \Omega\rangle \in M_0 \cap \prod_{i<\Omega} W_i$, where $W_i = (\vec{f}^{M_0})^\Omega_i(\kappa)$. Let's try to understand what W_i is. It's $j(\vec{f}^{N_0})^\Omega_i(\kappa)$, i.e., $[(\vec{f}^{N_0})^\Omega_i]_{U_\Omega}$. But, again because $\delta \geq \kappa^+$, $\vec{f}^{N_0} = \vec{f}$. So $W_i = [f^\Omega_i]_{U_\Omega}$, which by the properties of the $\vec{f}$ sequence, means that $W_i = U_i$. Thus, we are dealing with a sequence $\langle X_i \mid i < \Omega\rangle \in M_0 \cap \prod_{i<\Omega} U_i$. Note that $\vec{X} \in M_0 \subseteq \mathrm{V}$ (we're working in $\mathrm{V}[F]$). Since N_0 is the transitive collapse of $C^N_\delta(\mathrm{ran}(\sigma))$, and δ is the smallest size a dense subset of $\mathbb{M}$ can have (in V), we have the following "density" property of N_0 in V: given any sequence $\langle Y_i \mid i < \omega_1^{\mathrm{V}}\rangle \in \mathrm{V} \cap \prod_{i<\omega_1^{\mathrm{V}}} U_i$, there is a sequence $\langle Z_i \mid i < \omega_1^{\mathrm{V}}\rangle \in N_0 \cap \prod_{i<\omega_1^{\mathrm{V}}} U_i$ such that for all sufficiently large $i < \omega_1$, $Z_i \subseteq Y_i$. This is because (temporarily arguing in V) there is a dense subset D of $\mathbb{M}$ that has size δ and is a member of $C^N_\delta(\mathrm{ran}(\sigma))$, so that $D \subseteq C^N_\delta(\mathrm{ran}(\sigma))$, and it follows that $D \in N_0$, since $\delta \geq \kappa^+$. So there is a condition $\langle t, Z\rangle \in D \subseteq N_0$, $\langle t, T\rangle \leq \langle \emptyset, \vec{Y}\rangle$. It follows that for all $i < \omega_1^{\mathrm{V}}$ with $\mathrm{dom}(t) \subseteq i$, $Z_i \subseteq Y_i$. Applying this density property to $\vec{X}$, let $\vec{Z} \in N_0$ be such a sequence. By the fact that c' is $\mathbb{M}^{N_0}{\restriction}\Omega$-generic over N_0, it follows that for all sufficiently large $\xi < \Omega$, $c'(\xi) \in Z_\xi \subseteq X_\xi$. This verifies the first of the two genericity conditions. The second one is immediate, as stated above.

Now, knowing that c' is $(\mathbb{M}^{M_0})^-_{\langle\Omega,\kappa\rangle}$-generic over M_0, we can let $d \in \mathrm{V}[F]$ be $(\mathbb{M}^{M_0})^+_{\langle\Omega,\kappa\rangle}$-generic over $M_0[c']$, and then set $c'' = c' \cup \{\langle\Omega,\kappa\rangle\} \cup d$. It is now easy to check that c'' is $\mathbb{M}^{M_0}$-generic over M_0.

Finally, note that $\sigma_2 : \bar{N} \longrightarrow M_0$ is cofinal. To see this, note that $\sigma_0 : \bar{N} \longrightarrow N_0$ is cofinal (even $\bar{c}(\omega)$-cofinal). As noted earlier, this implies that the cofinality of $\mathrm{On} \cap N_0$ is ω, since $\bar{N}$ is countable. As a result, $j{\restriction}N_0 : N_0 \longrightarrow M_0$ is also cofinal.

Taken together, this shows that $\langle H_{(2^\delta)^+}, \sigma_2, c''\rangle$ is a model of $\mathcal{L}_2$. $\Box_{(2)}$

The main thing that's wrong with $\mathcal{L}_2$, of course, is that the target model is M_0 rather than N_0 (ultimately, we want it to be N). So let $\mathcal{L}_3$ be the ZFC^--theory on $L_{\alpha(N_0)}(N_0)$ in which this flaw is fixed. It has extra constants $\dot\sigma$, $\dot c$ and the following additional axioms, again with the differences to $\mathcal{L}_2$ highlighted:

1. $\dot\sigma : \underline{\bar{N}} \longrightarrow \underline{\mathbf{N_0}}$ is a cofinal, elementary embedding.
2. $\dot\sigma(\underline{\bar{s}}, \underline{\bar{\mathbb{M}}}, \underline{\bar\theta}, \underline{\bar\delta}) = \underline{s}^{N_0}, \underline{\mathbb{M}}^{N_0}, \underline{\theta}^{N_0}, \underline{\delta}^{N_0}$.
3. $\dot\sigma``\underline{\bar c} \subseteq \dot c$.
4. $\dot c$ is generic over $\underline{N_0}$ for $\underline{\mathbb{M}}^{\mathbf{N_0}}$.

