MORE ON HOD-SUPERCOMPACTNESS

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Abstract. We explore Woodin's Univerality Theorem and consider to what extent large cardinal properties are transferred into HOD (and other inner models). We also separate the concepts of supercompactness, supercompactness in HOD and being HOD-supercompact. For example, we produce a model where a proper class of supercompact cardinals are not HOD-supercompact but are supercompact in HOD. Additionally we introduce a way to measure the degree of HOD-supercompactness of a supercompact cardinal, and we develop methods to control these degrees simultaneously for a proper class of supercompact cardinals. Finally, we also produce a model in which the unique supercompact cardinal is also the only strongly compact cardinal, no cardinal is supercompact up to an inaccessible cardinal, level by level inequivalence holds and the unique supercompact cardinal is not HOD-supercompact.

1. Introduction

The following dichotomy follows from Jensen's Covering Lemma for L which says exactly one of the following holds: (1) L computes the singular cardinals and their successors correctly, or; (2) every uncountable cardinal is inaccessible in L. That is, either L is "close to V" or L is "far from V".

Canonical inner models other than L have been defined and shown to satisfy corresponding dichotomies, all of these inner models are contained in HOD, the class of hereditarily ordinal definable sets. The following result of Woodin, known as the HOD Dichotomy, extends the dichotomy of core models to HOD itself and is, in a sense, the abstract generalization of the L dichotomy.

Theorem 1.1. [20, Theorem 2.34]

Assume that δ is an extendible cardinal. Then exactly one of the following holds.

(1) For every singular cardinal $\gamma > \delta$, γ is singular in HOD *and $(\gamma^+)^{HOD} = \gamma^+$.*

(2) Every regular cardinal greater than δ is measurable in HOD*.*

This result of Woodin expresses the idea that either HOD is"close to V" or else HOD is "far from V". Woodin's HOD Conjecture proposes that HOD is close to V in a particular way; namely that (1) of the HOD dichotomy holds.

More formally, Woodin's HOD Hypothesis [20, Definition 2.42] postulates the existence of a a proper class of regular cardinals which have the property of not being ω-strongly measurable in HOD. Here, an uncountable regular cardinal λ is

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defined to be *ω-strongly measurable in* HOD iff there is $\kappa < \lambda$ such that $(2^\kappa)^{\mathsf{HOD}} < \lambda$ and there is no partition $\langle S_\alpha | \alpha < \kappa \rangle$ of $\mathrm{cof}(\omega) \cap \lambda$ into stationary sets such that $\langle S_\alpha | \alpha < \kappa \rangle \in \mathsf{HOD}$ [20, See definition 2.35]. If a regular cardinal is ω-strongly measurable in HOD, then it is measurable there [22, Lemma 10]. The HOD Conjecture [20, Definition 2.48] is just one sentence, expressing that ZFC + "There is a supercompact cardinal" proves the HOD Hypothesis. For a brief overview of the theorems see Section 3 in [4] and for more details see [20].

Woodin has convincing arguments that the HOD Conjecture is true. Indeed, there seem to be no current methods to produce a model of set theory with a supercompact cardinal in which the HOD Hypothesis fails. The pursuit of this model, however, and the exploration of to what extent large cardinal properties are exhibited in HOD, is a fruitful area of interest. One can consider to what extent HOD and V can be forced to disagree and then conversely, what are the limits of their disagreement. For example, in [6], a model is produced where the α^+ of HOD is strictly less than α^+ for every infinite cardinal α. In [5], Cheng, Friedman and Hamkins produce a variety of models where large cardinals in V are forced not to exhibit their large cardinal properties in HOD, for example, a model of a proper class of supercompact cardinals that are not even weakly compact in HOD. They leave a family of open questions relating to further forcing that HOD have no other large cardinals of a smaller type. However, in the case of a model with a supercompact cardinal, Woodin has proven that if there is a supercompact cardinal, there is a measurable cardinal in HOD[21]. Additionally under the assumption of the HOD Hypothesis, we get the following fact which we will discuss further in Section 2, that any HOD-supercompact cardinal is supercompact in HOD (see [19, Theorem 193]. Note that in this paper, what is referred to as the HOD Conjecture is in later papers, referred to as the HOD Hypothesis).

In this paper will explore both aspects of this question, that is, which large cardinal concepts are necessarily expressed in HOD and then, how far apart can we force HOD from V.

This paper is organized in the following way: In Section 2, we survey Woodin's results on when large cardinal properties are transferred into HOD, including defining HOD-supercompactness. This relates directly to Section 3 where we separate the implications between supercompactness, HOD-supercompactness and supercompactness in HOD. In Section 4 we introduce the way to measure the degree of HOD-supercompactness of a supercompact cardinal and we also develop methods to control these degrees simultaneously for a proper class of supercompact cardinals. In Section 5 we produce a model in which the unique supercompact cardinal is also the only strongly compact cardinal, no cardinal is supercompact up to an inaccessible cardinal, level by level inequivalence holds and the unique supercompact cardinal is not HOD-supercompact.

2. HOD-large cardinals and downward transference

Let us begin with Woodin's results on when large cardinal properties are transferred into inner models of ZFC. In his work on the HOD Conjecture, Woodin isolates the concept of of an inner model N being a *weak extender model for δ is supercompact*: N is an inner model of ZFC, and for every $\gamma > \delta$, there is a δ-complete normal fine measure U on $\mathcal{P}_\delta(\gamma)$ such that $N \cap \mathcal{P}_\delta(\gamma) \in U$ and $U \cap N \in N$, see [20, Definition 2.5]. His Universality Theorem encapsulates striking phenomena of downward transference of large cardinal properties from V to a weak extender model.

Theorem 2.1 (Universality, [20, Theorem 2.15])**.** *Suppose that N is a weak extender model for δ is supercompact and $\gamma > \delta$ is a cardinal in N. Suppose that $j : H(\gamma^+)^N \longrightarrow H(j(\gamma)^+)^N$ with $cp(j) \geq \delta$. Then $j \in N$.*

The following theorem is a consequence of the Universality Theorem:

Theorem 2.2 ([20, Theorem 2.28])**.** *Suppose N is a weak extender model for δ is supercompact, and $\kappa \geq \delta$ is supercompact. Then κ is supercompact in N.*

Thus, if one wants to use the Universality Theorem in order to conclude that certain large cardinal properties in V, which are witnessed by the existence of certain embeddings, are transferred down to N, these embeddings have to move N correctly, motivating the following definition.

Definition 2.3 ([19, Definition 132])**.** Let N be an inner model, and let $j : V \longrightarrow M$ be an elementary embedding, where M is transitive. Then we write $j(N)$ for $\bigcup_{\alpha\in\mathrm{On}} j(V_\alpha \cap N)$.

A cardinal κ is N-supercompact if for every $\alpha > \kappa$, there is an elementary embedding $j : V \longrightarrow M$, where M is transitive, such that $\mathrm{crit}(j) = \kappa$, $\alpha < j(\kappa)$, ${}^{V_\alpha}M \subseteq M$ and

$$j(N) \cap V_\alpha = N \cap V_\alpha$$

Note that if N is an inner model defined by some formula in some parameter, say $N = \{x \mid \varphi(x, p)\}$, and $j : V \longrightarrow M$ is as in the previous definition, then $j(N) = \{x \mid M \models \varphi(x, j(p))\}$. In particular, if N is definable without parameters (for example, if $N = \mathsf{HOD}$), then $j(N) = N^M$ in the usual sense. Thus, in the case that $N = \mathsf{HOD}$, the formula displayed in the previous definition is equivalent to saying

$$\mathsf{HOD}^M \cap V_\alpha = \mathsf{HOD} \cap V_\alpha$$

.

It follows from Theorems 2.1 and 2.2 that if there is an extendible cardinal δ, then δ is HOD-supercompact [19, Lemma 188], and under the assumption of the HOD Hypothesis, HOD is a weak extender model for δ is supercompact [19, Lemma 193]. In particular, under these assumptions, it follows that supercompact cardinals greater than δ are supercompact in HOD. The HOD Hypothesis allows one to conclude this, with just the assumption of a HOD-supercompact cardinal without assuming the existence of an extendible cardinal. This gives us the fact mentioned in Section 1.

Fact 2.4 ([19, Theorem 193])**.** *Under the* HOD *Hypothesis, any* HOD*-supercompact cardinal is supercompact in* HOD*.*

In order to discuss downward transference phenomena at weaker large cardinal concepts, let us make the following definition.

Definition 2.5. Let N be an inner model. Let κ be a cardinal and X a set. Then κ is (N, X)*-measurable* if there is a $j : V \longrightarrow M$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \mathrm{rnk}(X)$ and

$$j(N) \cap X = N \cap X.$$

We shall mostly be interested in the case where X is of the form $\mathcal{P}(\alpha)$ or V_α. Note that κ is λ-strong iff κ is (V, V_λ)-measurable. Similarly, κ is λ-supercompact iff κ is $(V, {}^\lambda V_\alpha)$-measurable, for every α.

Recall the concept of ineffability (introduced in [12], see also [11, Exercise 17.25]).

Definition 2.6. An inaccessible cardinal κ is *ineffable* if for every sequence $\langle a_\alpha \mid \alpha < \kappa\rangle$ such that $a_\alpha \subseteq \alpha$, for every $\alpha < \kappa$, it follows that there is a set $A \subseteq \kappa$ such that

$$\{\alpha < \kappa \mid A \cap \alpha = a_\alpha\}$$

is stationary in κ.

Lemma 2.7. *Suppose that N is an inner model such that κ is $(N, \mathcal{P}(\kappa))$-measurable. Then κ is ineffable in N.*

Proof. First note that since κ is inaccessible in V, it is also inaccessible in N. Let $\vec{a} = \langle a_\alpha \mid \alpha < \kappa\rangle \in N$ be as in the definition of ineffability, and let $j : V \longrightarrow M$ be an elementary embedding witnessing that κ is $(N, \mathcal{P}(\kappa))$-measurable. Let $\vec{a}' = j(\vec{a}) = \langle a'_\alpha \mid \alpha < j(\kappa)\rangle$, and set

$$A = a'_\kappa.$$

We claim that $B = \{\alpha < \kappa \mid A \cap \alpha = a_\alpha\}$ is stationary in κ. To see this, let $C \subseteq \kappa$ be club. Then $j(C) \cap \kappa = C$, so κ is a limit point of $j(C)$, and since $j(C)$ is club in $j(\kappa) > \kappa$, it follows that $\kappa \in j(C)$. Hence, in M, the statement "there is an $\alpha \in j(C)$ such that $j(A) \cap \alpha = a'_\alpha$" is true, for this is witnessed by κ – note that $j(A) \cap \kappa = A = a'_\kappa$. Hence, if we pull this statement back to V, it follows that there is an $\alpha \in C$ such that $A \cap \alpha = a_\alpha$. The point is now that since $\vec{a} \in N$, it follows that $\vec{a}' \in j(N)$, so that in particular, $A = a'_\alpha \in j(N)$. And since j witnesses that κ is $(N, \mathcal{P}(\kappa))$-measurable, we have that $A \in \mathcal{P}(\kappa) \cap j(N) = \mathcal{P}(\kappa) \cap N$. Since the stationarity of B goes down to N, this shows that κ is ineffable in N. □

Recall that for $n < \omega$, an inaccessible cardinal κ is Π^1_n-*indescribable* iff for every $A_0, A_1, \ldots, A_m \subseteq V_\kappa$ (where $m < \omega$) and any Π^1_n sentence φ (in the language of set theory with extra predicate symbols for $A_0, A_1, \ldots, A_m$), if $\langle V_\kappa, \in, A_0, \ldots, A_m\rangle \models \varphi$, then there is a $\bar{\kappa} < \kappa$ such that $\langle V_{\bar{\kappa}}, \in, A_0 \cap V_{\bar{\kappa}}, \ldots, A_m \cap V_{\bar{\kappa}}\rangle \models \varphi$. An inaccessible cardinal that is Π^1_n-indescribable for every $n < \omega$ is just called *indescribable*.

