Hierarchies of (virtual) resurrection axioms

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Abstract

I analyze the hierarchies of the bounded resurrection axioms and their "virtual" versions, the virtual bounded resurrection axioms, for several classes of forcings (the emphasis being on the subcomplete forcings). I analyze these axioms in terms of implications and consistency strengths. For the virtual hierarchies, I provide level-by-level equiconsistencies with an appropriate hierarchy of virtual partially super-extendible cardinals. I show that the boldface resurrection axioms for subcomplete or countably closed forcing imply the failure of Todorčević's square at the appropriate level. I also establish connections between these hierarchies and the hierarchies of bounded and weak bounded forcing axioms.

1 Introduction

In [Fuc16a], I began a systematic study of hierarchies of forcing axioms, with a focus on their versions for the class of subcomplete forcings. Here, I continue this study, moving from the usual forcing axioms to the resurrection axioms, but still focusing mostly on subcomplete forcings, although not exclusively. Subcomplete forcing was introduced by Jensen in [Jen09b]. It is a class of forcings iterable with revised countable support that doesn't add reals, preserves stationary subsets of ω_1, but may change cofinalities to be countable. Examples of subcomplete forcings include all countably closed forcings, Namba forcing (assuming CH), Příkrý forcing (see [Jen14]), generalized Příkrý forcing (see [Min17]), and the Magidor forcing to collapse the cofinality of a measurable cardinal of sufficiently high Mitchell order to ω_1 (see [Fuc16b]). For an excellent overview article on subcomplete forcing, see [Jen14].

The weakest axiom considered in [Fuc16a] is the bounded forcing axiom for a class Γ of forcings, which was characterized by Bagaria ([Bag00]) as saying that whenever $\mathbb{P} \in \Gamma$, then $H_{\omega_2} \prec_{\Sigma_1} H_{\omega_2}^{\mathbb{P}}$. There are several natural ways of strengthening this axiom. One is to consider the hierarchy of bounded or weak bounded forcing axioms, and this was done in [Fuc16a]. Another option is to consider the maximality principle for Γ, see [SV01], [Ham03], [Fuc08], [Fuc09], which says that every sentence that can be forced to be true by a forcing in Γ in such a way that it stays true in every further forcing extension by a forcing in Γ, is already true - since Σ_1 sentences, once true, persist to any outer model, this generalizes Bagaria's characterization of the bounded forcing axiom in a very natural way, and there are natural parametric versions of the maximality principles. However, the maximality principles are not really axioms, but rather axiom schemes, and thus seem somehow remote from the topic of forcing axioms. An alternative, very similarly

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motivated strengthening of the bounded forcing axiom for Γ is the resurrection axiom. Various variants of both the maximality principle and the bounded resurrection axiom for subcomplete forcings were considered in [Min17]. The resurrection axioms were originally introduced in [HJ14], and their boldface versions originate in [HJ]. Although the original formulation was different, motivated by Bagaria's characterization of the bounded forcing axiom for Γ, the appropriate version of the "most bounded" version of the resurrection axiom for the forcing classes I am mostly interested in is that for every $\mathbb{P} \in \Gamma$, there is a $\dot{\mathbb{Q}} \in \Gamma^{\mathrm{V}^{\mathbb{P}}}$ such that $H_{\omega_2} \prec H_{\omega_2}^{\mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}}}$. In this form, the axiom is also interesting for the class of countably closed forcings (whereas the traditional forcing axioms for countably closed forcing are outright provable in ZFC). The unbounded resurrection axiom for countably closed forcing was also considered in [Tsa15].

It was observed by Tsaprounis [Tsa15] that one may view this resurrection axiom as a bounded resurrection axiom, where the unbounded resurrection axiom says that for every cardinal $\kappa \geq \omega_2$ and every $\mathbb{P} \in \Gamma$, there is a $\dot{\mathbb{Q}} \in \Gamma^{\mathrm{V}^{\mathbb{P}}}$ such that in $\mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}}$, there is a λ and an elementary embedding $j : H_\kappa \prec H_\lambda^{\mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}}}$. Tsaprounis makes some additional requirements regarding the critical point of this embedding and the size of the image of the critical point under j which make sense for the classes of forcing notions he had in mind, but these additional properties actually follow automatically for these classes, and not making these requirements results in a more general concept. Obviously, there is a hierarchy here, starting at $\kappa = \omega_2$, and growing in strength as κ increases through the cardinals, with the unbounded resurrection axiom looming above. The consistency strengths grow very quickly in this hierarchy. Less obvious is maybe the hierarchy of the virtual versions of these resurrection axioms. I formulate the virtual unbounded resurrection axiom as before, except that the embedding is virtual, i.e., it is not required to exist in $\mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}}$, but in a further forcing extension (by an arbitrary forcing - so this forcing does not have to be in $\Gamma^{\mathrm{V}^{\mathbb{P}*\dot{\mathbb{Q}}}}$). Of course, for each cardinal $\kappa \geq \omega_2$, there is the obvious virtual bounded resurrection axiom $\mathsf{vRA}_\Gamma(H_\kappa)$. The difference between the usual and the virtual resurrection axioms occurs beyond $\kappa = \omega_2$, and it turns out that there is a hierarchy of virtual large cardinals (virtually super α-extendible) that pins down exactly the consistency strengths of the virtual resurrection axioms. I also explore the relationships between these hierarchies of forcing principles, and their interactions with the hierarchies of the (weak) bounded forcing axioms, in terms of implications, their effects on the failure of (weak) square principles, and their consistency strengths.

The paper is organized as follows. First, in Section 2, I introduce the hierarchy of resurrection axioms for subcomplete, proper, or countably closed forcing, leading from the resurrection axiom at H_{ω_2} up to the unbounded resurrection axiom. In Section 3, I explore the bottom of this hierarchy, the H_{ω_2} level, in terms of consistency strength and consequences with regards to stationary reflection, failure of square principles, and the continuum. I show that the (boldface) resurrection axiom for subcomplete forcing for H_{ω_2} implies the failure Todorčević's square principle $\square(\omega_2)$, and even the failure of the weaker square principle $\square(\omega_2, \omega)$. I introduce these principles in detail in this section. These effects continue up the hierarchy, as is shown in Section 4. There, I also explore the relationships between the hierarchy of resurrection axioms and the hierarchy of bounded forcing axioms. In Section 5, I then proceed to discuss the virtual versions of the resurrection axioms. I establish that the exact consistency strengths of the axioms in the virtual resurrection hierarchy are measured by the hierarchy of the virtually super α-extendible cardinals, in Lemmas 5.10 and 5.12, and Corollary 5.13 establishes that the consistency strength of the unbounded virtual resurrection axiom is given by the existence of a virtually extendible cardinal. Theorem 5.15 summarizes the connections between the large cardinals and the virtual resurrection axioms. In Section 6, I analyze how the hierarchies of the virtual resurrection axioms and of the weak bounded forcing axioms relate, in terms of implications and consistency strengths. Figure 6 (on page 37) gives an overview of all of these results: relationships between the hierarchies of forcing axioms and resurrection axioms, their consequences in terms of the failure of square principles, and their consistency strengths.

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2 A hierarchy of bounded resurrection axioms

The resurrection axioms for various forcing classes were originally introduced by Hamkins and Johnstone in [HJ14], and more recently, they added "boldface" variants of these axioms in [HJ]. Here is the definition, with notation that deviates from the original, to allow flexibility for variations to come.

Definition 2.1. Let Γ be a forcing class. Then $\mathsf{RA}_\Gamma(H_{2^\omega})$ says that whenever $\mathbb{P} \in \Gamma$ and $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over V, then there is a $\mathbb{Q} \in \Gamma^{\mathrm{V}[G]}$ such that if $H \subseteq \mathbb{Q}$ is $\mathbb{Q}$-generic over $\mathrm{V}[G]$, then

$$\langle H_{2^\omega}, \in\rangle \prec \langle (H_{2^\omega})^{\mathrm{V}[G][H]}, \in\rangle$$

To avoid a possible confusion, 2^ω is taken *de dicto* here, meaning that on the right hand side of the displayed formula, 2^ω, as well as the entire term H_{2^ω}, are interpreted in $\mathrm{V}[G][H]$.

In the boldface variant of the axiom, $\undertilde{\mathsf{RA}}_\Gamma(H_{2^\omega})$, one is allowed to add a predicate to the structure H_{2^ω}. So this axiom says that whenever $R \subseteq H_{2^\omega}$, $\mathbb{P} \in \Gamma$ and $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over V, then there is a $\mathbb{Q} \in \Gamma^{\mathrm{V}[G]}$ such that if $H \subseteq \mathbb{Q}$ is $\mathbb{Q}$-generic over $\mathrm{V}[G]$, then there is an $R' \subseteq (H_{2^\omega})^{\mathrm{V}[G][H]}$, $R' \in \mathrm{V}[G][H]$, such that

$$\langle H_{2^\omega}, \in, R\rangle \prec \langle (H_{2^\omega})^{\mathrm{V}[G][H]}, \in, R'\rangle$$

In this definition, as well as in the remainder of this paper, when saying that Γ is a forcing class, I mean that Γ is a class term, that is, it is of the form $\{x \mid \varphi(x, c)\}$, where $\varphi(x, y)$ is a formula in the language of set theory and c is a parameter. Even though there may be different formulas in a fixed model of set theory which define the same forcing class, I will always assume that φ is chosen canonically for the particular class at hand. For example, if Γ is supposed to stand for the class of proper forcing, then φ will not use a parameter, and it has to be chosen in such a way that ZFC proves that $\{x \mid \varphi(x)\}$ is the class of all proper forcing notions. Here, I will focus on the classes of countably closed, subcomplete, proper and semi-proper forcings. No parameters are needed to define any of these classes.