(3) $\mathcal{L}_3$ *is consistent.*

Proof of (3). This is because $j(\mathcal{L}_3) = \mathcal{L}_2$: $j(N_0) = M_0$, and $j(``\dot\sigma(\bar s) = \underline{s}") = ``\dot\sigma(\bar s) = \underline{j(s)}"$, et cetera. $\mathcal{L}_2$ is consistent, so M believes that $\mathcal{L}_2$ is consistent, so $\mathcal{L}_3 = j^{-1}(\mathcal{L}_2)$ is consistent, by elementarity. $\Box_{(3)}$

There are two issues that still need to be addressed: firstly, the closure in N of δ union the range of the embedding we're looking for should be equal to $C^N_\delta(\mathrm{ran}(\sigma))$, and secondly, it should exist in a Magidor-generic extension of V, rather than in $\mathrm{V}[F]$. To address these issues, we adopt an approach similar to the one Jensen used in order to prove the subcompleteness of Příkrý forcing, see [5, p. 128ff.].

We add one more requirement to the theory. Namely, let $\mathcal{L}_4$ be like $\mathcal{L}_3$, with the additional axiom expressing that $\dot\sigma$ is $(\bar\kappa^+)^{\bar N}$-cofinal (not only cofinal). So $\mathcal{L}_4$ is the ZFC^--theory on $L_{\alpha(N_0)}(N_0)$ with extra constants $\dot\sigma$, $\dot c$ and the following additional axioms, with the differences to $\mathcal{L}_3$ highlighted:

1. $\dot\sigma : \underline{\bar{N}} \longrightarrow \underline{N_0}$ is a $\overline{(\bar\kappa^+)^{\mathbf{\bar N}}}$**-cofinal**, elementary embedding.
2. $\dot\sigma(\underline{\bar{s}}, \underline{\bar{\mathbb{M}}}, \underline{\bar\theta}, \underline{\bar\delta}) = \underline{s}^{N_0}, \underline{\mathbb{M}}^{N_0}, \underline{\theta}^{N_0}, \underline{\delta}^{N_0}$.
3. $\dot\sigma``\underline{\bar c} \subseteq \dot c$.
4. $\dot c$ is generic over $\underline{N_0}$ for $\underline{\mathbb{M}}^{N_0}$.

(4) $\mathcal{L}_4$ *is consistent.*

Proof of (4). In $\mathrm{V}[F]$, let $\mathcal{A}$ be a model of $\mathcal{L}_3$. Let $\sigma_3 = \dot\sigma^{\mathcal{A}}$, $\tilde c = \dot c^{\mathcal{A}}$. So $\sigma_3 : \bar N \longrightarrow N_0$ is cofinal and elementary, $\tilde c$ is $\mathbb{M}^{N_0}$-generic over N_0 and $\sigma_3``\bar c \subseteq \tilde c$. Thus, σ_3 can be extended canonically to an elementary embedding $\sigma_3' : \bar N[\bar c] \longrightarrow N_0[\tilde c]$, so that $\sigma_3'(\bar c) = \tilde c$. Let $\sigma^* : \bar N[\bar c] \longrightarrow_{\sigma_3'{\restriction}(H_{\bar\kappa^+})^{\bar N}} N_5[c^*]$ be the liftup of $\bar N[\bar c]$ via $\sigma_3'{\restriction}H_{\bar\kappa^+}$.

Note that $\bar{N}$ is of the form $L_\tau^{\bar{A}} = \langle L_\tau[\bar{A}], \in, \bar{A} \cap L_\tau[\bar{A}]\rangle$. As a result, $\bar{N}$ is easily definable in $\bar{N}[\bar{c}]$ (without having to appeal to the definability of ground models in forcing extensions, and worrying about the fact that $\bar{N}$ may not be a ZFC model).