It is well-known that Π^1_1-indescribability is equivalent to weak compactness. Ineffability implies weak compactness (see [11, Exercise 17.26]), and in fact, it is not hard to see that it even implies Π^1_2-indescribability. Since ineffability is a Π^1_3 property, the least ineffable cardinal is not Π^1_3-indescribable, and hence, ineffability does not imply Π^1_3-indescribability. Thus, the following lemma adds something new.

Lemma 2.8. *Suppose that N is an inner model such that κ is $(N, \mathcal{P}(\kappa))$-measurable. Then κ is indescribable in N.*

Proof. Let $j : V \longrightarrow M$ be an elementary embedding, M transitive, $\kappa = \mathrm{crit}(j)$, such that $\mathcal{P}(\kappa) \cap j(N) = \mathcal{P}(\kappa) \cap N$. Since κ is inaccessible in N, there is in N a bijection $f : \kappa \longrightarrow V_\kappa \cap N$. It follows that $\mathcal{P}(V_\kappa) \cap j(N) = \mathcal{P}(V_\kappa) \cap N$, because if $X \subseteq V_\kappa \cap j(N)$, then $\bar{X} = f^{-1}``X = j(f)^{-1}``X \in \mathcal{P}(\kappa) \cap j(N) = \mathcal{P}(\kappa) \cap N$, so $X = f``\bar{X} \in N$; the reverse direction holds generally.

Now let $A_0, \ldots, A_m \subseteq V_\kappa \cap N$, and let φ be a Π^1_n sentence in the language of set theory with extra predicate symbols for $A_0, \ldots, A_m$, such that in N, it is the case that $\langle V_\kappa, \in, A_0, \ldots, A_m\rangle \models \varphi$ (so the domain of this model is $V_\kappa \cap N$ and "$\models$" refers to second order satisfaction). Then, in M, it is true that this same statement holds when relativized to $j(N)$. Note here that $A_i = j(A_i) \cap V_\kappa$, so that in particular, $A_i \in j(N)$, for $i \leq m$. The point here is that $\mathcal{P}(V_\kappa) \cap j(N) = \mathcal{P}(V_\kappa) \cap N$, so that second order satisfaction over $\langle V_\kappa \cap N, \in, A_0, \ldots, A_m\rangle$ is absolute between N and $j(N)$. Thus, in M, it is true that there is a $\bar{\kappa} < j(\kappa)$ such that in $j(N)$, it is the case that $\langle V_{\bar{\kappa}}, \in, j(A_0) \cap V_{\bar{\kappa}}, \ldots, j(A_m) \cap V_{\bar{\kappa}}\rangle \models \varphi$, as witnessed by $\bar{\kappa} = \kappa$. Pulling back via j shows that in V, there is a $\bar{\kappa} < \kappa$ such that in N, it is the case that $\langle V_{\bar{\kappa}}, \in, A_0 \cap V_{\bar{\kappa}}, \ldots, A_m \cap V_{\bar{\kappa}}\rangle \models \varphi$, showing that κ is indescribable in N, as desired. □

Thus, since the ineffability of κ is second order expressible over V_κ, Lemmas 2.7 and 2.8 show, using standard arguments, that if κ is $(N, \mathcal{P}(\kappa))$-measurable, then in N, it is an ineffable stationary limit of ineffable cardinals (meaning that the set of ineffable cardinals below κ is stationary in κ), and of stationary limits of ineffable cardinals, etc.

Question 2.9. Are there large cardinal properties stronger than ineffability that are transferred down to N, given an $(N, \mathcal{P}(\kappa))$-measurable cardinal? How about if κ is $(N, \mathcal{P}(\kappa^+))$-measurable, etc.?

Gödel's constructible universe L, when relativized to a transitive model of set theory, depends only on that model's ordinal height, and as a result, if κ is measurable, it is automatically (L, X)-measurable, for any set X. Hence, in full generality, just assuming (N, X)-measurability of a cardinal κ, one cannot prove that κ retains any large cardinal properties in N, beyond what is consistent with the axiom of constructibility.

Recall the following concept, due to Kunen.

Definition 2.10 (Kunen [13, Definition 1.1])**.** Let $M \models \mathsf{ZFC}$ be transitive (either a set or a proper class), let $\kappa \in M$, and let $U \subseteq \mathcal{P}(\kappa) \cap M$. U is an *M-ultrafilter* if $\langle M, \in, U\rangle \models$"$U$ is a normal ultrafilter on $\mathcal{P}(\kappa)$",[1] and if U is *weakly amenable* to M, meaning that if $\langle x_\xi \mid \xi < \kappa\rangle \in M$ is a sequence of subsets of κ, then the set $\{\xi < \kappa \mid x_\xi \in U\}$ is in M.

Observation 2.11. *If κ is $(N, \mathcal{P}(\kappa))$-measurable, then there is a normal ultrafilter U on κ such that $U \cap N$ is an N-ultrafilter.*

Proof. Let $j : V \longrightarrow M$ be an elementary embedding witnessing that κ is $(N, \mathcal{P}(\kappa))$-measurable, and let $U = \{X \subseteq \kappa \mid \kappa \in j(X)\}$ be the normal ultrafilter on κ derived from j. Clearly then, $\langle N, \in, U \cap N)\rangle \models$"$U \cap N$ is a normal ultrafilter on κ." To see that $U \cap N$ is weakly amenable to N, let $\vec{x} = \langle x_\xi \mid \xi < \kappa\rangle \in N$ be a sequence of subsets of κ. Then $a = \{\xi < \kappa \mid x_\xi \in U\} \in N$, because $\vec{Y} = j(\vec{X}) \in j(N)$, and so, $a = \kappa \cap \{\xi < j(\kappa) \mid \kappa \in Y_\xi\} \in j(N) \cap \mathcal{P}(\kappa) = N \cap \mathcal{P}(\kappa)$. □

Note that the N-ultrafilter $U \cap N$ in the previous observation yields a well-founded ultrapower of N, since it is the restriction of an actual normal ultrafilter (in V). It is not hard to see that if an ordinal κ carries an N-ultrafilter U such that the ultrapower of N by U is well-founded, then κ is ineffable and indescribable

[1] This means that for all $x \in M \cap \mathcal{P}(\kappa)$, either $x \in U$ or $\kappa \setminus x \in U$, if $x \subseteq y \subseteq \kappa$, $x \in U$ and $y \in M$, then $y \in U$, $\emptyset \notin U$, $\kappa \in U$, if $\langle x_i \mid i < \beta\rangle \in M \cap {}^\beta U$, where $\beta < \kappa$, then $\bigcap_{i<\beta} x_i \in U$, and if $\langle x_i \mid i < \kappa\rangle \in M \cap {}^\kappa U$, then the diagonal intersection, $\{\alpha < \kappa \mid \forall\beta < \alpha\ \alpha \in x_\beta\} \in U$.

in N, using the arguments in the proofs of Lemmas 2.7 and 2.8. This makes it an assumption that's interesting in its own right.

Let us close this section with the observation that sufficiently large cardinals imply the existence of supercompact cardinals that satisfy a considerable degree of HOD-supercompactness.

Definition 2.12. A cardinal δ is *Woodinized supercompact* if for every $X \subseteq V_\delta$, there is a $\kappa < \delta$ such that for every $\lambda < \delta$, there is a λ-supercompactness embedding $j : V \longrightarrow M$ such that $X \cap V_\lambda = j(X) \cap V_\lambda$.

It is part of the folklore that δ is Woodinized supercompact iff it is a Vopěnka cardinal, see [18, Corollary 10.6] for a proof. Of course, the assumption of a Woodinized supercompact cardinal is much more than is needed for the conclusion of the following observation.

Observation 2.13. *If δ is Woodinized supercompact, then V_δ is a model of the theory* ZFC + *"there is a proper class of* HOD*-supercompact cardinals."*

Proof. Let $\kappa < \delta$ be as in Definition 2.12 with respect to $X = \mathsf{HOD}^{V_\delta}$. Since the fine normal measures witnessing the existence of the required embeddings lie in V_δ, it follows that κ is HOD-supercompact in V_δ. It is easy to see that the set of such κ is unbounded in δ, completing the proof. □

3. Separations

In this section we will separate the notions of supercompactness, HOD-supercompactness and supercompactness in HOD. A reasonable place to begin is to ask, if κ is supercompact, is it HOD-supercompact? Sargsyan [17] answered this question in the negative in the following result.

Theorem 3.1 ([17])**.** *Suppose κ is a supercompact cardinal. Then there is a forcing extension of V in which κ is supercompact, but not* HOD*-supercompact.*

Later in this section, Theorem 3.9(1) will generalize and provide an alternate way of achieving this result while ensuring that κ remains supercompact in HOD.

Then, does HOD-supercompactness imply supercompactness in HOD? In general, for arbitrary inner models $N \subseteq V$ in place of HOD, this is not the case, as for example, every supercompact cardinal κ is (trivially) L-supercompact while not being supercompact in L.

Can we get the analogous version for HOD, i.e., a model where there is a cardinal κ that is HOD-supercompact but not supercompact in HOD? This turns out to be equivalent to forcing the failure of the HOD Hypothesis. That is because by Fact 2.4, if the HOD Hypothesis holds, then HOD-supercompactness implies supercompactness in HOD. So a model with a HOD-supercompact cardinal that's not supercompact in HOD would not satisfy the HOD Hypothesis.