Hamkins and Johnstone showed in the cases where Γ is the class of proper, semiproper forcings, that the resulting boldface resurrection axiom implies $2^\omega = \omega_2$, and they determined the consistency strengths of the (boldface) resurrection axioms to be a (strongly) uplifting cardinal. I will recall the definition of these large cardinal properties in the next section. They also showed that in the case where Γ is the class of countably closed forcings, their resurrection axiom implies CH, and that it trivially becomes equivalent to CH, since countably closed forcing can't change H_{ω_1}.

Instead of H_{2^ω}, I use a formulation of the resurrection axioms that is more suitable for countably closed and subcomplete forcings, as statements about H_{ω_2}, as in [Min17]. It will turn out that the resulting axioms for these forcing classes still imply CH but are not vacuous. This formulation is also suitable for the other classes of proper or semi-proper forcing, and I show in Observation 3.6 that the H_{ω_2} and H_{2^ω} versions of the boldface principles are equivalent, and the lightface principles are closely related. So I hope this change does not constitute an abuse of their original ideas.

I consider these resurrection axioms to be *bounded.* To motivate how to extend these resurrection axioms, and make them "less bounded", let us briefly think about the simplest case where Γ is the class of countably closed forcing notions. As explained above, in this case, the most suitable formulation of the lightface resurrection axiom is the one "at H_{ω_2}", saying that whenever G is generic for a countably closed forcing, there is a further countably closed forcing in $\mathrm{V}[G]$, such that if H is generic over $\mathrm{V}[G]$ for that forcing, then it follows that $\langle H_{\omega_2},\in\rangle \prec \langle H_{\omega_2}^{\mathrm{V}[G][H]},\in\rangle$. This principle is equiconsistent with an uplifting cardinal, as I will point out later. Notice that we cannot consistently replace ω_2 with ω_3 here, to make the axiom less bounded, since ω_2 may be collapsed to ω_1 in $\mathrm{V}[G]$, which means that the size of ω_2^{V} will be $\omega_1^{\mathrm{V}[G][H]}$, no matter how H is chosen. Thus, letting $\delta=\omega_2^{\mathrm{V}}$, the statement "$\delta$ is a cardinal" is true in $\langle H_{\omega_3},\in\rangle$, but it will not be true in $\langle H_{\omega_3}^{\mathrm{V}[G][H]},\in\rangle$, for any H. The parameter δ would have to be replaced with $\omega_2^{\mathrm{V}[G][H]}$! Thus, one is naturally led to generalize the concept to ω_3 by requiring the existence of an H as above such that in $\mathrm{V}[G][H]$, there is an *elementary embedding* j in $\mathrm{V}[G][H]$ from $\langle H_{\omega_3},\in\rangle$ to $\langle H_{\omega_3}^{\mathrm{V}[G][H]},\in\rangle$, which I will write as $j:\langle H_{\omega_3},\in\rangle \prec \langle H_{\omega_3}^{\mathrm{V}[G][H]},\in\rangle$. This embedding, in particular, would have to map ω_2^{V} to $\omega_2^{\mathrm{V}[G][H]}$. This indeed generalizes the H_{ω_2} case: looking back, the elementary embedding in that case was the identity, and in fact, whenever we're in the situation that there is an elementary embedding $j:\langle H_{\omega_2},\in\rangle \prec \langle H_{\omega_2}^{\mathrm{V}[G][H]},\in\rangle$, where $\omega_1^{\mathrm{V}[G][H]}=\omega_1^{\mathrm{V}}$ (as will be the case whenever G and H are generic for one of the classes mentioned before, since they all preserve ω_1, meaning that no forcing in any of these classes can collapse ω_1), then it follows easily that j is the identity. This is why in the formulation of the generalized resurrection axioms, where ω_2 can be replaced with any cardinal κ, I will always require the existence of elementary embeddings, even though in the case $\kappa=\omega_2$, it will follow that this embedding is the identity, when the forcing class under consideration preserves ω_1.

In fact, what is needed in order to conclude that the embedding is the identity on H_{ω_2} is that Γ preserves ω_1 *and* that whenever $\mathbb{P}\in\Gamma$ and G is generic for $\mathbb{P}$, then in $\mathrm{V}[G]$, it is still the case that every forcing in $\Gamma^{\mathrm{V}[G]}$ preserves $\omega_1^{\mathrm{V}[G]}=\omega_1^{\mathrm{V}}$. I will express this by saying that Γ is Γ-*necessarily* ω_1-*preserving,* employing terminology from modal logic as in [Ham03]. Similarly, I will say that Γ is Γ-*necessarily stationary set preserving* if every forcing in Γ preserves stationary subsets of ω_1, and this remains true in any forcing extension by a forcing in Γ. In general, a property holds Γ-*necessarily* if it holds in V and its forcing extensions by forcings in Γ.

Tsaprounis considered the unbounded resurrection axioms in [Tsa15]. The following definition introduces a hierarchy of resurrection axioms, starting with the original lightface/boldface axioms at the bottom, and leading up to these unbounded ones at the top. I will first give the definition, and then comment on apparent differences between it and the presentation in [Tsa15].

Definition 2.2. Let $\kappa\ge\omega_2$ be a cardinal, and let Γ be a forcing class. The *resurrection axiom for* Γ *at* H_κ, $\mathsf{RA}_\Gamma(H_\kappa)$, says that whenever G is generic over V for some forcing $\mathbb{P}\in\Gamma$, there is a $\mathbb{Q}\in\Gamma^{\mathrm{V}[G]}$ and a λ such that whenever H is $\mathbb{Q}$-generic over $\mathrm{V}[G]$, then in $\mathrm{V}[G][H]$, λ is a cardinal and there is an elementary embedding

$$j:\langle H_\kappa^{\mathrm{V}},\in\rangle \prec \langle H_\lambda^{\mathrm{V}[G][H]},\in\rangle$$

The boldface resurrection axiom for Γ at H_κ, $\underset{\sim}{\mathsf{RA}}_\Gamma(H_\kappa)$, says that for every $A\subseteq\kappa$ and every G as above, there is a $\mathbb{Q}$ as above such that for every H as above, in $\mathrm{V}[G][H]$, there are a B and a j such that

$$j:\langle H_\kappa^{\mathrm{V}},\in,A\rangle \prec \langle H_\lambda^{\mathrm{V}[G][H]},\in,B\rangle,$$

and such that if κ is regular, then λ is regular in $\mathrm{V}[G][H]$.

The *unbounded resurrection axiom for* Γ, UR_Γ, asserts that $\mathsf{RA}_\Gamma(H_\kappa)$ holds for every cardinal $\kappa\ge\omega_2$.

If Γ is the class of subcomplete forcings, then $\mathsf{RA}_{\mathsf{SC}}(H_\kappa)$, $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_\kappa)$ and $\mathsf{UR}_{\mathsf{SC}}$ stands for $\mathsf{RA}_\Gamma(H_\kappa)$, $\underset{\sim}{\mathsf{RA}}_\Gamma(H_\kappa)$ and UR_Γ, and similarly, for these axioms about the class of countably closed forcings, I write $\mathsf{RA}_{\sigma\text{-closed}}(H_\kappa)$, $\underset{\sim}{\mathsf{RA}}_{\sigma\text{-closed}}(H_\kappa)$ and $\mathsf{UR}_{\sigma\text{-closed}}$.

Let me state part of the discussion preceding this definition as a simple observation, to avoid a possible confusion about this point.

Observation 2.3. *Let Γ be Γ-necessarily ω_1-preserving. Then $\mathsf{RA}_\Gamma(H_{\omega_2})$ is equivalent to the statement that whenever G is generic over V for a forcing $\mathbb{P}$ from Γ, then there is a forcing notion $\mathbb{Q} \in \Gamma^{\mathrm{V}[G]}$ such that whenever H is $\mathbb{Q}$-generic over $\mathrm{V}[G]$, we have that*

$$\langle H_{\omega_2}, \in\rangle \prec \langle H_{\omega_2}^{\mathrm{V}[G][H]}, \in\rangle$$

A similar equivalence holds for $\underset{\sim}{\mathsf{RA}}_\Gamma(H_{\omega_2})$: in this case as well, the embedding required to exist in Definition 2.2 can be equivalently replaced with the identity.