It follows then that the liftup of $\bar{N}[\bar{c}]$ must be of the form $N_5[c^*]$. Note also that $(\bar{\kappa}^+)^{\bar{N}}$ is still a regular cardinal in $\bar{N}[\bar{c}]$, since $\bar{\mathbb{M}}$ has the $(\bar{\kappa}^+)^{\bar{N}}$-c.c. in $\bar{N}$. Let $k^* : N_5[c^*] \longrightarrow N_0[\tilde{c}]$ be the canonical embedding, so $k^* \circ \sigma^* = \sigma_3'$. All of these embeddings are cofinal. Let $s^{N_5} = \sigma^*(\bar{s})$, $\mathbb{M}^{N_5} = \sigma^*(\bar{\mathbb{M}})$, $\theta^{N_5} = \sigma^*(\bar{\theta})$ and $\delta^{N_5} = \sigma^*(\bar{\delta})$.

Let $\bar{\sigma}^* = \sigma^* \restriction \bar{N}$. Then $\bar{\sigma}^* : \bar{N} \longrightarrow N_5$ is an elementary embedding and $\bar{\sigma}^*``\bar{c} \subseteq c^*$, where c^* is $\mathbb{M}^{N_5}$-generic over N_5. I claim that, moreover, $\bar{\sigma}^*$ is $(\bar{\kappa}^+)^{\bar{N}}$-cofinal. To this end, let $a \in N_5$. Then $a \in \sigma^*(b)$, for some $b \in \bar{N}[\bar{c}]$ such that, letting β be the cardinality of b in $\bar{N}[\bar{c}]$, β is at most $\bar{\kappa}$. This is because $\bar{\sigma}^* : \bar{N}[\bar{c}] \longrightarrow N_5[c^*]$ is $(\bar{\kappa}^+)^{\bar{N}}$-cofinal. We may assume that $b \subseteq \bar{N}$. Let $f : \beta \longrightarrow b$ enumerate b, $f \in \bar{N}[\bar{c}]$. By the $(\bar{\kappa}^+)^{\bar{N}}$-c.c., there is a function $g \in \bar{N}$, $g : \beta \longrightarrow \bar{N}$, such that for every $\gamma < \beta$, $f(\gamma) \in g(\gamma)$, and the cardinality of $g(\gamma)$ is at most $\bar{\kappa}$ in $\bar{N}$. Let $\bar{b} = \bigcup_{\gamma<\beta} g(\gamma)$. Then $b \subseteq \bar{b}$ and the $\bar{N}$-cardinality of $\bar{b}$ is at most $\bar{\kappa}$. Of course, now we have that $a \in \sigma^*(b) \subseteq \sigma^*(\bar{b}) = \bar{\sigma}^*(\bar{b})$.

So let $\mathcal{L}_5$ be the ZFC^--theory on $L_{\alpha(N_5)}(N_5)$ in the language with the extra constants $\dot{\sigma}$, $\dot{c}$ and the following additional axioms, with the differences to $\mathcal{L}_4$ highlighted:

1. $\dot{\sigma} : \underline{\bar{N}} \longrightarrow \underline{\mathbf{N_5}}$ is a $\underline{(\bar{\kappa}^+)^{\bar{N}}}$-cofinal, elementary embedding.
2. $\dot{\sigma}(\underline{\bar{s}}, \underline{\bar{\mathbb{M}}}, \underline{\bar{\theta}}, \underline{\bar{\delta}}) = \underline{\mathbf{s}}^{\mathbf{N_5}}, \underline{\mathbb{M}}^{\mathbf{N_5}}, \underline{\theta}^{\mathbf{N_5}}, \underline{\delta}^{\mathbf{N_5}}$.
3. $\dot{\sigma}``\underline{\bar{c}} \subseteq \dot{c}$.
4. $\dot{c}$ is generic over $\underline{\mathbf{N_5}}$ for $\underline{\mathbb{M}}^{\mathbf{N_5}}$.

This theory $\mathcal{L}_5$ is consistent, since $\langle H_\mu, \bar{\sigma}^*, c^*\rangle$ is a model, for sufficiently large μ. Now, since $k^* \restriction N_5 : N_5 \longrightarrow N_0$ cofinally, and $\mathcal{L}_5$ is Σ_1 over $L_{\alpha(N_5)}(N_5)$ in the parameter N_5, and some other parameters in N_5, the theory over $L_{\alpha(N_0)}(N_0)$ which has the same Σ_1-definition in $L_{\alpha(N_0)}(N_0)$, with N_5 replaced by N_0 and the parameters moved by k^*, is consistent, by Lemma 2.15. But that theory is $\mathcal{L}_4$. $\square_{(4)}$