We saw previously that κ being supercompact does not imply that κ is HOD-supercompact, but does it imply that κ is supercompact in HOD? Let us look at the following theorem:

Theorem 3.2 ([5, Theorem 10])**.** *There is a class forcing $\mathbb{P}$ such that*

1. *All supercompact cardinals of the ground model are preserved and no new ones are created.*
2. *There are no supercompact cardinals in the* HOD *of the extension.*

(3) *The supercompact cardinals of the extension are not weakly compact in the* HOD *of the extension.*

Putting this theorem together with our Lemma 2.7 results in:

Corollary 3.3. *The class forcing $\mathbb{P}$ of the previous theorem has the additional property that no supercompact cardinal in the extension is* HOD*-supercompact there. In fact, if κ is supercompact in the extension, then it is not even* $(\mathsf{HOD}, \mathcal{P}(\kappa))$*-measurable there.*

This is because otherwise, such a κ would be ineffable in the HOD of the extension, but it is not even weakly compact there. In particular, in this way, we can get a model of set theory in which there is a proper class of supercompact cardinals, none of which is HOD-supercompact, thus answering the question implied in [2, Remark 3]. Note that the model of Theorem 3.2 can be easily seen to show that Fact 2.4 cannot be improved to the statement that under the HOD Hypothesis, every supercompact cardinal is supercompact in HOD. This is because the model in the theorem satisfies the HOD Hypothesis, i.e., it has a proper class of regular cardinals that are not ω-strongly measurable in HOD. This can be verified by looking at its proof, and it follows from its statement in case there is a proper class of supercompact cardinals: none of these are even weakly compact in HOD.

This brings us to the next case, that is, a model where κ is supercompact and supercompact in HOD, but not HOD-supercompact. To that end, in the following, we use the concept of N-strongness.

Definition 3.4. Let N be a set or a proper class. Then κ is N*-strong* if for every $\lambda > \kappa$, there is an elementary embedding $j : V \longrightarrow M$, where M is transitive, such that $\mathrm{crit}(j) = \kappa$, $\lambda < j(\kappa)$, $V_\lambda \subseteq M$ and

$$j(N) \cap V_\lambda = N \cap V_\lambda$$

Observation 3.5. *Suppose κ is N-strong. Then the following are equivalent:*

(1) $V_\kappa \subseteq N$,
(2) $V = N$.

Proof. It suffices to show that (1) implies (2). Let $a \in N$, and pick $\lambda > \mathrm{rnk}(a) \cup \kappa$. Let $j : V \longrightarrow M$ be a λ-strongness embedding that verifies that κ is $(N, \{a\})$-strong. Then by elementarity, $j(V_\kappa) \subseteq j(N)$. But $j(\kappa) > \lambda$, and $V_\lambda = V_\lambda^M$, so $a \in V_\lambda = V_\lambda^M \subseteq j(V_\kappa) \subseteq j(N)$. So $a \in V_\lambda \cap j(N) = V_\lambda \cap N$, that is, $a \in N$. □

Observation 3.6. *Suppose κ is supercompact and indestructible under $<\kappa$-directed closed forcing (or strong and indestructible under $\leq\kappa$-strategically closed forcing). Then the following are equivalent:*

(1) *κ is* HOD*-supercompact (or* HOD*-strong),*
(2) $V = \mathsf{HOD}$.

Proof. Clearly, (2) implies (1), so it suffices to prove the converse. It was noted in [2] that if κ is an indestructible supercompact cardinal, then $V_\kappa \subseteq \mathsf{HOD}$, because for any set $a \in V_\kappa$, there is a $<\kappa$-directed closed forcing that codes it into the continuum function beyond κ, and this forcing preserves the supercompactness of κ, so that the fact that a is coded is reflected below κ. Clearly, indestructible strongness suffices for this argument to go through. Now, Observation 3.5 applies and yields the result. □

To keep the statement of the following theorem relatively free from technicalities, let us introduce the following terminology.

Definition 3.7. We say that V *is securely coded* if whenever g is generic over V for some set-sized forcing notion $\mathbb{P}$, then $V \subseteq \mathsf{HOD}^{V[g]}$.

Thus, if V is securely coded, then not only do we have that $V = \mathsf{HOD}$, because V is a forcing extension of itself by trivial forcing, so that we get that $\mathsf{HOD} \subseteq V \subseteq \mathsf{HOD}$, but moreover, every set in V is ordinal definable in any set-forcing extension of V. One way to achieve this property is by forcing the continuum coding axiom CCA, which states that every set is encoded into the continuum function, see [16].

When analyzing $\mathsf{HOD}^{V[g]}$, where $V[g]$ is a forcing extension of V, an indispensable tool for us will be the concept of almost homogeneity: a notion of forcing $\mathbb{P}$ is called *almost homogeneous* if for every $p, q \in \mathbb{P}$, there is an automorphism π of $\mathbb{P}$ such that $\pi(p)$ is compatible with q (see [14, P. 244, ex. (E1)]). The crucial fact for us is:

Fact 3.8. *(Folklore) Let $\mathbb{P}$ be an almost homogeneous notion of forcing, and let g be $\mathbb{P}$-generic. Then*

$$\mathsf{HOD}^{V[g]} \subseteq \mathsf{HOD}^V_{\{\mathbb{P}\}}.$$

Proof. To clarify, $\mathsf{HOD}^V_{\{\mathbb{P}\}}$ is the class of all sets that are hereditarily definable in V from ordinals and the parameter $\mathbb{P}$.

Since $\mathbb{P}$ is almost homogeneous, it follows that whenever $\varphi(\check{a}_0, \ldots, \check{a}_{n-1})$ is a formula in the forcing language for $\mathbb{P}$, either $\Vdash_{\mathbb{P}} \varphi(\check{a}_0, \ldots, \check{a}_{n-1})$ or $\Vdash \neg\varphi(\check{a}_0, \ldots, \check{a}_{n-1})$, (see [14, P. 245, ex. (E1)]). We will use this in the following. Suppose the claimed inclusion does not hold, and let a be an $\in$-minimal counterexample. It follows that a is a subset of $\mathsf{HOD}^V_{\{\mathbb{P}\}}$. Since $a \in \mathsf{HOD}^{V[g]}$, it is of the form $a = \{b \mid \varphi(b, \alpha)\}^{V[g]}$. Then in V, $a = \{b \mid \ \Vdash_{\mathbb{P}} \varphi(\check{b}, \check{\alpha}\}$. This definition of a uses only $\mathbb{P}$ and α as parameters. So $a \in \mathsf{OD}^V_{\{\mathbb{P}\}}$. But since $a \subseteq \mathsf{HOD}^V_{\{\mathbb{P}\}}$, as noted above, this means that $a \in \mathsf{HOD}^V_{\{\mathbb{P}\}}$ after all, a contradiction. □

Note that in particular, if $\mathbb{P}$ is almost homogeneous and ordinal definable, then if g is $\mathbb{P}$ generic, it follows that $\mathsf{HOD}^{V[g]} \subseteq \mathsf{HOD}^V$. This is true of the forcing notions $\mathrm{Add}(\kappa, \delta)$ to add δ Cohen subsets of κ: they are almost homogeneous (see [14, P. 245, ex. (E2)]), and it is obvious that they are ordinal definable. We will frequently make use of this in the following.

We are now ready to show that supercompactness together with supercompactness in HOD does not imply HOD-supercompactness. Note that this was not achieved by Corollary 3.3, since in the model of that corollary, no supercompact cardinal is supercompact in HOD. Let us investigate the effects of adding Cohen subsets to a model with a supercompact cardinal, in various settings.

Theorem 3.9. *Let κ be a supercompact cardinal.*

1. *Suppose κ is indestructible and $g \subseteq \kappa$ is generic for $\mathrm{Add}(\kappa, 1)$. Then in $V[g]$, κ is supercompact but not HOD-supercompact. If, in addition, V is securely coded, then κ is supercompact in $\mathsf{HOD}^{V[g]} = V$.*
2. *Suppose V is securely coded and $g \subseteq \bar{\kappa}$ is generic for $\mathrm{Add}(\bar{\kappa}, 1)$, where $\bar{\kappa} < \kappa$. Then in κ is HOD-supercompact in $V[g]$, but $V[g] \neq \mathsf{HOD}^{V[g]} = V$.*

Before proving this theorem, let us remark that it is well-known that if κ is supercompact, then there is a set forcing extension $V[g]$ in which κ's supercompactness is indestructible under $<\kappa$-directed closed forcing (see [15]), and one can then perform a class forcing that codes itself into the continuum function above κ, thus obtaining a model that is securely coded and in which κ is indestructibly supercompact, see the proof of Theorem 4.10.

Proof. For part (1): Since κ is indestructible, it remains supercompact, and indestructibly so, in $V[g]$. Clearly, $g \notin \mathsf{HOD}^{V[g]}$, since $\mathrm{Add}(\kappa, 1)$ is almost homogenous. Hence, $\mathsf{HOD}^{V[g]} \neq V[g]$, and it follows by Observation 3.6 that κ is not HOD-supercompact in $V[g]$ (because that observation shows that if κ were HOD-supercompact in $V[g]$, then $V[g]$ would have to be equal to $\mathsf{HOD}^{V[g]}$). If V is securely coded, then, by the almost homogeneity of $\mathrm{Add}(\kappa, 1)$, it follows that $V = \mathsf{HOD}^{V[g]}$, as $V \subseteq \mathsf{HOD}^{V[g]} \subseteq V$. So, since κ is supercompact in V, it is supercompact in $\mathsf{HOD}^{V[g]}$.

For part (2): As before, it follows that $V = \mathsf{HOD}^{V[g]} \neq V[g]$. κ remains supercompact in $V[g]$, because the forcing is small. To see that κ is HOD-supercompact in $V[g]$, let λ be given, and let $j : V \longrightarrow M$ be a δ-supercompactness embedding with critical point κ, where $\delta > |V_\lambda|$, so that $V_\lambda = V_\lambda^M$. Then j lifts to a δ-supercompactness embedding $j' : V[g] \longrightarrow M[g]$ with critical point κ. By elementarity, M is securely coded, and so, $\mathsf{HOD}^{M[g]} = M$. We get:

$$\mathsf{HOD}^{V[g]} \cap V_\lambda = V_\lambda = M \cap V_\lambda = \mathsf{HOD}^{M[g]} \cap V_\lambda,$$

showing that κ is HOD-supercompact in $V[g]$. □

We will analyze in more detail the result of adding a Cohen subset to a regular cardinal above a supercompact cardinal in Theorem 4.4.

Let us now summarize what we know about the implications between HOD-supercompactness, supercompactness, and supercompactness in HOD.

Lemma 3.10. *Suppose there is a supercompact cardinal. Then there exist class forcing notions $\mathbb{P}$ (possibly trivial) to force each of the following to hold:*

(1) ZFC+ *there is a cardinal that is supercompact,* HOD*-supercompact, and supercompact in* HOD*. In addition, we can ensure that $V \neq$* HOD *holds in the forcing extension.*
(2) ZFC+ *there is a cardinal that is not supercompact, (hence not* HOD*-supercompact), and supercompact in* HOD*.*
(3) ZFC+ *there is a cardinal that is supercompact, not* HOD*-supercompact and supercompact in* HOD*.*
(4) ZFC+ *there is a cardinal that is supercompact, not* HOD*-supercompact and not supercompact in* HOD*.*

It is trivial to obtain a model of ZFC *in which there is a cardinal that is not supercompact, not* HOD*-supercompact and not supercompact in* HOD*.*

Proof. Part (1) follows from Theorem 3.9, part (2) and the following remark.