The clause about the cofinalities of κ and λ in the definition of $\underset{\sim}{\mathsf{RA}}_\Gamma(H_\kappa)$, while natural, may seem a little ad hoc. But note that $\mathsf{RA}_\Gamma(H_{\kappa^+})$ implies this form of $\underset{\sim}{\mathsf{RA}}_\Gamma(H_\kappa)$. Note also that in the case that κ is a successor cardinal, it follows that λ is a successor cardinal in $\mathrm{V}[G][H]$, without imposing any requirements about the cofinalities of κ and λ, so in that case, it wouldn't be necessary to add this clause. The purpose of adding this requirement in the general case is the desire to have principles which generalize the effects that $\underset{\sim}{\mathsf{RA}}_\Gamma(H_{\omega_2})$ has on the failure of square principles, and this is where these clauses are used (see the proofs of Lemma 4.4, Lemma 4.5 and Theorem 4.7). The minimal assumption needed for these proofs to go through is that if $\mathrm{cf}^{\mathrm{V}}(\kappa) > \omega_1$, then $\mathrm{cf}^{\mathrm{V}[G][H]}(\lambda) > \omega_1$ as well.

I would like to address an apparent difference between Definition 2.2 and the one given in [Tsa15] by Tsaprounis. There, the definition of UR_Γ posits that what I call $\mathsf{RA}_\Gamma(H_\kappa)$ hold for all $\kappa > \max\{\omega_2, 2^\omega\}$, and additional requirements are imposed on the elementary embedding j, namely that $\mathrm{crit}(j) = \max\{\omega_2, 2^\omega\}$ and $j(\mathrm{crit}(j)) > \kappa$. First, for all the forcing classes I am interested in, $\mathsf{RA}_\Gamma(H_{\omega_2})$ implies that $2^\omega \le \omega_2$. In the case of proper or semi-proper forcing, this follows from Observation 3.5, which says that $\mathsf{RA}_\Gamma(H_{\omega_2})$ implies the bounded forcing axiom for Γ, which, in turn, implies that $2^\omega = \omega_2$, by [Moo05]. In the case of subcomplete or countably closed forcing, this follows from Fact 3.1, which says that in this case, $\mathsf{RA}_\Gamma(H_{\omega_2})$ implies $\diamondsuit$, and thus CH. Thus, in the cases which are of interest here, $\max(\omega_2, 2^\omega) = \omega_2$. I can not make a requirement about the critical point of j, since I allow the case that j is the identity, which occurs if $\kappa = \omega_2$. But notice that all the classes of forcing I am interested in allow us to collapse any uncountable cardinal we want to ω_1, even over any extension of V by a forcing in Γ. As a result, the additional requirements about j made in Tsaprounis' definition can be met for free. Namely, assume that $\kappa > \omega_2$ is a cardinal for which $\mathsf{RA}_\Gamma(H_\kappa)$ holds, as defined above. Let G be generic for some forcing notion $\mathbb{P}$ in Γ. We can now pick G' to be generic over $\mathrm{V}[G]$ for the collapse of κ to ω_1, let's call this forcing $\mathbb{P}' = (\dot{\mathbb{P}}')^G$. In each of the cases of interest here, it follows that $\mathbb{P} * \dot{\mathbb{P}}'$ is still in Γ. By $\mathsf{RA}_\Gamma(H_\kappa)$, applied to $\mathbb{P} * \dot{\mathbb{P}}'$ and $G * G'$, there is an H generic for some forcing in $\Gamma^{\mathrm{V}[G*G']}$, such that in $\mathrm{V}[G * G'][H]$, there is an elementary $j : \langle H_\kappa, \in\rangle \prec \langle H_\lambda^{\mathrm{V}[G*G'][H]}, \in\rangle$, for some $\mathrm{V}[G * G'][H]$-cardinal λ. It follows easily that the critical point of j has to be ω_2, since ω_1 is preserved, so that $j(\omega_1) = \omega_1$, and since κ is collapsed to ω_1 in $\mathrm{V}[G * G']$, it follows that $j(\omega_2) = \omega_2^{\mathrm{V}[G*G'][H]} > \kappa$.

Thus, dropping these requirements about the critical point of j and the size of its image under j resulted in a concept that captures the original resurrection axioms as well as the intermediate stages on the way to the unbounded one, for the classes of forcing under consideration here.

I would now like to make a comment on the monotonicity of $\mathsf{RA}_\Gamma(H_\kappa)$. Certainly, increasing κ yields a potentially stronger principle, that is, if $\kappa < \kappa'$, then $\mathsf{RA}_\Gamma(H_{\kappa'})$ implies $\mathsf{RA}_\Gamma(H_\kappa)$, since if we have reached an extension $\mathrm{V}[G][H]$ in which there is an elementary $j' : \langle H_{\kappa'}, \in\rangle \prec \langle H_{\lambda'}^{\mathrm{V}[G][H]}, \in\rangle$, then letting j be the restriction of j' to H_κ and $\lambda = j'(\lambda)$, it follows that $j : \langle H_\kappa, \in\rangle \prec \langle H_\lambda^{\mathrm{V}[G][H]}, \in\rangle$, since H_κ is a class definable in $H_{\kappa'}$ from κ, and $H_\lambda^{\mathrm{V}[G][H]}$ is definable from λ in $\langle H_{\lambda'}, \in\rangle$ using the same definition, and since if κ is regular in V, then it is regular in $H_{\kappa'}^{\mathrm{V}}$, so that $\lambda = j'(\kappa)$ is regular in $H_{\lambda'}^{\mathrm{V}[G][H]}$, which implies that it is regular in $\mathrm{V}[G][H]$. However, we do not have monotonicity in the parameter Γ. Increasing Γ results in a wider variety of challenges G (in Definition 2.2), which seems to make the concept stronger, but on the other hand there is a wider variety of potential answers H to choose from in order to meet the challenge and resurrect, which seems to make the concept weaker. As an example, I have already mentioned that $\mathsf{RA}_{\sigma\text{-closed}}(H_{\omega_2})$ implies CH, but we shall see in Observation 3.6 that $\mathsf{RA}_{\text{proper}}(H_{\omega_2})$ implies $2^\omega = \omega_2$, even though the class of countably closed forcing notions is contained in the class of proper forcing notions.

Note that in the definition of the boldface principle $\underset{\sim}{\mathsf{RA}}_\Gamma(H_\kappa)$, I only allowed predicates which are subsets of κ, not of H_κ. The reason for this is that I want this principle to be intermediate between $\mathsf{RA}_\Gamma(H_\kappa)$ and $\mathsf{RA}_\Gamma(H_{\kappa^+})$, which is obvious using this definition of the concept since every subset of κ is a member of H_{κ^+}. Moreover, in applications, the predicates I used so far could always be coded as subsets of κ. Let me now continue with a simple observation on the cofinalities of κ and λ in Definition 2.2.

Observation 2.4. *Suppose κ is a singular cardinal and $\underset{\sim}{\mathsf{RA}}_\Gamma(H_\kappa)$ holds. Then for every $A \subseteq \kappa$ and every G generic for a forcing in Γ, there is a $\mathbb{Q} \in \Gamma^{\mathrm{V}[G]}$ such that if H is generic for $\mathbb{Q}$ over $\mathrm{V}[G]$, then in $\mathrm{V}[G][H]$, there are a B, a cardinal λ and an elementary embedding j such that*

$$j : \langle H_\kappa^{\mathrm{V}}, \in, A\rangle \prec \langle H_\lambda^{\mathrm{V}[G][H]}, \in, B\rangle,$$

with $j(\mathrm{cf}^{\mathrm{V}}(\kappa)) = \mathrm{cf}^{\mathrm{V}[G][H]}(\lambda)$.

Proof. Let $\bar{\kappa} = \mathrm{cf}(\kappa)$, and let $F : \bar{\kappa} \longrightarrow \kappa$ be monotone and cofinal. Clearly, F can be easily coded as a subset of κ. Let A and G be as stated. By $\underset{\sim}{\mathsf{RA}}_\Gamma(H_\kappa)$, let $\mathbb{Q}$, H, F', B be such that

$$j : \langle H_\kappa^{\mathrm{V}}, \in, A, F\rangle \prec \langle H_\lambda^{\mathrm{V}[G][H]}, \in, B, F'\rangle$$

in $\mathrm{V}[G][H]$. Let $\bar{\lambda} = \mathrm{cf}^{\mathrm{V}[G][H]}(\lambda)$. Then $F' : j(\bar{\kappa}) \longrightarrow \lambda$ is monotone and cofinal, so $\bar{\lambda} \le j(\bar{\kappa})$. By elementarity, $j(\bar{\kappa})$ is regular in $H_\lambda^{\mathrm{V}[G][H]}$ and hence in $\mathrm{V}[G][H]$. It follows that $\bar{\lambda} = j(\bar{\kappa})$, because if $\bar{\lambda} < j(\bar{\kappa})$, then a cofinal function $g : \bar{\lambda} \longrightarrow \lambda$ would induce a cofinal function from $\bar{\lambda}$ to $j(\bar{\kappa})$, contradicting that $j(\bar{\kappa})$ is regular in $\mathrm{V}[G][H]$. □

It was shown in [Tsa15, Theorems 2.3 and 2.4] that one can force UR_Γ over a model with an extendible cardinal, where Γ is the class of ccc, σ-closed, proper, or stationary set preserving forcings. The same argument shows the consistency of the axiom for the class of subcomplete forcings.