Now we can finish the proof as the proof of subcompleteness of Příkrý forcing in [5, p. 131 f.]. Let $\mathfrak{B} \in \mathrm{V}[F]$ be a model of $\mathcal{L}_4$. Let $\tilde{\sigma}^* = \dot{\sigma}^{\mathfrak{B}}$ and $d = \dot{c}^{\mathfrak{B}}$. Then we have: $\tilde{\sigma}^* : \bar{N} \longrightarrow N_0$ is $(\bar{\kappa}^+)^{\bar{N}}$-cofinal, $\tilde{\sigma}^*``\bar{c} \subseteq d$, and d is $\mathbb{M}^{N_0}$-generic over N_0. Since $\vec{U}_i^{N_0} = U_i \cap N_0$ and since for every $\vec{X} \in \prod_{i<\omega_1^{\mathrm{V}}} U_i$ in V, there is a $\vec{Y} \in \prod_{i<\omega_1^{\mathrm{V}}} U_i$ in N_0 such that for all sufficiently large $i < \omega_1$, $Y_i \subseteq X_i$, and since $(\vec{f}^{N_0})_i^j = f_i^j$ (see the proof of (2)), it follows that d is not only N_0-generic, but even V-generic for $\mathbb{M}$. Let $\tilde{\sigma} = k_0 \circ \tilde{\sigma}^*$. Then $\tilde{\sigma} : \bar{N} \longrightarrow N$, $\tilde{\sigma}$ moves the parameters correctly, $\tilde{\sigma}``\bar{c} \subseteq d$ and

(5) $C_\delta^N(\mathrm{ran}(\tilde{\sigma})) = C_\delta^N(\mathrm{ran}(\sigma))$.

Proof of (5). First, note that $N_0 = C_\delta^{N_0}(\mathrm{ran}(\tilde{\sigma}^*))$, because σ^* is $(\bar{\kappa}^+)^{\bar{N}}$-cofinal, which means that every element of N_0 is of the form $\sigma^*(f)(\xi)$, for some $\xi < \sigma^*((\bar{\kappa}^+)^{\bar{N}}) \leq \delta$. As in [5, p. 131], it follows that

$$C_\delta^N(\mathrm{ran}(\sigma)) = k_0``N_0 = k_0``C_\delta^{N_0}(\mathrm{ran}(\tilde{\sigma}^*)) = C_\delta^N(\mathrm{ran}(\tilde{\sigma})).$$

$\square_{(5)}$

It was crucial in the proof of the previous fact that σ^* is $(\bar{\kappa}^+)^{\bar{N}}$-cofinal. This is why the consistency of $\mathcal{L}_4$ was needed.

So $\tilde{\sigma}$ is exactly like the embedding we are looking for, except that we have to find such an embedding in an $\mathbb{M}$-generic extension of V, not in V$[F]$. To do this, working in V$[d]$, let μ be regular with $N \in H_\mu$. Let $\tilde{M} = \langle H_\mu, N, d, \theta, \mathbb{M}, s, \sigma \rangle$, and let $\mathcal{L}_6$ be the ZFC^--theory on $\tilde{M}$ with extra constant $\dot{\sigma}$ and the following additional axioms:

1. $\dot{\sigma} : \underline{\bar{\mathbf{N}}} \longrightarrow \underline{\mathbf{N}}$ is an elementary embedding.
2. $\dot{\sigma}(\underline{\bar{s}}, \underline{\bar{\mathbb{M}}}, \underline{\bar{\theta}}, \underline{\bar{\delta}}) = \underline{s}, \underline{\mathbb{M}}, \underline{\theta}, \underline{\delta}$.
3. $\dot{\sigma}$"$\underline{\bar{c}} \subseteq \underline{d}$.
4. $C^N_\delta(\mathrm{ran}(\dot{\sigma})) = C^N_\delta(\mathrm{ran}(\sigma))$.

Clearly, $\mathcal{L}_6$ is consistent, as witnessed by $\tilde{\sigma} \in$ V$[F]$. Now, in V$[d]$, let $\pi : \bar{M} \longrightarrow \tilde{M}$ be elementary with $N, d, \theta, \mathbb{M}, s, \sigma \in \mathrm{ran}(\pi)$, $\bar{M}$ countable and transitive. Let $\mathcal{L}_7$ be the "preimage" of $\mathcal{L}_6$ under π. More precisely, $\mathcal{L}_7$ is the theory over $\bar{M}$ that has the same Σ_1 definition over $\bar{M}$ as $\mathcal{L}_6$ had over $\tilde{M}$, with all the parameters moved by π^{-1}. Since $\mathcal{L}_6$ is consistent, so is $\mathcal{L}_7$. Since $\mathcal{L}_7$ is countable, it has a model $\bar{\mathfrak{A}}$ (even in V). Let $\bar{\sigma} = \dot{\sigma}^{\bar{\mathfrak{A}}}$. Note that $\tilde{M}$ sees that $\bar{N}$ is countable, which implies that $\pi^{-1}(\bar{N}) = \bar{N}$ and $\pi^{-1}(\bar{c}) = \bar{c}$. It follows that