For (2), as mentioned in the remark after the statement of Theorem 3.9, if κ is supercompact, then there is a set forcing extension $V[g]$ in which κ is indestructible under $<\kappa$-directed closed forcing (see [15]), and one can then perform a class forcing that codes itself into the continuum function above κ, thus obtaining a model that is securely coded and in which κ is indestructibly supercompact. Let us call this model which is securely coded and where κ is indestructibly supercompact, $\bar{V}$. Force over $\bar{V}$ to collapse κ to ω with $\mathrm{Col}(\omega,\kappa)$. In this forcing extension, $\bar{V}[c]$, κ is clearly no longer supercompact, and hence not HOD-supercompact. Since collapse forcing is almost homogeneous and $\bar{V}$ is securely coded, by Fact 3.8 we get that $\mathsf{HOD}^{\bar{V}[c]} = \bar{V}$, so that κ is supercompact in $\mathsf{HOD}^{\bar{V}[c]}$. Let us remark that there is some flexibility here. For example, instead of forcing with $\mathrm{Col}(\omega,\kappa)$, we could have destroyed the supercompactness of κ by adding a homogeneous κ-Souslin tree - this forcing is also almost homogeneous, so that the above argument goes through, but in the forcing extension, κ will still be inaccessible.

For (3): This is exactly the model obtained in the proof of Theorem 3.9, part (1), where we start in a model that is securely coded and κ is supercompact.

Part (4) follows from Corollary 3.3.

Finally, to obtain a model of ZFC in which there is a cardinal that is not supercompact, not HOD-supercompact and not supercompact in HOD, we can work in $V = L$, or any model that has no inner model with a supercompact. □

Note that as stated previously, using Fact 2.4, a model where κ is HOD-supercompact but not supercompact in HOD requires forcing the failure of the HOD Hypothesis. This covers the remaining possible constellations.

4. Controlling the degree of HOD-supercompactness

In analogy to Definition 2.5, we can define a version suitable for supercompact cardinals.

Definition 4.1. Let N be an inner model. Let κ be a cardinal and X a set. Then κ is (N,X)*-supercompact* if for every λ, there is a λ-supercompactness embedding $j : V \longrightarrow M$ with $j(\kappa) > \mathrm{rnk}(X)$ and

$$j(N) \cap X = N \cap X.$$

Let's say that the *N-supercompactness degree of κ,* $\mathsf{deg}_{N\text{-}\mathsf{SC}}(\kappa)$, is the least α such that κ is *not* $(N,\mathcal{P}(\alpha))$-supercompact, if there is such an α, and let $\mathsf{deg}_{N\text{-}\mathsf{SC}}(\kappa) = \infty$ otherwise (i.e., if κ is N-supercompact).

Clearly, one can define analogues of this for other large cardinal notions as well. In particular, it is clear what it should mean that a strong cardinal is (N,X)-strong.

The choice of $\mathcal{P}(\alpha)$ to measure the degree of N-supercompactness may seem somewhat arbitrary, but it is actually quite natural. It is easy to see that if κ is $(N,\mathcal{P}(\alpha))$-supercompact and $|\beta| = \alpha$, then κ is also $(N,\mathcal{P}(\beta))$-supercompact. It also follows in this case that κ is $(N,\mathcal{P}(\alpha))$-supercompact iff it is (N,H_{α^+})-supercompact, since members of H_{α^+} are naturally coded by subsets of α.

If the definition of the inner model N is sufficiently local, then one direction of the equality demanded in the definition of (N,X)-supercompactness is vacuous. The same proof works to show that the same is true of (N,X)-strongness.

Observation 4.2. *Let κ be a supercompact cardinal, and let X be a set. Then for any λ, there is an elementary embedding $j : V \longrightarrow M$ with ${}^{\lambda}M \subseteq M$, $j(\kappa) > \mathrm{rnk}(X)$ and*

$$\mathsf{HOD} \cap X \subseteq j(\mathsf{HOD}) \cap X.$$

Proof. Let X and λ be given. By reflection, we may choose a $\lambda' > \mathrm{rnk}(X)$ such that $\mathsf{HOD} \cap V_{\lambda'} = \mathsf{HOD}^{V_{\lambda'}}$, and let $j : V \longrightarrow M$ be a $(\kappa, |V_{\lambda'}|)$-supercompactness embedding. Then $V_{\lambda'} = V^M_{\lambda'}$, and so, every $a \in \mathsf{TC}(X)$ that's ordinal definable in V is also ordinal definable in M (in fact, in $V^M_{\lambda'}$). In particular, $\mathsf{HOD} \cap X \subseteq j(\mathsf{HOD}) \cap X$. □

The methods of Section 2 allowed us to construct models of set theory in which there is a proper class of supercompact cardinals whose HOD-supercompactness fails as early as possible, i.e., such that whenever κ is supercompact, then $\mathsf{deg}_{\mathsf{HOD\text{-}SC}}(\kappa) = \kappa$ (and in fact, κ is not even $(\mathsf{HOD}, \mathcal{P}(\kappa))$-measurable). Here, we devise a technique to produce models where the failure of HOD-supercompactness occurs later. The following theorem is a natural extension of Theorem 3.9, giving more information on the failure of HOD-supercompactness. To facilitate its formulation, let's introduce the following terminology.

Definition 4.3. For an ordinal λ, let $\mathcal{P}_{\mathsf{bd}}(\lambda)$ be the collection of bounded subsets of λ, and let us say that $\mathsf{deg}_{\mathsf{HOD\text{-}SC}}(\kappa) = \lambda$ *in the strict sense* if κ is $(\mathsf{HOD}, \mathcal{P}_{\mathsf{bd}}(\lambda))$-supercompact but not $(\mathsf{HOD}, \mathcal{P}(\lambda))$-supercompact.

Another way to say this, in the case where λ is regular, is that κ is $(\mathsf{HOD}, H_\lambda)$-supercompact but not $(\mathsf{HOD}, H_{\lambda^+})$-supercompact. This is because for any two inner models M and N, we have that $H^M_\lambda = H^N_\lambda$ iff $\mathcal{P}_{\mathsf{bd}}(\lambda)^M = \mathcal{P}_{\mathsf{bd}}(\lambda)^N$.

Theorem 4.4. *Suppose V is securely coded and κ is an indestructible supercompact cardinal. Let $\lambda \geq \kappa$ be a regular cardinal and let g be generic for* $\mathrm{Add}(\lambda, 1)$*. Then in $V[g]$, the following is true: κ is supercompact, κ is supercompact in* HOD *and* $\mathsf{deg}_{\mathsf{HOD\text{-}SC}}(\kappa) = \lambda$ *in the strict sense.*

Proof. It follows as in the proof of Theorem 3.9 that $\mathsf{HOD}^{V[g]} = V$, and that κ is supercompact in $V[g]$ as well as in $\mathsf{HOD}^{V[g]}$.

(1) *In $V[g]$, κ is not $(\mathsf{HOD}, \mathcal{P}(\lambda))$-supercompact.*

To see this, working in $V[g]$, let $j : V[g] \longrightarrow M$ be a θ-supercompactness embedding with critical point κ, where $\theta > \lambda$. Then $\mathcal{P}(\lambda)^{V[g]} = \mathcal{P}(\lambda)^M$.

By elementarity, since $V[g]$ is a forcing extension of its HOD (namely V) by g, it follows that M is a forcing extension of its HOD by $j(g)$, so $M = \mathsf{HOD}^M[j(g)]$, where $j(g)$ is generic over HOD^M for $j(\mathrm{Add}(\lambda, 1))$. Note that $j(\mathrm{Add}(\lambda, 1)) = \mathrm{Add}(j(\lambda), 1)^M = \mathrm{Add}(j(\lambda), 1)^{\mathsf{HOD}^M}$. Working in M, $\mathrm{Add}(j(\lambda), 1)$ adds no bounded subsets of $j(\lambda)$, and so, it does not add a subset of λ (since $\lambda < \theta \leq j(\kappa) \leq j(\lambda)$). Thus,

$$\mathcal{P}(\lambda)^M = \mathcal{P}(\lambda)^{\mathsf{HOD}^M} = \mathcal{P}(\lambda) \cap \mathsf{HOD}^M.$$

But then, viewing g as a subset of λ, we have that $g \in \mathcal{P}(\lambda)^{V[g]} = \mathcal{P}(\lambda)^M = \mathcal{P}(\lambda) \cap \mathsf{HOD}^M$, yet $g \notin \mathsf{HOD}^{V[g]} = V$. This shows (1).

(2) *In $V[g]$, κ is $(\mathsf{HOD}, \mathcal{P}_{\mathsf{bd}}(\lambda))$-supercompact.*

To see this, let $\theta \geq \lambda$, and let $j : V[g] \longrightarrow M$ be as above. Then, as before,

$$\mathcal{P}(\lambda)^{V[g]} = \mathcal{P}(\lambda)^M = \mathcal{P}(\lambda) \cap \mathsf{HOD}^M.$$

But now,

$$\mathcal{P}_{\mathsf{bd}}(\lambda)^{V[g]} = \mathcal{P}_{\mathsf{bd}}(\lambda)^{\mathsf{HOD}^{V[g]}},$$

because $\mathrm{Add}(\lambda, 1)$ does not add a bounded subset to λ, and thus, $\mathcal{P}_{\mathsf{bd}}(\lambda)^{V[g]} = \mathcal{P}_{\mathsf{bd}}(\lambda)^V = \mathcal{P}_{\mathsf{bd}}(\lambda)^{\mathsf{HOD}^{V[g]}}$. Putting the two displayed formulas together results in

$$\mathcal{P}_{\mathsf{bd}}(\lambda) \cap \mathsf{HOD}^M = \mathcal{P}_{\mathsf{bd}}(\lambda) \cap \mathsf{HOD}^{V[g]}.$$

This completes the proof of (2), and of the theorem. □

We now turn to the problem of realizing the effect of the previous theorem simultaneously for a possibly proper class of supercompact cardinals. Before we do, let's observe a limitation to the freedom in which one may manipulate the degrees of HOD-supercompactness of several supercompact cardinals.

Lemma 4.5. *Let $\kappa_0 < \kappa_1$ be supercompact cardinals such that κ_0 is $(\mathsf{HOD}, \mathcal{P}_{\mathsf{bd}}(\kappa_1))$-supercompact. Then, if κ_1 is $(\mathsf{HOD}, \mathcal{P}_{\mathsf{bd}}(\lambda))$-supercompact, so is κ_0.*

Proof. Let θ be given. We have to produce a θ-supercompactness embedding with critical point κ_0 such that the HOD of the target model has the same bounded subsets of λ as the HOD of V. Clearly, we may assume that $\theta > \lambda$. We may assume too that $\lambda \geq \kappa_1$, since for $\lambda < \kappa_1$, we already know that κ_0 is $(\mathsf{HOD}, \mathcal{P}(\lambda))$-supercompact, since we assumed that κ_0 is $(\mathsf{HOD}, \mathcal{P}_{\mathsf{bd}}(\kappa_1)$-supercompact.