Fact 2.5. *If κ is an extendible cardinal, then there is an iteration of subcomplete forcings, contained in V_κ, satisfying the κ-c.c., such that $\mathsf{UR}_{\mathsf{SC}}$ holds in the generic extension.*

3 The bottom of the hierarchy

I'll first focus on the resurrection axioms for countably closed or subcomplete forcing at H_{ω_2}, that is, $\mathsf{RA}_{\sigma\text{-closed}}(H_{\omega_2})$, $\underset{\sim}{\mathsf{RA}}_{\sigma\text{-closed}}(H_{\omega_2})$, $\mathsf{RA}_{\mathsf{SC}}(H_{\omega_2})$ and $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})$. It was shown in [Min17] that $\mathsf{RA}_{\mathsf{SC}}(H_{\omega_2})$ implies Jensen's combinatorial principle $\diamondsuit$. The same is true of $\mathsf{RA}_{\sigma\text{-closed}}(H_{\omega_2})$ (by a simpler argument).

Fact 3.1 ([Min17, Proposition 4.2.15])**.** $\mathsf{RA}_{\mathsf{SC}}(H_{\omega_2})$/$\mathsf{RA}_{\sigma\text{-}closed}(H_{\omega_2})$ *imply* $\diamondsuit$*.*

Proof. Adding a Cohen subset A of ω_1 also adds a $\diamondsuit$-sequence, see [Kun80, Theorem 8.3], and $\diamondsuit$ remains true in any further forcing extension by a forcing that's subcomplete in $\mathrm{V}[A]$ (see [Jen09a, Chapter 3, page 7, Lemma 4]). By assumption, there is an H which is generic over $\mathrm{V}[A]$ for a subcomplete forcing, such that $\langle H_{\omega_2}, \in\rangle \prec \langle H_{\omega_2}^{\mathrm{V}[A][H]}, \in\rangle$. The principle $\diamondsuit$ can be expressed over H_{ω_2}, and it holds in the latter model, so it holds in the former as well. □

In general, any statement of the form $\varphi^{H_{\omega_2}}$ that's implied by the maximality principle for subcomplete or countably closed forcing is also a consequence of the corresponding resurrection axioms, and it was observed in [Min17] and in [Fuc08] that these maximality principles imply $\diamondsuit$.

So while the forcing axioms for subcomplete forcing considered in [Fuc16a] were just compatible with CH, the principles under consideration now actually imply it (and more). The consistency strength of the resurrection axioms at the bottom of the hierarchy is precisely determined as follows.

Definition 3.2. An inaccessible cardinal κ is *uplifting* if there are arbitrarily large inaccessible cardinals γ such that $\langle \mathrm{V}_\kappa, \in\rangle \prec \langle \mathrm{V}_\gamma, \in\rangle$. It is *strongly uplifting* if for every $A \subseteq \mathrm{V}_\kappa$, there are arbitrarily large (inaccessible) γ such that there is a $B \subseteq \mathrm{V}_\gamma$ with $\langle \mathrm{V}_\kappa, \in, A\rangle \prec \langle \mathrm{V}_\gamma, \in, B\rangle$.

These cardinals were introduced in [HJ14] and [HJ]. In the definition of strongly uplifting, the inaccessibility of γ does not need to be required explicitly, see [HJ, Theorem 3].

Fact 3.3 (Minden)**.** $\mathsf{RA}_{\mathsf{SC}}(H_{\omega_2})$/$\mathsf{RA}_{\sigma\text{-}closed}(H_{\omega_2})$ *are equiconsistent with the existence of an uplifting cardinal, and* $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})$/$\underset{\sim}{\mathsf{RA}}_{\sigma\text{-}closed}(H_{\omega_2})$ *are equiconsistent with a strongly uplifting cardinal.*

Proof. The claims regarding the lightface resurrection principles and the existence of an uplifting cardinal can be found in [Min17, Theorems 4.2.12, 4.3.13]. Minor modifications of the proofs show the claims regarding the boldface resurrection principles and the existence of strongly uplifting cardinals. In more detail, the proof of [Min17, Theorem 4.3.6] contains a forcing construction which achieves slightly more than $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})$, but starts from slightly more than a strongly uplifting cardinal. One can easily simplify the construction to start from just a strongly uplifting cardinal and yield only $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})$. For the converse, the proof of [Min17, Theorem 4.3.7] contains an argument showing that $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})$ implies that ω_2 is strongly uplifting in L. The same arguments show the results concerning $\underset{\sim}{\mathsf{RA}}_{\sigma\text{-closed}}(H_{\omega_2})$. □

I will now explore a connection to the bounded forcing axiom, $\mathsf{BFA}(\Gamma)$. This axiom was originally introduced in [GS95] in a combinatorial way that was then shown by Bagaria to be equivalent to the following property, which I will take as its definition, since it is more useful in the present context.

Theorem 3.4 ([Bag00, Thm. 5])**.** *The bounded forcing axiom* $\mathsf{BFA}(\Gamma)$ *for a forcing class* Γ *is equivalent to* $\Sigma_1(H_{\omega_2})$*-absoluteness for forcing notions* $\mathbb{P}$ *in* Γ*. The latter means that whenever* $\varphi(x)$ *is a* Σ_1*-formula and* $a \in H_{\omega_2}$*, then* $\mathrm{V} \models \varphi(a)$ *iff for every* $\mathbb{P}$*-generic* g*,* $\mathrm{V}[g] \models \varphi(a)$*.*

If a forcing class Γ has the very natural property that for every forcing $\mathbb{P} \in \Gamma$ and every condition $p \in \mathbb{P}$, the restriction $\mathbb{P}_{\leq p}$ of $\mathbb{P}$ to conditions below p is also in Γ, then this characterization of $\mathsf{BFA}(\Gamma)$ can be equivalently expressed by saying that whenever G generic for some $\mathbb{P} \in \Gamma$, then

$$\langle H_{\omega_2}, \in\rangle \prec \langle H_{\omega_2}^{\mathrm{V}[G]}, \in\rangle$$

This is the case for all classes of forcing under consideration here, and it is obvious that $\mathsf{RA}_\Gamma(H_{\omega_2})$ implies this generic absoluteness property. This is recorded in the following observation, and I will later give a proof of the more general Lemma 4.3.

Observation 3.5. *$\mathsf{RA}_\Gamma(H_{\omega_2})$ implies $\mathsf{BFA}(\Gamma)$.*

This observation allows us to compare the current version of the resurrection axioms at the level H_{ω_2} to the original ones from [HJ14], which use H_{2^ω}, in the case of proper or semi-proper forcing. In the proof, and in the rest of the paper, when κ is a regular cardinal and X is a set, I will write $\mathrm{Col}(\kappa, X)$ for the forcing notion to collapse X to κ, that is the poset consisting of functions of the form $f : \alpha \longrightarrow X$, where $\alpha < \kappa$, ordered by reverse inclusion. Also, I say that a forcing is $<\kappa$-closed if every decreasing sequence of length less than κ has a lower bound in $\mathbb{P}$. Thus, $\mathrm{Col}(\kappa, X)$ is $<\kappa$-closed.

Observation 3.6. *Let Γ be either the class of proper or of semi-proper forcings. Then*

1. *$\mathsf{RA}_\Gamma(H_{\omega_2})$ is equivalent to $\mathsf{RA}_\Gamma(H_{2^\omega}) + \neg\mathsf{CH}$.*
2. *$\utilde{\mathsf{RA}}_\Gamma(H_{\omega_2})$ is equivalent to $\utilde{\mathsf{RA}}_\Gamma(H_{2^\omega})$.*

Proof. Let's prove 1 first. For the direction from left to right, by Observation 3.5, $\mathsf{RA}_\Gamma(H_{\omega_2})$ implies that $\mathsf{BFA}(\Gamma)$ holds, and this implies by [Moo05] that $2^\omega = \omega_2$. Let G be generic for $\mathbb{P} \in \Gamma$. By $\mathsf{RA}_\Gamma(H_{\omega_2})$, let H be generic for a $\mathbb{Q} \in \Gamma^{\mathrm{V}[G]}$, such that

$$\langle H_{2^\omega}, \in\rangle = \langle H_{\omega_2}, \in\rangle \prec \langle H_{\omega_2}^{\mathrm{V}[G][H]}, \in\rangle$$

We're done if $\mathrm{V}[G][H] \models 2^\omega = \omega_2$. Note that it cannot be that $\mathrm{V}[G][H] \models 2^\omega = \omega_1$, because this could be expressed in $H_{\omega_2}^{\mathrm{V}[G][H]}$, so it would have to be true in V, which it is not. The only other option is that $\mathrm{V}[G][H] \models 2^\omega \geq \omega_3$. But then, if I is generic over $\mathrm{V}[G][H]$ for $\mathbb{R} = \mathrm{Col}(\omega_2, 2^\omega)^{\mathrm{V}[G][H]}$, a forcing in $\Gamma^{\mathrm{V}[G][H]}$ that's $<\omega_2$-closed there, it follows that $H_{\omega_2}^{\mathrm{V}[G][H]} = H_{\omega_2}^{\mathrm{V}[G][H][I]}$, and $\mathrm{V}[G][H][I] \models 2^\omega = \omega_2$. Thus, letting $\mathbb{R} = \dot{\mathbb{R}}^H$, it follows that $H * I$ is generic over $\mathrm{V}[G]$ for the forcing $\mathbb{Q} * \dot{\mathbb{R}}$, which is in $\Gamma^{\mathrm{V}[G]}$, and we have that $\langle H_{2^\omega}, \in\rangle \prec \langle H_{2^\omega}^{\mathrm{V}[G][H*I]}, \in\rangle$.