1. $\bar{\sigma} : \bar{N} \longrightarrow \pi^{-1}(N)$ is elementary.
2. $\bar{\sigma}(\bar{s}, \bar{\mathbb{M}}, \bar{\theta}, \bar{\delta}) = \pi^{-1}(s), \pi^{-1}(\mathbb{M}), \pi^{-1}(\theta), \pi^{-1}(\delta)$.
3. $\bar{\sigma}$"$\bar{c} \subseteq \pi^{-1}(d)$.
4. $C^{\pi^{-1}(N)}_{\pi^{-1}(\delta)}(\mathrm{ran}(\bar{\sigma})) = C^{\pi^{-1}(N)}_{\pi^{-1}(\delta)}(\mathrm{ran}(\pi^{-1}(\sigma)))$.

Let $\sigma' = \pi \circ \bar{\sigma}$. Then $\sigma' \in$ V$[d]$ is finally as wished. Namely, σ' has the following properties:

1.' $\sigma' : \bar{N} \longrightarrow N$ is elementary.
2.' $\sigma'(\bar{s}, \bar{\mathbb{M}}, \bar{\theta}, \bar{\delta}) = s, \mathbb{M}, \theta, \delta$.
3.' (σ')"$\bar{c} \subseteq d$.
4.' $C^N_\delta(\mathrm{ran}(\sigma')) = C^N_\delta(\mathrm{ran}(\sigma))$.

Here, 1.' and 2.' follow immediately from 1. and 2., respectively. To check 3.', let $i < \Omega$. Then $\sigma'(\langle i, \bar{c}(i)\rangle) = \pi(\bar{\sigma}(\langle i, \bar{c}(i)\rangle)) = \pi(\langle i, \pi^{-1}(d)(i)\rangle) = \langle i, d(i)\rangle$, since $\bar{\sigma}$"$\bar{c} \subseteq \pi^{-1}(d)$ and $\bar{\sigma}\restriction\Omega = \pi\restriction\Omega = \mathrm{id}$.

Finally, let's verify point 4.' in detail. The inclusion from left to right is clear because $\mathrm{ran}(\bar{\sigma}) \subseteq C^{\pi^{-1}(N)}_{\pi^{-1}(\delta)}(\mathrm{ran}(\pi^{-1}(\sigma)))$, which implies that $\mathrm{ran}(\pi \circ \bar{\sigma}) \subseteq C^N_\delta(\mathrm{ran}(\sigma))$. This, in turn, implies the desired inclusion immediately.

For the opposite direction, let $c \in C^N_\delta(\mathrm{ran}(\sigma))$. Writing f^N for the canonical Skolem function of N, there is an $n < \omega$, an $\alpha < \delta$ and an $a \in \bar{N}$ such that

$$c = f^N(n, \langle \alpha, \sigma(a)\rangle)$$

We have that $C^{\pi^{-1}(N)}_{\pi^{-1}(\delta)}(\mathrm{ran}(\bar{\sigma})) = C^{\pi^{-1}(N)}_{\pi^{-1}(\delta)}(\mathrm{ran}(\pi^{-1}(\sigma)))$, and $\pi^{-1}(\sigma)(a)$ belongs to the set on the right hand side of this equation. This means that there is an $m < \omega$, and $\tilde{\alpha} < \pi^{-1}(\delta)$ and a $b \in \bar{N}$ such that $\pi^{-1}(\sigma)(a) = f^{\pi^{-1}(N)}(m, \langle \tilde{\alpha}, \bar{\sigma}(b)\rangle)$. So applying π to this fact gives

$$\sigma(a) = f^N(m, \langle \pi(\tilde{\alpha}), \sigma'(b)\rangle)$$

Substituting this into the equation above gives

$$c = f^N(n, \langle \alpha, f^N(m, \langle \pi(\tilde{\alpha}), \sigma'(b)\rangle)\rangle)$$

Since $\pi(\tilde{\alpha}) < \delta$, this is in $C^N_\delta(\mathrm{ran}(\sigma'))$, as wished.

So we have found an embedding σ' in $\mathrm{V}[d]$ with all the properties demanded by subcompleteness. Thus, there is a condition in the $\mathbb{M}$-generic filter associated to d that forces the existence of such an embedding, concluding the proof. □

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