Let $j_1 : V \longrightarrow M$ be a θ-supercompactness embedding with critical point κ_1 such that $\mathsf{HOD} \cap \mathcal{P}_{\mathsf{bd}}(\lambda) = \mathsf{HOD}^M \cap \mathcal{P}_{\mathsf{bd}}(\lambda)$. In particular, $j_1(\kappa_1) > \theta > \lambda$. Since $j_1(\kappa_0) = \kappa_0$, by elementarity, it is true in M that κ_0 is $(\mathsf{HOD}, \mathcal{P}_{\mathsf{bd}}(j_1(\kappa_1)))$-supercompact. Since $j_1(\kappa_1) > \theta > \lambda$, there is in M a θ-supercompactness embedding $j_0 : M \longrightarrow N$ with critical point κ_0 such that $\mathsf{HOD}^M \cap \mathcal{P}(\lambda) = \mathsf{HOD}^N \cap \mathcal{P}(\lambda)$. But then,

$$j_0 \circ j_1 : V \longrightarrow N$$

is a θ-supercompactness embedding with critical point κ_0, since ${}^{\theta}N = ({}^{\theta}N) \cap M$, as M is θ-closed in V, and further, $({}^{\theta}N) \cap M \subseteq N$, as N is θ-closed in M. Thus, ${}^{\theta}N \subseteq N$. Moreover, we have that $\mathsf{HOD} \cap \mathcal{P}_{\mathsf{bd}}(\lambda) = \mathsf{HOD}^M \cap \mathcal{P}_{\mathsf{bd}}(\lambda) = \mathsf{HOD}^N \cap \mathcal{P}_{\mathsf{bd}}(\lambda)$. Thus, $j_0 \circ j_1$ witnesses that κ is $(\mathsf{HOD}, \mathcal{P}_{\mathsf{bd}}(\lambda))$-supercompact. □

Clearly, the argument of the previous proof did not have anything to do with HOD, and one can easily generalize it to arbitrary classes X. For example, if $\kappa_0 < \kappa_1$ are supercompact cardinals, κ_0 is (X, H_{κ_1})-supercompact and κ_1 is (X, H_θ)-supercompact, for some $\theta > \kappa_1$, then κ_0 is also (X, H_θ)-supercompact.

Corollary 4.6. *Let $\kappa_0 < \kappa_1$ be supercompact cardinals such that $\mathsf{deg}_{\mathsf{HOD\text{-}SC}}(\kappa_0) \geq \kappa_1$. Then $\mathsf{deg}_{\mathsf{HOD\text{-}SC}}(\kappa_0) \geq \mathsf{deg}_{\mathsf{HOD\text{-}SC}}(\kappa_1)$.*

The following definition is designed to simplify the formulation for the upcoming theorems. It expresses a sparsity property of a set S of ordinals, and in the following remark, we will elaborate on cases in which it is satisfied.

Definition 4.7. Let $S \subseteq \mathrm{On}$ be a class of ordinals, and let κ be a supercompact cardinal with $\kappa \leq \sup S$. Then S is *scattered across* κ if for all sufficiently large θ, if $j : V \longrightarrow M$ is a θ-supercompactness embedding with critical point κ, then $\min(j(S) \setminus \kappa) > \min(S \setminus \kappa)$.

Note that whether or not S is scattered across κ depends only on the initial segment $S \cap \delta$, where $\delta = \min(S \setminus \kappa) + 1$.

Remark 4.8. Suppose S is a class of regular cardinals such that for every $\lambda \in S$, there is a supercompact cardinal $\kappa \leq \lambda$ such that $[\kappa, \lambda) \cap S = \emptyset$. Then, if $\kappa \leq \sup S$ is a supercompact cardinal, each of the following conditions implies that S is scattered across κ:

(1) $S \cap \kappa$ is bounded in κ.

(2) For all sufficiently large θ, if $j : V \longrightarrow M$ is a θ-supercompactness embedding with critical point κ, then in M, there are no supercompact cardinals in the interval $[\kappa, \min(S \setminus \kappa)]$.
(3) The interval $(\kappa, \min(S \setminus \kappa)]$ does not contain a supercompact cardinal, and for all sufficiently large θ, if $j : V \longrightarrow M$ is a θ-supercompactness embedding with critical point κ, then κ is not supercompact in M.

Proof of remark. Let $\lambda = \min(S \setminus \kappa)$.

For (1), let $\theta \geq \lambda$, and let $j : V \longrightarrow M$ be a θ-supercompactness embedding with critical point κ. Then $j(S) = j(S \cap \kappa) \cup j(S \setminus \kappa) = (S \cap \kappa) \cup j(S \setminus \kappa)$, and so,

$$\min(j(S) \setminus \kappa) = \min(j(S \setminus \kappa)) = j(\min(S \setminus \kappa)) = j(\lambda) > \theta \geq \lambda.$$

For (2), suppose $S \cap \kappa$ is unbounded in κ; otherwise, we can argue as in (1). Let $j : V \longrightarrow M$ be a θ-supercompactness embedding with critical point κ, where θ is large enough that we know that in M, there is no supercompact cardinal in the interval $[\kappa, \lambda]$. We have to show that $[\kappa, \lambda] \cap j(S) = \emptyset$. Suppose, towards a contradiction, that there is an $\alpha \in [\kappa, \lambda] \cap j(S)$. By assumption on S, there is a $\kappa' \leq \alpha$ such that κ' is supercompact in M and $[\kappa', \alpha) \cap j(S) = \emptyset$. It follows that $\kappa' < \kappa$, because there are no M-supercompact cardinals in $[\kappa, \alpha]$. But $S \cap \kappa = j(S) \cap \kappa$ is unbounded in κ, so there is a $\beta \in j(S) \cap (\kappa', \kappa)$, and in particular, in $[\kappa', \alpha)$, a contradiction.

For (3), we may again assume that $S \cap \kappa$ is unbounded in κ, by (1). Since $(\kappa, \lambda]$ does not contain a supercompact cardinal, we may choose, for every $\gamma \in (\kappa, \lambda]$, an ordinal θ_γ such that γ is not θ_γ-supercompact. Let $\theta \geq \sup_{\gamma \in (\kappa, \lambda]} 2^{\theta_\gamma^{<\kappa}}$ be sufficiently large (in the sense of the statement of (3)), and let $j : V \longrightarrow M$ be a θ-supercompactness embedding with critical point κ. Then in M, κ is not supercompact, and moreover, $[\kappa, \lambda]$ does not contain an M-supercompact cardinal, or else, if $\gamma > \kappa$ were a counterexample, then it would be θ_γ-supercompact in V, a contradiction. Assume that $\lambda' = \min(j(S) \setminus \kappa) \leq \lambda$. Let $\bar{\kappa} \leq \lambda'$ be supercompact in M such that $[\bar{\kappa}, \lambda') \cap j(S) = \emptyset$. Then, as before, $\bar{\kappa} < \kappa$, but since $j(S) \cap \kappa$ is unbounded in κ, this implies that $[\bar{\kappa}, \kappa) \cap j(S) \neq \emptyset$, and so, $[\bar{\kappa}, \lambda') \cap j(S) \neq \emptyset$, a contradiction. □

Thus, for example, if there is no supercompact cardinal of Mitchell order 1, $\mathfrak{S}$ is the class of all supercompact cardinals, and $F : \mathfrak{S} \longrightarrow \mathrm{On}$ is a function such that for every $\kappa \in \mathfrak{S}$, $F(\kappa) \geq \kappa$ is a regular cardinal such that $(\kappa, F(\kappa)]$ contains no supercompact cardinal, then $S = \mathrm{ran}(F)$ is scattered across every $\kappa \in \mathfrak{S}$ (by (3) above). Note that $\mathfrak{S}$ may contain supercompact limits of supercompact cardinals. A special case of the following theorem says that in this situation, if V is securely coded and every supercompact cardinal is indestructible, then we can force to reach a model in which every $\kappa \in \mathfrak{S}$ is still supercompact and $F(\kappa)$ is the HOD-supercompactness degree of κ (in the strict sense).

The formulation of the next theorem is somewhat technical, but a more stratified version follows.

Theorem 4.9. *Suppose that V is securely coded. Let S be a class of regular cardinals such that for every $\lambda \in S$, there is a supercompact cardinal $\kappa \leq \lambda$ such that $[\kappa, \lambda) \cap S = \emptyset$. Let $\mathbb{Q} = \mathbb{Q}_S$ be the Easton support product of the forcing notions* $\mathrm{Add}(\lambda, 1)$*, for $\lambda \in S$. Then, if g is generic for $\mathbb{Q}$, we have that*

(1) any indestructibly supercompact cardinal κ is supercompact in $V[g]$,

(2) $V = \mathsf{HOD}^{V[g]}$.

Moreover, if $\kappa \le \sup S$ *is such that in* $V[g]$, κ *is supercompact and* S *is scattered across* κ*, then, letting* $\lambda = \min(S \setminus \kappa)$*:*

(3) in $V[g]$, $\mathsf{deg}_{\mathsf{HOD\text{-}SC}}(\kappa) \le \lambda$,

(4) if κ *is indestructibly supercompact in* V*, then in* $V[g]$, $\mathsf{deg}_{\mathsf{HOD\text{-}SC}}(\kappa) = \lambda$ *in the strict sense.*

Note: When we say that in $V[g]$, S is scattered across κ, we mean the class S, as defined in V. Since $V = \mathsf{HOD}^{V[g]}$, this is easily expressible in $V[g]$. Thus, what we mean is that in $V[g]$, it is the case that S^{HOD} is scattered across κ.

Proof of Theorem 4.9. For $\lambda \in S$, let us write $\mathbb{Q}_\lambda$ for $\mathrm{Add}(\lambda, 1)$, and for an ordinal α, let us also use the obvious notation $\mathbb{Q}_{<\alpha}$ for $\mathbb{Q}_{S\cap\alpha}$, $\mathbb{Q}_{>\alpha}$ for $\mathbb{Q}_{S\setminus(\alpha+1)}$, $\mathbb{Q}_{\le\alpha}$ for $\mathbb{Q}_{S\cap(\alpha+1)}$ and $\mathbb{Q}_{\ge\alpha}$ for $\mathbb{Q}_{S\setminus\alpha}$.

Let g be generic for $\mathbb{Q}$, and let $g_{<\alpha}$, g_α, $g_{>\alpha}$ be the canonical factors of g, keeping in mind that g may be a proper class.

Fix a cardinal κ, and let $\lambda = \min(S \setminus \kappa)$, if defined. Note that if λ is undefined, then $S \subseteq \kappa$ and $\mathbb{Q}_{\ge\kappa}$ is trivial forcing, and if λ is defined, then $\mathbb{Q}_{\ge\kappa} = \mathbb{Q}_{\ge\lambda}$ is $<\lambda$-directed closed.

(1) *If* κ *is indestructibly supercompact, then* κ *is supercompact in* $V[g]$.