For the direction from right to left, first observe that $\mathsf{RA}_\Gamma(H_{2^\omega}) + \neg\mathsf{CH}$ implies that $2^\omega = \omega_2$, because otherwise if $2^\omega \geq \omega_3$, then one could let G be generic for $\mathrm{Col}(\omega_1, \omega_2)$, which is in Γ, since it is countably closed. But then, letting $\delta = \omega_2^{\mathrm{V}}$, the statement "$\delta$ is a cardinal" is true in $\langle H_{2^\omega}^{\mathrm{V}}, \in\rangle$, but not in $\langle H_{2^\omega}^{\mathrm{V}[G][H]}, \in\rangle$ for any further forcing extension $\mathrm{V}[G][H]$. Now, if G is generic for some $\mathbb{P} \in \Gamma$, then by $\mathsf{RA}_\Gamma(H_{2^\omega})$, we can let H be generic over $\mathrm{V}[G]$ for some $\mathbb{Q} \in \Gamma^{\mathrm{V}[G]}$, such that $\langle H_{\omega_2}, \in\rangle = \langle H_{2^\omega}, \in\rangle \prec \langle H_{2^\omega}^{\mathrm{V}[G][H]}, \in\rangle$. Since $2^\omega = \omega_2$, it follows that $\langle H_{2^\omega}, \in\rangle$ believes that there is exactly one uncountable cardinal, and so the same is true in $\langle H_{2^\omega}^{\mathrm{V}[G][H]}, \in\rangle$, which means that $\mathrm{V}[G][H]$ believes that $2^\omega = \omega_2$. Thus, $\langle H_{\omega_2}, \in\rangle \prec \langle H_{\omega_2}^{\mathrm{V}[G][H]}, \in\rangle$, as desired.

Now, let's turn to 2. For the direction from left to right, let's assume $\utilde{\mathsf{RA}}_\Gamma(H_{\omega_2})$. To show that $\utilde{\mathsf{RA}}_\Gamma(H_{2^\omega})$ holds, let $A \subseteq H_{2^\omega}$. Let $\mathbb{P} \in \Gamma$, and let G be $\mathbb{P}$-generic over V. We have seen that already the lightface principle $\mathsf{RA}_\Gamma(H_{\omega_2})$ implies $\mathsf{BFA}(\Gamma)$. By [Moo05], $\mathsf{BFA}(\Gamma)$ implies $2^\omega = 2^{\omega_1} = \omega_2$. In particular, H_{ω_2} has cardinality ω_2. Recall that $\utilde{\mathsf{RA}}_\Gamma(H_{\omega_2})$ only allows the use of predicates which are subsets of ω_2, so we have to code A as a subset of ω_2. So let $F : \omega_2 \longrightarrow H_{\omega_2}$ be a bijection, and let $E = \{\langle\alpha, \beta\rangle \mid F(\alpha) \in F(\beta)\}$ (using Gödel pairs, E can easily be coded as a subset of ω_2). Let $\bar{A} = F^{-1}``A$. By $\utilde{\mathsf{RA}}_\Gamma(H_{\omega_2})$, let $\mathbb{Q} \in \Gamma^{\mathrm{V}[G]}$, let H be $\mathbb{Q}$-generic over $\mathrm{V}[G]$, and let $E', \bar{A}'$ be such that

$$\langle H_{\omega_2}, \in, E, \bar{A}\rangle \prec \langle H_{\omega_2}^{\mathrm{V}[G][H]}, \in, E', \bar{A}'\rangle$$

Since the cofinality of ω_2 is greater than ω, it can be expressed in $\langle H_{\omega_2}, \in, E, \bar{A}\rangle$ that E is extensional and well-founded, so that the corresponding statement is true in $\langle H_{\omega_2}^{\mathrm{V}[G][H]}, \in, E', \bar{A}'\rangle$. It can moreover be expressed that the transitive collapse of $\langle \omega_2, E\rangle$ is equal to H_{ω_2}. Hence, the same is true in $\langle H_{\omega_2}^{\mathrm{V}[G][H]}, \in, E', \bar{A}'\rangle$. So, letting F' be the Mostowski collapse, which is in $\mathrm{V}[G][H]$, it follows that

$$F' : \langle \omega_2^{\mathrm{V}[G][H]}, E', \bar{A}'\rangle \to \langle H_{\omega_2}^{\mathrm{V}[G][H]}, \in, A'\rangle$$

is an isomorphism, where $A' = (F')``\bar{A}'$. A simple computation now shows that

$$F' \circ F^{-1} : \langle H_{\omega_2}, \in, A\rangle \prec \langle H_{\omega_2}^{\mathrm{V}[G][H]}, \in, A'\rangle$$

Since $\omega_1^{\mathrm{V}} = \omega_1^{\mathrm{V}[G][H]}$, it follows that $F' \circ F^{-1} = \mathrm{id}$, so that

$$\langle H_{\omega_2}, \in, A\rangle \prec \langle H_{\omega_2}^{\mathrm{V}[G][H]}, \in, A'\rangle$$

Again, $\omega_2 = 2^\omega$ in V, and in $\mathrm{V}[G][H]$, we clearly have that $2^\omega \geq \omega_2$. In the case that $2^\omega \geq \omega_3$ in $\mathrm{V}[G][H]$, we can let I be $\mathrm{Col}(\omega_2, 2^\omega)^{\mathrm{V}[G][H]}$-generic over $\mathrm{V}[G][H]$ to get

$$\langle H_{2^\omega}, \in, E, \bar{A}\rangle \prec \langle H_{2^\omega}^{\mathrm{V}[G][H*I]}, \in, A'\rangle$$

For the converse, assume $\underset{\sim}{\mathsf{RA}}_\Gamma(H_{2^\omega})$. To prove $\underset{\sim}{\mathsf{RA}}_\Gamma(H_{\omega_2})$, let $A \subseteq \omega_2$, let $\mathbb{P} \in \Gamma$, and let G be $\mathbb{P}$-generic over V. It was shown in [HJ, Theorem 17] that $\underset{\sim}{\mathsf{RA}}_\Gamma(H_{2^\omega})$ implies $2^\omega = \omega_2$. So we can apply $\underset{\sim}{\mathsf{RA}}_\Gamma(H_{2^\omega})$ to get a $\mathbb{Q} \in \Gamma^{\mathrm{V}[G]}$ be such that if H is $\mathbb{Q}$-generic over $\mathrm{V}[G]$, then there is an $A' \in \mathrm{V}[G][H]$ such that

$$\langle H_{\omega_2}, \in, A\rangle = \langle H_{2^\omega}, \in, A\rangle \prec \langle H_{2^\omega}^{\mathrm{V}[G][H]}, \in, A'\rangle$$

As before, it follows that $2^\omega = \omega_2$ in $\mathrm{V}[G][H]$, so we are done. □

I will need some facts on the preservation of stationary sets by forcing.