Proof of (1). Since κ is indestructible, it follows that κ is supercompact in $W = V[g_{\ge\kappa}]$. If $S \cap \kappa$ is bounded in κ, then $\mathbb{Q}_{<\kappa}$ has size less than κ, and it follows that κ is supercompact in $W[g_{<\kappa}] = V[g]$. If $S \cap \kappa$ is unbounded in κ, then, working in W, there are arbitrarily large $\theta > \lambda$ with $\mathrm{cf}(\theta) > \kappa$, $\theta^{<\kappa} = \theta$ and $2^\theta = \theta^+$, since the singular cardinal hypothesis holds in W, as κ is supercompact there. Fixing such a θ, we can find in W a θ-supercompactness embedding $j : W \longrightarrow M$ with critical point κ such that in M, $[\kappa, \theta]$ contains no supercompact cardinal. Let $\lambda' = \min(j(S) \setminus \kappa)$. Then in M, there is a supercompact cardinal $\kappa' \le \lambda'$ such that $[\kappa', \lambda') \cap j(S) = \emptyset$. Thus, $\theta < \kappa' \le \lambda'$, since $[\kappa, \theta]$ contains no supercompact cardinal of M. This means that $\min(j(S) \setminus \kappa) > \theta$, and so, $j(\mathbb{Q}_{<\kappa})$ factors as $\mathbb{Q}_{<\kappa} \times j(\mathbb{Q}_{<\kappa})_{>\theta}$. But in this situation, j lifts to a θ-supercompactness embedding in $V[g]$, as in [5, proof of Theorem 10]: in M, the second factor, $j(\mathbb{Q}_{<\kappa})_{>\theta}$, is $\le\theta$-closed, $j(\kappa)$-c.c. and has size $j(\kappa)$. So in M, there are $j(\kappa)^{<j(\kappa)} = j(\kappa)$ many maximal antichains in $j(\mathbb{Q}_{<\kappa})_{>\theta}$. But in W, the cardinality of $j(\kappa)$ is $2^\theta = \theta^+$ and M is closed under θ-sequences. This means that one can construct in $V[g_{\ge\kappa}]$ a filter in $j(\mathbb{Q}_{<\kappa})_{>\theta}$ that's generic over M, by enumerating the maximal antichains in $j(\mathbb{Q}_{<\kappa})_{>\theta}$ that M sees in order type θ^+ and building a decreasing sequence of conditions in $j(\mathbb{Q}_{<\kappa})_{>\theta}$ that meets these antichains one by one. Calling the M-generic filter generated by this sequence g^*, one sees that j lifts to $j' : W[g_{<\kappa}] \longrightarrow M[g_{<\kappa}][g^*]$, and j' witnesses that κ is θ-supercompact in $W[g_{<\kappa}] = V[g]$. □

(2) $\mathsf{HOD}^{V[g]} = V$.

Proof of (2). We will use here the fact that a product of almost homogeneous forcing notions is also almost homogeneous. This is easy to see: let $\langle \mathbb{P}_i \mid i < \theta \rangle$ be a sequence of almost homogeneous forcing notions, and let $\mathbb{P} = \prod_{i<\theta} \mathbb{P}_i$, with any support (all we need is that the set of supports allowed forms an ideal on θ). Now, given conditions $p, q \in \mathbb{P}$, where $p = \langle p_i \mid i < \theta \rangle$ and $q = \langle q_i \mid i < \theta \rangle$, we have that for every $i < \theta$, there is an automorphism π_i of $\mathbb{P}_i$ such that $\pi_i(p_i)$ is compatible with q_i. Clearly then, if we define $\pi : \mathbb{P} \longrightarrow \mathbb{P}$ by $\pi(\langle r_i \mid i < \theta\rangle) = \langle \pi_i(r_i) \mid i < \theta\rangle$, then π is an automorphism of $\mathbb{P}$, and $\pi(p)$ is compatible with $\pi(q)$: for any $i < \theta$, there is an $r_i \in \mathbb{P}_i$ with $r_i \le q_i, \pi_i(p_i)$. If both p_i and q_i are the trivial condition of $\mathbb{P}_i$, then we may choose r_i to be trivial as well. Then, the support of $r = \langle r_i \mid i < \theta\rangle$ is contained in the union of the support of p_i and the support of q_i, and is thus allowed in the formation of $\mathbb{P}$. So $r \in \mathbb{P}$ witnesses that $\pi(p)$ and q are compatible.

Turning to the proof, the claimed equality follows now immediately if S is a set, and hence $\mathbb{Q}$ is a set-sized forcing notion that's almost homogeneous, because V is securely coded. Namely, using Fact 3.8, it follows that $V \subseteq \mathsf{HOD}^{V[g]} \subseteq V$.

So suppose S is a proper class.

We will use the following fact: if κ' is a strong cardinal, then $\mathsf{HOD} \cap V_{\kappa'} = \mathsf{HOD}^{V_{\kappa'}}$. The direction from right to left is trivial. For the other direction, suppose $a \in V_{\kappa'}$ and $a \in \mathsf{HOD}$. Then there is some λ' such that $a \in \mathsf{HOD}^{V_{\lambda'}}$. Let $j : V \to M$ be a λ'-strong embedding with critical point κ'. Then in M, it is the case that there is a $\bar{\lambda} < j(\kappa')$ such that a is in $\mathsf{HOD}^{V_{\bar{\lambda}}}$ (as evidenced by λ'). Pulling back, this implies that there is a $\bar{\lambda} < \kappa'$ such that $a \in \mathsf{HOD}^{V_{\bar{\lambda}}}$. So $a \in \mathsf{HOD}^{V_{\kappa'}}$.

We will use this fact in proving that $V \subseteq \mathsf{HOD}^{V[g]}$. So let $a \in V$. Since a can be coded by a set of ordinals, we may assume it is a set of ordinals. Let $\mu < \nu$ be the next two members of S greater than $\sup(a)$. Let $\kappa' \le \nu$ be a supercompact cardinal such that $[\kappa', \nu) \cap S = \emptyset$. Then $\mu < \kappa' \le \nu$. It follows that $\mathbb{Q}_{<\kappa'}$ has size less than κ', and that κ' is still supercompact, and hence strong, in $V[g_{<\kappa'}]$. Since V is securely coded, we know that $a \in \mathsf{HOD}^{V[g_{<\kappa'}]}$. By the above reflection fact, applied in $V[g_{<\kappa'}]$, it follows that $a \in \mathsf{HOD}^{V_{\kappa'}^{V[g_{<\kappa'}]}}$. But $V_{\kappa'}^{V[g_{<\kappa'}]} = V_{\kappa'}^{V[g]}$, and so, $a \in \mathsf{HOD}^{V_{\kappa'}^{V[g]}}$, and in particular, $a \in \mathsf{HOD}^{V[g]}$.

For the converse, let $a \in \mathsf{HOD}^{V[g]}$, where a is a set of ordinals. By reflection, a is ordinal definable in some $V_\gamma^{V[g]}$, and hence in $V_\gamma^{V[g_{<\alpha}]}$, for some sufficiently large α. In particular, $a \in \mathsf{HOD}^{V[g_{<\alpha}]}$. But $g_{<\alpha}$ is generic over V for $\mathbb{Q}_{<\alpha}$, a set sized almost homogeneous forcing notion. As before, it follows by Fact 3.8 that $a \in V$. □

(3) *Suppose that κ is supercompact in $V[g]$ and S is scattered across κ in $V[g]$. Then κ is not $(\mathsf{HOD}, \mathcal{P}(\lambda))$-supercompact in $V[g]$.*

Proof of (3). Let $W = V[g]$. Since S is scattered across κ in W, there is a $\theta^* \ge \lambda$ such that whenever $\theta \ge \theta^*$ and $j : W \longrightarrow M$ is a θ-supercompactness embedding with critical point κ, then $\lambda' := \min(j(S) \setminus \kappa) > \lambda$. Since M is θ-closed in W, it follows that $\mathcal{P}(\lambda)^W = \mathcal{P}(\lambda)^M$. In particular, $g_\lambda \in M$.

Assume that in this situation, $\mathcal{P}(\lambda) \cap \mathsf{HOD}^W = \mathcal{P}(\lambda) \cap \mathsf{HOD}^M$. Since $\mathsf{HOD}^W = V$, this means that

$$\mathcal{P}(\lambda) \cap V = \mathcal{P}(\lambda) \cap \mathsf{HOD}^M.$$

To derive a contradiction, first, note that

$$\mathcal{P}(\lambda) \cap \mathsf{HOD}^M[j(g)_{\ge\kappa}] = \mathcal{P}(\lambda) \cap \mathsf{HOD}^M = \mathcal{P}(\lambda) \cap V.$$

The first equality holds because $\lambda' = \min(j(S) \setminus \kappa) > \lambda$, so that $j(g)_{\ge\kappa} = j(g)_{\ge\lambda'}$ does not add subsets of λ over HOD^M, and the second equality holds by assumption. But note further that $g_{<\kappa} = j(g)_{<\kappa}$, so that we get that

$$\mathcal{P}(\lambda) \cap \mathsf{HOD}^M[j(g)_{\ge\kappa}][j(g)_{<\kappa}] = \mathcal{P}(\lambda) \cap V[g_{<\kappa}],$$

because any subset of λ in $\mathsf{HOD}^M[j(g)_{\geq\kappa}][j(g)_{<\kappa}]$ is of the form $\tau^{g_{<\kappa}}$, for some nice $\mathbb{Q}_{<\kappa}$-name τ for a subset of λ in $\mathsf{HOD}^M[j(g)_{\geq\kappa}]$. Such a τ is of the form $\bigcup_{\alpha<\lambda}\{\check{\alpha}\}\times A_\alpha$, where each A_α is some antichain in $\mathbb{Q}_{<\kappa}$. Clearly then, $\tau \in H_{\lambda^+}^{\mathsf{HOD}^M[j(g)_{\geq\kappa}]} = H_{\lambda^+}^V$, so that $\tau^{g_{<\kappa}} \in V[g_{<\kappa}]$. The converse uses the same argument.

But by elementarity, we know that $M = \mathsf{HOD}^M[j(g)]$, so that by the equality displayed above,

$$\mathcal{P}(\lambda) \cap M = \mathcal{P}(\lambda) \cap V[g_{<\kappa}].$$

This is a contradiction, because $g_\lambda \in \mathcal{P}(\lambda) \cap M$, but $g_\lambda \notin \mathcal{P}(\lambda) \cap V[g_{<\kappa}]$. □

Note that (3) shows that in $V[g]$, $\mathsf{deg}_{\mathsf{HOD\text{-}SC}}(\kappa) \leq \lambda$, as claimed. It remains to prove claim (4). To this end, assume that κ is indestructibly supercompact in V and S is scattered across κ in $V[g]$. By (1), κ is supercompact in $V[g]$, and by (3), κ is not $(\mathsf{HOD}, \mathcal{P}(\lambda))$-supercompact in $V[g]$. The next claim will state that κ is $(\mathsf{HOD}, \mathcal{P}_{\mathsf{bd}}(\lambda))$-supercompact in $V[g]$, and this will complete the proof, because it follows then that in $V[g]$, $\mathsf{deg}_{\mathsf{HOD\text{-}SC}}(\kappa) \leq \lambda$ in the strict sense, which is what claim (4) says.

(4) *κ is $(\mathsf{HOD}, \mathcal{P}_{\mathsf{bd}}(\lambda))$-supercompact in $V[g]$.*

Proof of (4). Since $\mathbb{Q}_{\geq\kappa} = \mathbb{Q}_{S\setminus\kappa}$ can be defined from $S \setminus \kappa$ as $\mathbb{Q}$ is defined from S, (2) applies to this forcing notion as well, so that

$$\mathsf{HOD}^{V[g_{\geq\kappa}]} = V = \mathsf{HOD}^{V[g]}.$$

Moreover, since κ is indestructibly supercompact in V it follows that κ is supercompact in $V[g_{\geq\kappa}]$.

As in the proof of (1), working in $V[g_{\geq\kappa}]$, there are arbitrarily large $\theta > \lambda$ with $\mathrm{cf}(\theta) > \kappa$, $\theta^{<\kappa} = \theta$ and $2^\theta = \theta^+$. Fixing such a θ, let $j : V[g_{\geq\kappa}] \longrightarrow M$ be a θ-supercompactness embedding in $V[g_{\geq\kappa}]$ with critical point κ such that in M, $[\kappa, \theta]$ contains no supercompact cardinal. Let $\lambda' = \min(j(S) \setminus \kappa)$. It follows as in the proof of (1) that $\lambda < \theta < \lambda'$, $j(\mathbb{Q}_{<\kappa})$ factors as $\mathbb{Q}_{<\kappa} \times j(\mathbb{Q}_{<\kappa})_{>\theta}$ and j lifts to a θ-supercompactness embedding in $V[g]$. Let $j' : V[g_{\geq\kappa}][g_{<\kappa}] \longrightarrow M[j'(g_{<\kappa})]$ be the lifted embedding. As in (1), it witnesses that κ is θ-supercompact in $V[g]$, but we shall now show that it actually witnesses that κ is $(\mathsf{HOD}, \mathcal{P}_{\mathsf{bd}}(\lambda))$-supercompact in $V[g]$.

By elementarity, we have that $M = \mathsf{HOD}^M[j(g_{\geq\kappa})]$. We have

$$\mathcal{P}(\lambda) \cap \mathsf{HOD}^M = \mathcal{P}(\lambda) \cap M,$$

because $j(\mathbb{Q}_{\geq\kappa})$ is $<\lambda'$-closed in M, and $\lambda' > \lambda$. Moreover,

$$\mathcal{P}(\lambda) \cap M = \mathcal{P}(\lambda) \cap V[g_{\geq\kappa}],$$

because M is θ-closed in $V[g_{\geq\kappa}]$ and $\theta \geq \lambda$. And

$$\mathcal{P}_{\mathsf{bd}}(\lambda) \cap V[g_{\geq\kappa}] = \mathcal{P}_{\mathsf{bd}}(\lambda) \cap V,$$

because $\mathbb{Q}_{\geq\kappa}$ is $<\lambda$-closed in V. Since $V = \mathsf{HOD}^{V[g_{\geq\kappa}]}$, all of this taken together shows that

$$\mathcal{P}_{\mathsf{bd}}(\lambda) \cap \mathsf{HOD}^M = \mathcal{P}_{\mathsf{bd}}(\lambda) \cap \mathsf{HOD}^{V[g_{\geq\kappa}]}.$$

We have that $\mathsf{HOD}^{V[g_{\geq\kappa}]} = \mathsf{HOD}^{V[g_{\geq\kappa}][g_{<\kappa}]}$. By the elementarity of j', this implies that $\mathsf{HOD}^M = \mathsf{HOD}^{M[j'(g_{<\kappa})]}$. Combining the previous displayed equality with these two equalities results in

$$\mathcal{P}_{\mathsf{bd}}(\lambda) \cap \mathsf{HOD}^{M[j'(g_{<\kappa})]} = \mathcal{P}_{\mathsf{bd}}(\lambda) \cap \mathsf{HOD}^{V[g]}.$$

□

This completes the proof of the theorem. □

Theorem 4.10. *Any model of set theory $\bar{V}$ has a proper class forcing extension $V = \bar{V}[G]$, such that the following holds. In V, suppose S is a class of regular cardinals such that for every $\lambda \in S$, there is a supercompact cardinal $\kappa \leq \lambda$ such that $[\kappa, \lambda) \cap S = \emptyset$. Let $\mathbb{Q} = \mathbb{Q}_S$ be the Easton support product of the forcing notions* $\mathrm{Add}(\lambda, 1)$*, for $\lambda \in S$. Then, if g is $\mathbb{Q}$-generic over V, we have that*

(a) $\bar{V}$, $V = \mathsf{HOD}^{V[g]}$ and $V[g]$ have the same supercompact cardinals,
(b) if in $V[g]$, $\kappa \leq \sup S$ is supercompact and S is scattered across κ, [2] *then in $V[g]$, $\mathsf{deg}_{\mathsf{HOD\text{-}SC}}(\kappa) = \lambda$ in the strict sense, where $\lambda = \min(S \setminus \kappa)$.*

Proof. If in $\bar{V}$, there are unboundedly many supercompact cardinals, then we can let $\mathbb{P}$ be Cohen forcing, followed by the global Laver preparation that renders every supercompact cardinal of $\bar{V}$ indestructible. Otherwise, we let $\mathbb{P}$ be Cohen forcing, followed by the global Laver preparation, followed by the self-encoding forcing above the supremum of the supercompact cardinals, as described in [16, Thm. 11]. If G is generic for $\mathbb{P}$, then $V = \bar{V}[G]$ is securely coded, since V satisfies the continuum coding axiom (either because there is a proper class of indestructible supercompact cardinals, or because it was explicitly forced to hold, by the self-encoding forcing).

By the main result of [10], the class of supercompact cardinals in the sense of $\bar{V}$ is the same as in the sense of V.

Now, working in V, let S be a class of regular cardinals as described, let $\mathbb{Q} = \mathbb{Q}_S$, and let g be $\mathbb{Q}$-generic over V. Applying Theorem 4.9 in V shows that every supercompact cardinal of V is supercompact in $V[g]$, since every supercompact cardinal of V is indestructible in V. Moreover, every supercompact cardinal of $V[g]$ is supercompact in $\bar{V}$, since the combined forcing $\mathbb{P} * \dot{\mathbb{Q}}$ has a closure point (in the sense of [10, Def. 12]) at ω, and hence, by [10, Lemma 13 and Corollary 26] it is supercompact in V. Thus, $\bar{V}, V$ and $V[g]$ all have the same supercompact cardinals. Moreover, by Theorem 4.9, we have that $\mathsf{HOD}^{V[g]} = V$, and so, every supercompact cardinal of $V[g]$ is supercompact in $\mathsf{HOD}^{V[g]}$. This shows part (a) of the theorem.

For part (b), suppose in $V[g]$, $\kappa \leq \sup S$ is supercompact and S is scattered across κ. Letting $\lambda = F(\kappa)$, it then follows from Theorem 4.9 that $\mathsf{deg}_{\mathsf{HOD\text{-}SC}}(\kappa) = \lambda$ in the strict sense, because κ is indestructible in V. □

An intriguing question that's naturally raised by Lemma 4.5 and Theorem 4.10 is as follows.

Question 4.11. Is the function that assigns to every supercompact cardinal its HOD-supercompactness degree necessarily weakly monotonic?

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[2] To be more precise, what we mean is the following: if $\delta = \min(S \setminus \kappa)$, and $s = S \cap (\delta + 1)$ (in V), then in $V[g]$, s is scattered across κ – see the remark after Definition 4.7. Equivalently, since it will be the case that $\mathsf{HOD}^{V[g]} = V$, one could formulate this by asking that in $V[g]$, S^{HOD} (that is, S^V) is scattered across κ.

5. On the failure of HOD-supercompactness when level by level inequivalence holds

Recall that a model V of ZFC containing at least one supercompact cardinal satisfies *level by level inequivalence between strong compactness and supercompactness* (which we shall henceforth abbreviate as *level by level inequivalence*) iff for every non-supercompact measurable cardinal δ, there is a cardinal $\lambda > \delta$ such that $V \models$ "δ is λ-strongly compact yet δ is not λ-supercompact". This notion is studied in [1] (as well as elsewhere) and is dual to the notion of *level by level equivalence between strong compactness and supercompactness* introduced by the first author and Shelah in [3].

In [2, Theorem 8], the first and second authors showed the consistency, relative to the appropriate hypotheses, of the theory ZFC + "Level by level equivalence between strong compactness and supercompactness holds" + "The least supercompact cardinal κ is not HOD-supercompact". We now establish an analogue of this theorem for level by level inequivalence by proving the following.

Theorem 5.1. *Suppose $V \models ZFC + GCH +$ "$\kappa < \lambda$ are least such that κ is λ-supercompact and λ is inaccessible". There is then a partial ordering $\mathbb{P} \in V$, a submodel $V' \subseteq V^{\mathbb{P}}$ of ZFC, and $\kappa_0 < \kappa$ such that $V' \models$ "κ_0 is supercompact and is the only strongly compact cardinal" + "No cardinal is supercompact up to an inaccessible cardinal" + "Level by level inequivalence holds". In V', κ_0 is not HOD-supercompact.*

Note that we say κ *is supercompact up to the inaccessible cardinal* γ iff for every cardinal $\eta < \gamma$, κ is η-supercompact. Also, we take as notation that for any ordinal δ, δ' is the least inaccessible cardinal greater than δ. Suppose now $\kappa > \omega$ is a regular cardinal. A partial ordering $\mathbb{P}(\omega, \kappa)$ that will be used in the proof of Theorem 5.1 is the partial ordering for adding a non-reflecting stationary set of ordinals of cofinality ω to κ. Specifically, $\mathbb{P}(\omega, \kappa)$ is defined as $\{p \mid$ For some $\alpha < \kappa$, $p : \alpha \longrightarrow \{0, 1\}$ is a characteristic function of S_p, a subset of α not stationary at its supremum nor having any initial segment which is stationary at its supremum, such that $\beta \in S_p$ implies $\beta > \omega$ and $\text{cf}(\beta) = \omega\}$, ordered by $q \leq p$ iff $q \supseteq p$ and $S_p = S_q \cap \sup(S_p)$, i.e., S_q is an end extension of S_p. Additional details about this partial ordering may be found in [3].

Before proving Theorem 5.1, we first establish the following lemma key to its proof.