Fact 3.7. *Suppose Γ is a forcing class such that the bounded forcing axiom for Γ, $\mathsf{BFA}(\Gamma)$, holds, in the sense that for every $\mathbb{P}$ in Γ, if G is generic for $\mathbb{P}$ over V, then $\langle H_{\omega_2}, \in\rangle \prec_{\Sigma_1} \langle H_{\omega_2}^{\mathrm{V}[G]}, \in\rangle$. Then every $\mathbb{P} \in \Gamma$ preserves stationary subsets of ω_1.*

Proof. Let $\kappa = \omega_1$. If $S \subseteq \kappa$ were stationary in V but not in $\mathrm{V}[G]$, then the statement "there is a club subset of κ that's disjoint from S" would be a Σ_1 statement about κ and S true in $\langle H_{\omega_2}, \in\rangle^{\mathrm{V}[G]}$ but false in $\langle H_{\omega_2}, \in\rangle$. □

Fact 3.8. *If a forcing $\mathbb{P}$ preserves stationary subsets of ω_1, then it preserves stationary subsets of any θ with $\mathrm{cf}(\theta) = \omega_1$.*

Proof. Suppose $S \subseteq \theta$ is stationary. Let $f : \omega_1 \longrightarrow \theta$ be normal and cofinal. Then $\bar{S} = f^{-1}``S$ is stationary in $\omega_1$. Now, if $G$ is $\mathbb{P}$-generic and $D \in \mathrm{V}[G]$ is closed and unbounded in $\theta$, then $\bar{D} = f^{-1}``D$ is closed and unbounded in ω_1, so since $\mathbb{P}$ preserves stationary subsets of ω_1, there is $\alpha \in \bar{S} \cap \bar{D}$, so that $f(\alpha) \in S \cap D$, showing that $\mathbb{P}$ preserves the stationarity of S. □

Fact 3.9. *Suppose $\mathrm{cf}(\kappa) \geq \omega_1$. Then countably closed forcing preserves the stationarity of any stationary subset of κ consisting of ordinals of cofinality ω.*

Proof. I think this is due to Baumgartner, but lacking a reference, I will sketch the proof. By an argument similar to the one given in the proof of Fact 3.8, we may assume that κ is regular. Suppose $\mathbb{P}$ is countably closed, $S \subseteq \kappa$ is stationary, and assume, towards a contradiction, that some $\mathbb{P}$-name $\dot{C}$ is forced by a condition $p \in \mathbb{P}$ to be a club subset of κ disjoint from S. Let $M = \langle H_\theta, \in, \mathbb{P}, p, \dot{C}, S, <\rangle$, where θ is sufficiently large and regular, and $<$ is a well-ordering of H_θ. Since S is stationary, there is an $X \prec M$ such that $X \cap \kappa = \bar{\kappa} \in S$. Letting $\langle \kappa_n \mid n < \omega\rangle$ be increasing and cofinal in $\bar{\kappa}$, we can construct a decreasing sequence $\langle p_n \mid n < \omega\rangle$ in $\mathbb{P} \cap X$ below p such that for every $n < \omega$, there is a δ_n such that p_n forces that δ_n is the least member of $\dot{C}$ above κ_n. It follows that $\delta_n \in X$, and hence that $\kappa_n \le \delta_n < \bar{\kappa}$, for $n < \omega$, so that $\sup_{n<\omega} \delta_n = \bar{\kappa}$. Now any lower bound for $\langle p_n \mid n < \omega\rangle$ forces that $\bar{\kappa}$ is in $S \cap \dot{C}$, a contradiction. □

I will now turn to effects of resurrection axioms at H_{ω_2} on stationary reflection.

Definition 3.10. Let κ be an ordinal of uncountable cofinality. An ordinal $\gamma < \kappa$ of uncountable cofinality is a *reflection point* of a stationary set $S \subseteq \kappa$ if $S \cap \gamma$ is stationary in γ. It is a *simultaneous reflection point* of a sequence $\vec{S} = \langle S_\alpha \mid \alpha < \theta\rangle$ of stationary subsets of κ if it is a reflection point of each S_α, for $\alpha < \theta$.

Lemma 3.11. *Assume* $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})$ *or* $\underset{\sim}{\mathsf{RA}}_{\sigma\text{-}closed}(H_{\omega_2})$*. Then every sequence* $\vec{S} = \langle S_i \mid i < \omega_1\rangle$ *of stationary subsets of* ω_2 *each of which consist of ordinals of countable cofinality has a simultaneous reflection point. Actually, this is a consequence of* $\underset{\sim}{\mathsf{RA}}_\Gamma(H_{\omega_2})$ *whenever* Γ *contains a forcing of the form* $\mathrm{Col}(\omega_1, \theta)$*, for some* $\theta \ge \omega_2$*, and if* Γ*-necessarily,* Γ *is stationary set preserving.*

Proof. Let $\vec{S}$ be given, and let let $M = \langle H_{\omega_2}, \in, \tilde{S}\rangle$, where $\tilde{S} = \bigcup_{i<\omega_1} \{i\} \times S_i$, coded as a subset of ω_2. Let G be V-generic for $\mathrm{Col}(\omega_1, \omega_2)$. By Fact 3.9, each S_i is still stationary in $\mathrm{V}[G]$. Let $\mathbb{Q}$ be subcomplete (σ-closed) in $\mathrm{V}[G]$ and let H be $\mathbb{Q}$-generic over $\mathrm{V}[G]$ such that in $\mathrm{V}[G][H]$, there is a model $N = \langle H_{\omega_2}^{\mathrm{V}[G][H]}, \in, \tilde{T}\rangle$ such that $M \prec N$, by $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})/\underset{\sim}{\mathsf{RA}}_{\sigma\text{-closed}}(H_{\omega_2})$. Let $\kappa = \omega_2^{\mathrm{V}}$. Clearly, letting $T_i = \{\xi \mid \langle i, \xi\rangle \in \tilde{T}\}$ for $i < \omega_1$, it follows that $S_i = T_i \cap \kappa$, and N believes that each S_i is stationary in κ, since S_i is stationary in $\mathrm{V}[G]$, where the cofinality of κ is ω_1, so $\mathbb{Q}$ preserves the stationarity of S_i over $\mathrm{V}[G]$ by Fact 3.8. N also believes that the cofinality of κ is ω_1. By elementarity, M believes that there is a $\bar{\kappa}$ of cofinality ω_1 such that for every $i < \omega_1$, $S_i \cap \bar{\kappa}$ is stationary in $\bar{\kappa}$. Since H_{ω_2} contains every subset of $\bar{\kappa}$, M is right about that. □

Note that if every sequence $\vec{S}$ as in the previous lemma has a simultaneous reflection point, then the set of such reflection points is actually stationary, because given any club set C, one can consider the sequence $\vec{S'}$, where $S'_i = S_i \cap C$.

Definition 3.12 ([Fuc16a])**.** Let τ be a cardinal greater than ω_1. Then SFP_τ (the strong Friedman property at τ) is the following reflection principle: whenever $\langle A_i \mid i < \omega_1\rangle$ is a sequence of stationary subsets of τ such that each A_i consists of ordinals of countable cofinality, and $\langle D_i \mid i < \omega_1\rangle$ is a partition of ω_1 into stationary sets, then there is a normal (that is, increasing and continuous) function $f : \omega_1 \longrightarrow \tau$ such that for every $i < \omega_1$, we have that $f``D_i \subseteq A_i$.

It is easy to see that SFP_τ implies the simultaneous reflection described in Lemma 3.11, namely that every ω_1-sequence of stationary subsets of τ, each consisting of ordinals of countable cofinality, has a simultaneous reflection point (and this implies that each such sequence actually has stationarily many reflection points); see [Fuc16a, Obs. 2.8]. Jensen showed that the forcing axiom for the class of subcomplete forcing, denoted SCFA, implies that SFP_τ holds, for every regular $\tau > \omega_1$, see [Jen14, p. 154, Lemma 7.1]. I will show that SFP_{ω_2} follows from the weak version of the boldface resurrection axiom, going back to [HJ14], adapted to the present context.

Definition 3.13. Let Γ be a forcing class. The *weak resurrection axiom for* Γ *at* H_{ω_2}, $\mathsf{wRA}_\Gamma(H_{\omega_2})$, says that whenever G is generic for a forcing in Γ, there is a further forcing $\mathbb{Q} \in \mathrm{V}[G]$ (not necessarily in $\Gamma^{\mathrm{V}[G]}$) such that if H is generic for that forcing over $\mathrm{V}[G]$, then $\langle H_{\omega_2}, \in\rangle \prec \langle H_{\omega_2}^{\mathrm{V}[G][H]}, \in\rangle$. $\mathsf{w}\underset{\sim}{\mathsf{RA}}_\Gamma(H_{\omega_2})$ is defined similarly, allowing a predicate $A \subseteq \omega_2$, and guaranteeing the existence of a $B \subseteq \omega_2^{\mathrm{V}[G][H]}$ in $\mathrm{V}[G][H]$ such that $\langle H_{\omega_2}, \in, A\rangle \prec \langle H_{\omega_2}^{\mathrm{V}[G][H]}, \in, B\rangle$.

It is easy to see that the weak resurrection axiom at H_{ω_2} can only hold for a forcing class Γ that consists of stationary set preserving forcing notions; it actually implies $\mathsf{BFA}(\Gamma)$ (see Fact 3.7 in this context). Note also that the forcing $\mathbb{Q}$ in the definition necessarily preserves ω_1.

Lemma 3.14. $\mathsf{w}\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})$ *implies* SFP_{ω_2}.

Proof. Let $\langle A_i \mid i < \omega_1\rangle$ be a sequence of stationary subsets of ω_2 consisting of ordinals of countable cofinality. Let $\langle D_i \mid i < \omega_1\rangle$ be a partition of ω_1 into stationary subsets. In [Jen14, p. 154, Lemma 7.1], Jensen points out that the forcing $\mathbb{P}$ to add a normal function $f : \omega_1 \longrightarrow \omega_2^{\mathrm{V}}$ such that for every $i < \omega_1$, $f``D_i \subseteq A_i$ is subcomplete. It consists of countable initial segments of such a function, of successor length, ordered by reverse inclusion. Let $M = \langle H_{\omega_2}, \in, \langle A_i \mid i < \omega_1\rangle, \langle D_i \mid i < \omega_1\rangle\rangle$ (coding $\vec{A}$ as a subset of ω_2 in a straightforward way). By $\mathsf{w}\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})$, let $\mathbb{Q} \in \mathrm{V}[G]$ be a poset such that, letting H be $\mathrm{V}[G]$-generic for $\mathbb{Q}$, there is a structure $N = \langle H_{\omega_2}^{\mathrm{V}[G][H]}, \in, \langle B_i \mid i < \omega_1\rangle, \langle \tilde{D}_i \mid i < \omega_1\rangle\rangle$ in $\mathrm{V}[G][H]$ such that $M \prec N$. Note that since $M \prec N$, it follows that $\omega_1^{\mathrm{V}} = \omega_1^M = \omega_1^N = \omega_1^{\mathrm{V}[G][H]}$. Clearly, $D_i = \tilde{D}_i$ and $A_i = B_i \cap \omega_2^{\mathrm{V}}$, for all $i < \omega_1$. Since f is in $H_{\omega_2}^{\mathrm{V}[G][H]}$ the statement that there exists an ordinal λ and a normal function $h : \omega_1 \longrightarrow \lambda$ such that for every $i < \omega_1$, $h``D_i \subseteq B_i$ is true in N, and hence, the corresponding statement is true in M, with B_i replaced by A_i. □

I want to make a connection to Jensen's weak square principles now, so I will briefly recall their definitions. These principles go back to [Jen72, §5].

Definition 3.15. Let κ be a cardinal. A $\square_\kappa$*-sequence* is a sequence $\langle C_\alpha \mid \kappa < \alpha < \kappa^+,\ \alpha \text{ limit}\rangle$ of sets C_α club in α with $\mathrm{otp}(C_\alpha) \le \kappa$ such that for each limit point β of C_α, $C_\beta = C_\alpha \cap \beta$. $\square_\kappa$ is the principle saying that there is a $\square_\kappa$-sequence.

If λ is another cardinal, then a $\square_{\kappa,\lambda}$-sequence is a sequence $\langle \mathcal{C}_\alpha \mid \kappa < \alpha < \kappa^+,\ \alpha \text{ limit}\rangle$ such that each $\mathcal{C}_\alpha$ has size at most λ, and such that each $C \in \mathcal{C}_\alpha$ is club in α, has order-type at most κ and satisfies the coherency condition that for every limit point β of C, $C \cap \beta \in \mathcal{C}_\beta$. Again, $\square_{\kappa,\lambda}$ is the assertion that there is a $\square_{\kappa,\lambda}$-sequence. $\square_{\kappa,\kappa}$ is known as *weak square* and denoted by $\square^*_\kappa$. $\square_{\kappa,<\lambda}$ is defined like $\square_{\kappa,\lambda}$, except that each $\mathcal{C}_\alpha$ is required to have size less than λ.

Corollary 3.16. $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})$, $\underset{\sim}{\mathsf{RA}}_{\sigma\text{-}closed}(H_{\omega_2})$ *or* $\mathsf{w}\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})$ *imply the failure of* $\square_{\omega_1,\omega}$. *But* $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})$/$\underset{\sim}{\mathsf{RA}}_{\sigma\text{-}closed}(H_{\omega_2})$ *imply that* $\square^*_{\omega_1}$ *holds.*

Proof. It was shown in Lemma 3.11 $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})$/$\underset{\sim}{\mathsf{RA}}_{\sigma\text{-closed}}(H_{\omega_2})$ implies that every ω_1-sequence of stationary subsets of ω_2, each consisting of ordinals of countable cofinality, has a simultaneous reflection point. This form of stationary reflection implies the failure of $\square_{\omega_1,\omega}$, by [CM11, Lemma 2.2]. The principle $\mathsf{w}\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})$ implies SFP_{ω_2}, which, in turn, also implies this simultaneous stationary reflection principle, and hence the failure of $\square_{\omega_1,\omega}$. Finally, $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})$/$\underset{\sim}{\mathsf{RA}}_{\sigma\text{-closed}}(H_{\omega_2})$ imply $\diamondsuit$, by Fact 3.1, and hence CH, which implies $\square^*_{\omega_1}$; this latter implication is probably due to Jensen, but see [MLH13, Theorems 3.1, 3.2] for details. □

Observation 3.17. $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})$/$\underset{\sim}{\mathsf{RA}}_{\sigma\text{-}closed}(H_{\omega_2})$ *are consistent with* $\forall\lambda \ge \omega_2\ \square_\lambda$.

Proof. This is because one may force $\underset{\sim}{\mathsf{R}}\mathsf{A}_{\mathsf{SC}}(H_{\omega_2})/\underset{\sim}{\mathsf{R}}\mathsf{A}_{\sigma\text{-closed}}(H_{\omega_2})$ over L, if L has a strongly uplifting cardinal κ, see the references made in the proof of Fact 3.3. The forcing is κ-c.c., and if g is generic for it, then $\omega_2^{L[g]} = \kappa$. Hence, the $\square_\lambda$-sequences from L survive, for $\lambda \geq \kappa = \omega_2^{L[g]}$. □

So, we have precisely determined the extent of $\square$ principles under $\underset{\sim}{\mathsf{R}}\mathsf{A}_{\mathsf{SC}}(H_{\omega_2})/\underset{\sim}{\mathsf{R}}\mathsf{A}_{\sigma\text{-closed}}(H_{\omega_2})$. It is known that the proper forcing axiom implies failures of Todorčević's square principles ([Vel86], [Sch07]), and the next goal is to show that the boldface resurrection axioms for subcomplete or σ-closed forcing allow us to make that conclusion as well. The motivation for deriving failures of square principles is that these can be used to establish consistency strength lower bounds on the principles that imply them, and failures of Todorčević's forms of square principles in combination with simultaneous failures of the regular square principle are much higher in consistency strength ([Sch07]). The following definition introduces even weaker forms of Todorčević's variant of square that were also considered in [Wei10], [HLH16].

Definition 3.18. Let λ be a limit of limit ordinals. A sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \lambda,\ \alpha \text{ limit}\rangle$ is *coherent* if for every limit $\alpha < \lambda$, $\mathcal{C}_\alpha \neq \emptyset$ and for every $C \in \mathcal{C}_\alpha$, we have that C is club in α and for every limit point β of C, it follows that $C \cap \beta \in \mathcal{C}_\beta$. A *thread* through $\vec{\mathcal{C}}$ is a club set $T \subseteq \lambda$ such that for every limit point β of T less than λ, we have that $T \cap \beta \in \mathcal{C}_\beta$. If κ is a cardinal, then the principle $\square(\lambda, {<}\kappa)$ says that there is a $\square(\lambda, {<}\kappa)$-sequence, that is, a coherent sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \lambda,\ \alpha \text{ limit}\rangle$ such that each $\mathcal{C}_\alpha$ has size less than κ, and such that $\vec{\mathcal{C}}$ has no thread. I may write $\square(\lambda, \kappa)$ for the principle $\square(\lambda, {<}\kappa^+)$. The principle $\square(\lambda, 1)$ is denoted $\square(\lambda)$.

In the case where $\kappa = 1$, a $\square(\lambda, \kappa)$-sequence is of course taken to be a sequence of club sets, rather than a sequence of singletons of club sets. This case has been studied extensively by Todorčević, see [Tod10] for an overview. It is easy to see that if λ is a cardinal, then a $\square_{\lambda,\kappa}$ sequence is also a $\square(\lambda^+, \kappa)$ sequence. Namely, let $\vec{\mathcal{C}}$ be a $\square_{\lambda,\kappa}$ sequence. Then one can easily construct a coherent sequence $\vec{\mathcal{C}}'$ from $\vec{\mathcal{C}}$ by letting $\mathcal{C}'_\alpha = \{\alpha\}$ (that is, α is viewed as a subset of α here) for limit ordinals $\alpha \leq \lambda$, and setting $\mathcal{C}'_\alpha = \{C \setminus (\lambda+1) \mid C \in \mathcal{C}_\alpha\}$ for limit ordinals α with $\lambda < \alpha < \lambda^+$. This sequence still has the property that whenever $C \in \mathcal{C}'_\alpha$, then $\operatorname{otp}(C) \leq \lambda$. It follows that $\vec{\mathcal{C}}'$ is a $\square(\lambda^+, \kappa)$-sequence, because if T were a thread, then T would have to be closed unbounded in λ^+, but if we let γ be the $(\lambda+1)$-st limit point of T, then $T \cap \gamma \in \mathcal{C}'_\gamma$ has order type $\lambda + \omega$. As with the square principles introduced earlier, increasing κ makes it easier to satisfy them.

A version of the following lemma for the more familiar weak square principle $\square_{\lambda,<\lambda}$ was shown in [MLH13, Lemma 4.5].