Lemma 5.2. *For κ a regular cardinal, $\mathbb{P}(\omega, \kappa)$ is* cone homogeneous*, i.e., given any two conditions $p, q \in \mathbb{P}(\omega, \kappa)$, p and q can be extended to conditions p' and q' such that there is an isomorphism from $\{r \in \mathbb{P} \mid r \leq p'\}$ to $\{s \in \mathbb{P} \mid s \leq q'\}$.*

Proof. Suppose $p, q \in \mathbb{P}(\omega, \kappa)$. Let $\beta = \sup(S_p \cup S_q)$, where $\beta = 0$ if $S_p = S_q = \emptyset$ (i.e., if $p = q = 1$). Let β^* be the least ordinal of cofinality ω greater than $\max(\beta, \omega)$. Define p' and q' as the characteristic functions of $S_p \cup \{\beta^*\}$ and $S_q \cup \{\beta^*\}$ respectively. Clearly, $p' \leq p$ and $q' \leq q$. Further, for any $r \in \mathbb{P}(\omega, \kappa)$ and ordinal γ, write $S_r = S_r^{\geq \gamma} \cup S_r^{< \gamma}$, where $S_r^{\geq \gamma} = \{\alpha \in S_r \mid \alpha \geq \gamma\}$ and $S_r^{< \gamma} = \{\alpha \in S_r \mid \alpha < \gamma\}$. If we now let π defined on $\{r \in \mathbb{P} \mid r \leq p'\}$ be given by $\pi(r) =$ The characteristic function of $S_r^{\geq \beta^*} \cup S_{q'}^{< \beta^*}$, then π is well-defined and is the desired isomorphism onto $\{s \in \mathbb{P} \mid s \leq q'\}$. □

We turn now to the proof of Theorem 5.1.

Proof. Suppose V is as in the hypotheses of Theorem 5.1. Define $\mathbb{P}^0 = \langle\langle \mathbb{P}^0_\alpha, \dot{\mathbb{Q}}^0_\alpha\rangle \mid \alpha \le \kappa\rangle \in V$ as the Easton support iteration of length $\kappa+1$ such that $\mathbb{P}^0_0 = \{\emptyset\}$ and $\dot{\mathbb{Q}}^0_0 = \dot{\mathrm{A}}\mathrm{dd}(\omega, 1)$ (so $\mathbb{P}^0$ begins by adding a Cohen subset of ω). For $1 \le \delta \le \kappa$, $\dot{\mathbb{Q}}^0_\delta$ is a term for trivial forcing, except if $V \models$ "δ is a measurable cardinal". In this case, if $\delta < \kappa$, $\dot{\mathbb{Q}}^0_\delta = \dot{\mathrm{A}}\mathrm{dd}(\delta, 1) * \dot{\mathbb{R}}_\delta$, where $\dot{\mathbb{R}}_\delta$ is a term for the partial ordering coding the Cohen subset of δ added by $\mathrm{Add}(\delta, 1)$ into the continuum function above δ' in the manner of [17]. If $\delta = \kappa$, $\dot{\mathbb{Q}}^0_\kappa = \dot{\mathrm{A}}\mathrm{dd}(\kappa, 1)$ (so in particular, the Cohen subset of κ added at stage κ is not coded).

Define $V_1 = V^{\mathbb{P}^0}$. Let $j : V \longrightarrow M$ be an elementary embedding witnessing the λ-supercompactness of κ which is generated by a supercompact ultrafilter over $P_\kappa(\lambda)$. Since $V \models$ "$\lambda = \kappa'$", $M \models$ "$\lambda = \kappa'$". Hence, by the definition of $\mathbb{P}^0$, $j(\mathbb{P}^0) = \mathbb{P}^0 * \dot{\mathbb{R}}'$, where the first nontrivial stage in $\mathbb{R}'$ is well above λ. Thus, the proofs of [1, Lemmas 2.1 and 2.6] show that $V^{\mathbb{P}^0} \models$ "κ is λ-supercompact", i.e., $V_1 \models$ "κ is λ-supercompact".

Suppose $\delta \le \kappa$ is such that $V \models$ "δ is supercompact up to δ'". Write $\mathbb{P}^0 = \mathbb{P}^0_\delta * \dot{\mathrm{A}}\mathrm{dd}(\delta, 1) * \dot{\mathbb{R}}^\delta$. Sargsyan's arguments from [17] show that $V^{\mathbb{P}^0_\delta * \dot{\mathrm{A}}\mathrm{dd}(\delta,1)} \models$ "$V_{\delta'} \models$ 'δ is supercompact but not HOD-supercompact' ". Since $\Vdash_{\mathbb{P}^0_\delta * \dot{\mathrm{A}}\mathrm{dd}(\delta,1)}$ "$\dot{\mathbb{R}}^\delta$ is $<\delta'$-directed closed", $V^{\mathbb{P}^0_\delta * \dot{\mathrm{A}}\mathrm{dd}(\delta,1) * \dot{\mathbb{R}}^\delta} = V^{\mathbb{P}^0} \models$ "$V_{\delta'} \models$ 'δ is supercompact but not HOD-supercompact' " as well. It also clearly follows that $V^{\mathbb{P}^0} \models$ "δ is supercompact up to δ'". In addition, note that we can write $\mathbb{P}^0 = \mathrm{Add}(\omega, 1) * \dot{\mathbb{Q}}'$, where $\mathrm{Add}(\omega, 1)$ is nontrivial, $|\mathrm{Add}(\omega, 1)| = \omega$, and $\Vdash_{\mathrm{Add}(\omega,1)}$ "$\dot{\mathbb{Q}}'$ is $\le\aleph_1$-directed closed". Consequently, the Gap Forcing Theorem [8, 9] implies that for any γ, if $V_1 \models$ "γ is supercompact up to γ'", then $V \models$ "γ is supercompact up to γ'" as well. This means that the set A defined in V_1 as $A = \{\delta \le \kappa \mid \delta$ is supercompact up to δ' and $V_{\delta'} \models$ "δ is supercompact but not HOD-supercompact"$\}$ is composed precisely of those $\delta \le \kappa$ such that $V \models$ "δ is supercompact up to δ'".

Working now in V_1, define $\mathbb{P}^1 = \langle\langle \mathbb{P}^1_\alpha, \dot{\mathbb{Q}}^1_\alpha\rangle \mid \alpha < \kappa\rangle$ as the Easton support iteration of length κ such that $\mathbb{P}^1_0 = \{\emptyset\}$ and $\dot{\mathbb{Q}}^1_0 = \dot{\mathrm{A}}\mathrm{dd}(\omega, 1)$. For $0 \le \delta < \kappa$, $\dot{\mathbb{Q}}^1_\delta$ is a term for trivial forcing, except if $V_1 \models$ "δ is a measurable cardinal which is not supercompact up to δ'". In this case, $\dot{\mathbb{Q}}^1_\delta = \dot{\mathbb{P}}(\omega, \delta)$. Because $V_1 \models$ "κ is λ-supercompact", the arguments used in the proof of [1, Theorem 1.2] (specifically, the proofs of [1, Lemmas 2.6 – 2.8] and the intervening and following remarks) show that for some $\delta < \kappa$ which is supercompact up to δ' in $V_2 = V_1^{\mathbb{P}^1}$, $(V_{\delta'})^{V_2} \models$ "δ is supercompact and is the only strongly compact cardinal" + "No cardinal is supercompact up to an inaccessible cardinal" + "Level by level inequivalence holds". Thus, the proof of Theorem 5.1 will be complete once we have shown that $(V_{\delta'})^{V_2} \models$ "δ is not HOD-supercompact".

To do this, we first note that it is possible to write $\mathbb{P}^1 = \mathrm{Add}(\omega, 1) * \dot{\mathbb{Q}}''$, where $\Vdash_{\mathrm{Add}(\omega,1)}$ "$\dot{\mathbb{Q}}''$ is $\le\aleph_1$-strategically closed". Hence, by the Gap Forcing Theorem, $V_1 \models$ "δ is supercompact up to δ'". As we have previously observed, the Gap Forcing Theorem yields that $V \models$ "δ is supercompact up to δ'" and that $\delta \in A$.

Now, we assume towards a contradiction that $(V_{\delta'})^{V_2} \models$ "δ is HOD-supercompact". Suppose $\eta > \delta$, $\eta < \delta'$ is a strong limit cardinal in $(V_{\delta'})^{V_2}$. Following [17], let $k^* : (V_{\delta'})^{V_2} \longrightarrow N^*$ be an elementary embedding witnessing the η-supercompactness of δ such that $\mathsf{HOD}^{N^*} \cap V_\eta = \mathsf{HOD}^{(V_{\delta'})^{V_2}} \cap V_\eta = \mathsf{HOD}^{(V_\eta{}^{(V_{\delta'})^{V_2}})}$. Again by the Gap Forcing Theorem, since $(V_{\delta'})^{V_2} = ((V_{\delta'})^{V_1})^{\mathbb{P}^1_\delta}$, k^* must lift

$k : (V_{\delta'})^{V_1} \longrightarrow N$ which witnesses the η-supercompactness of δ in $(V_{\delta'})^{V_1}$. Here, $N \subseteq N^*$ and $N^* = N^{k(\mathbb{P}^1_\delta)}$. In particular, because $\delta \in A$, by the definition of $\mathbb{P}^0$, for x the Cohen subset of δ added by $\mathrm{Add}(\delta, 1)$, $x \in \mathsf{HOD}^N$. However, because $k(\mathbb{P}^1_\delta)$ does not erase the coding of x in N, $x \in \mathsf{HOD}^{N^*}$. As $x \in (V_\eta)^{N^*}$, $x \in \mathsf{HOD}^{N^*} \cap V_\eta = \mathsf{HOD}^{(V_{\delta'})^{V_2}} \cap V_\eta$, i.e., $x \in \mathsf{HOD}^{(V_{\delta'})^{V_2}}$. In addition, Lemma 5.2 and the definition of $\mathbb{P}^1_\delta$ yield that each component partial ordering of $\mathbb{P}^1_\delta$ is cone homogeneous. Consequently, by [7, Fact 1(2), Fact 1(3), and Lemma 6], $\mathbb{P}^1_\delta$ is cone homogeneous. Therefore, since $\mathbb{P}^1_\delta$ is ordinal definable in $(V_{\delta'})^{V_1}$, by the proof of [7, Lemma 3], $\mathsf{HOD}^{(V_{\delta'})^{V_2}} \subseteq \mathsf{HOD}^{(V_{\delta'})^{V_1}}$. This means that $x \in \mathsf{HOD}^{(V_{\delta'})^{V_1}}$. However, because $\delta \in A$, Sargsyan's argument of [17, Lemma 2.2] shows that $x \notin \mathsf{HOD}^{(V_{\delta'})^{V_1}}$. This contradiction, together with defining $\kappa_0 = \delta'$ and $\mathbb{P} = \mathbb{P}^0 * \dot{\mathbb{P}}^1$, complete the proof of Theorem 5.1. □

We conclude with the following questions related to Theorem 5.1. More specifically:

Question 5.3. Is it possible to prove an analogue of Theorem 5.1 in a universe in which the class of supercompact cardinals can be arbitrary?

Question 5.4. Is it possible to prove an analogue of Theorem 5.1 along the lines of Theorem 4.4?

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