Lemma 3.19. *Suppose λ is a regular uncountable cardinal. Then a ${<}\lambda$-closed forcing cannot add a new thread (i.e., a thread that didn't exist in* V*) to a coherent sequence of length λ^+ all of whose elements have size less than λ.*

Proof. Magidor's proof of [MLH13, Lemma 4.5] goes through verbatim. □

In the following, I will need to use the definition of subcompleteness, due to Jensen. While there are several versions in the literature, I use the one given in [Jen09a, §3, pp. 3]. I will frequently use models of the theory ZFC^-, which consists of the ZFC axioms, with Power Set and Replacement removed, and the Collection Scheme added. The Collection Scheme consists of all formulas of the form $\forall \vec{z}(\forall x \exists y \varphi(x, y, \vec{z}) \longrightarrow \forall u \exists v \forall x \in u \exists y \in v \varphi(x, y, \vec{z}))$, where $\varphi(x, y, \vec{z})$ is any formula in the language of set theory with all free variables shown, see [Jen14, P. 85]. If κ is regular, then H_κ is a model of ZFC^-.

Definition 3.20. A transitive set N (usually a model of ZFC^-) is *full* if there is an ordinal $\gamma > 0$ such that $L_\gamma(N) \models \mathsf{ZFC}^-$ and N is regular in $L_\gamma(N)$, meaning that if $x \in N$, $f \in L_\gamma(N)$ and $f : x \longrightarrow N$, then $\mathrm{ran}(f) \in N$.

The idea is that N can be put inside a transitive model of ZFC^- which thinks that the domain of N is equal to H_τ, where τ is the ordinal height of N. Following Jensen, if A is a set and τ is an ordinal, I will in the following write L^A_τ for the structure $\langle L_\tau[A], \in, A \cap L_\tau[A]\rangle$. When I say that a structure N of the form L^A_τ satisfies ZFC^-, then I mean ZFC^- in the language with a unary predicate symbol $\dot{A}$ that is interpreted by $\bar{A} = A \cap L_\tau[A]$ in N. Inside such a structure, the $L_\alpha[\bar{A}]$ hierarchy can be defined (for $\alpha < \tau$), with its canonical well-order. For $X \subseteq N$, I will write $\mathrm{Hull}^N(X)$ for the Skolem hull of X, using the canonical Skolem functions associated to this canonical well-ordering of the universe of N.

Definition 3.21. Let $\mathbb{P}$ be a poset and let $\delta(\mathbb{P})$ the minimal cardinality of a dense subset of $\mathbb{P}$. Then $\mathbb{P}$ is *subcomplete* if for all sufficiently large cardinals θ with $\mathbb{P} \in H_\theta$, any ZFC^- model $N = L^A_\tau$ with $\theta < \tau$ and $H_\theta \subseteq N$, any $\sigma : \bar{N} \prec N$ such that $\bar{N}$ is countable, transitive and full and such that $\mathbb{P}, \theta \in \mathrm{ran}(\sigma)$, any $\bar{G} \subseteq \bar{\mathbb{P}}$ which is $\bar{\mathbb{P}}$-generic over $\bar{N}$, and any $s \in \mathrm{ran}(\sigma)$, the following holds: letting $\sigma(\bar{s}, \bar{\theta}, \bar{\mathbb{P}}) = s, \theta, \mathbb{P}$, there is a condition $p \in \mathbb{P}$ such that whenever $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over V with $p \in G$, there is in V$[G]$ a σ' such that

1. $\sigma' : \bar{N} \prec N$ is an elementary embedding,
2. $\sigma'(\bar{s}, \bar{\theta}, \bar{\mathbb{P}}) = s, \theta, \mathbb{P}$,
3. $(\sigma')``\bar{G} \subseteq G$,
4. $\mathrm{Hull}^N(\delta(\mathbb{P}) \cup \mathrm{ran}(\sigma')) = \mathrm{Hull}^N(\delta(\mathbb{P}) \cup \mathrm{ran}(\sigma))$.

I will not use property 4. of the previous definition in what follows. That property is crucial for proving iteration theorems for subcomplete forcing, though, see [Jen14]. I will frequently consider forcing extensions of transitive set-sized models of ZFC^-. In this context, the forcing theorem remains valid, see [Jen14, pp. 88-89].

Lemma 3.22. *Let λ be an ordinal with $\mathrm{cf}(\lambda) = \omega_1$. Then subcomplete forcing cannot add a new thread to a coherent sequence of length λ all of whose elements have size less than 2^ω.*

Proof. Before beginning the proof, let me emphasize that the given coherent sequence is not assumed to be a $\square(\lambda, {<}2^\omega)$-sequence. It may have threads, but the point is that no new threads can be added, that is, no new club subsets of λ that cohere with the sequence can be adjoined by subcomplete forcing.

Let $\mathbb{P}$ be subcomplete, and let $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \lambda,\ \alpha \text{ limit}\rangle$ be a coherent sequence all of whose elements have size less than 2^ω. Let $f : \omega_1 \longrightarrow \lambda$ be a normal, cofinal function, and let $g : \mathcal{P}(\omega) \longrightarrow 2^\omega$ be a bijection. Suppose $\dot{b}$ is a $\mathbb{P}$-name such that $\mathbb{P}$ forces that $\dot{b}$ is a new thread through $\vec{\mathcal{C}}$ (that is, a thread that did not exist in V). Fix enumerations

$$\mathcal{C}_\alpha = \{C^\alpha_\nu \mid \nu < \kappa_\alpha\}$$

with $\kappa_\alpha < 2^\omega$, for every limit ordinal $\alpha < \lambda$. Let $N = L_\tau[A]$ with $H_\theta \subseteq N$, where θ is sufficiently large, $\theta < \tau$, and let $\sigma : \bar{N} \prec N$, where $\bar{N}$ is countable and full, such that $\theta, f, g, \mathbb{P}, \dot{b}, \vec{\mathcal{C}} \in \mathrm{ran}(\sigma)$. Let $\sigma(\bar{\theta}, \bar{f}, \bar{g}, \bar{\mathbb{P}}, \dot{\bar{b}}, \vec{\bar{\mathcal{C}}}) = \theta, f, g, \mathbb{P}, \dot{b}, \vec{\mathcal{C}}$, and let $\bar{G}$ be generic for $\bar{\mathbb{P}}$ over $\bar{N}$.

Let $\Omega = \omega_1^{\bar{N}} = \mathrm{crit}(\sigma)$. By subcompleteness, let $p \in \mathbb{P}$ be such that if G is generic for $\mathbb{P}$ over V with $p \in G$, then in V$[G]$, there is a σ' with $\sigma'(\bar{\theta}, \bar{f}, \bar{g}, \bar{\mathbb{P}}, \dot{\bar{b}}, \vec{\bar{\mathcal{C}}}) = \theta, f, g, \mathbb{P}, \dot{b}, \vec{\mathcal{C}}$ and $(\sigma')``\bar{G} \subseteq G$. Let $D = \mathrm{ran}(f)$ and $\bar{D} = \mathrm{ran}(\bar{f})$.

(1) *(a)* $\sigma\restriction\bar{D} = \sigma'\restriction\bar{D}$

(b) $\sigma\restriction(2^\omega)^{\bar{N}} = \sigma'\restriction(2^\omega)^{\bar{N}}$

Proof of (1). Clearly, $\sigma\restriction\Omega = \sigma'\restriction\Omega = \mathrm{id}\restriction\Omega$. So, for $\xi < \Omega$, $\sigma(\bar{f}(\xi)) = \sigma(\bar{f})(\sigma(\xi)) = \sigma'(\bar{f})(\sigma'(\xi)) = \sigma'(\bar{f}(\xi))$, showing (a). Similarly, $\sigma\restriction\mathcal{P}(\omega)^{\bar{N}} = \sigma'\restriction\mathcal{P}(\omega)^{\bar{N}} = \mathrm{id}\restriction\mathcal{P}(\omega)^{\bar{N}}$. So, for $x \in \mathcal{P}(\omega)^{\bar{N}}$, $\sigma(\bar{g}(x)) = \sigma(\bar{g})(\sigma(x)) = \sigma'(\bar{g})(\sigma'(x)) = \sigma'(\bar{g}(x))$, showing (b). $\square_{(1)}$

Let $\bar{\lambda} = \sup\bar{D}$, so that $\sigma'(\bar{\lambda}) = \lambda$, and set $\tilde{\lambda} = \sup\sigma``\bar{\lambda}$.

(2) $\dot{b}^G \cap \tilde{\lambda} \in \mathcal{C}_{\tilde{\lambda}}$

Proof of (2). Note that $\mathrm{cf}(\tilde{\lambda}) = \omega$, so $\tilde{\lambda} < \lambda$. To prove the claim, it suffices to show that $\tilde{\lambda}$ is a limit point of $\dot{b}^G$, because $\dot{b}^G$ is a thread through $\vec{\mathcal{C}}$. To see that $\tilde{\lambda}$ is a limit point of $\dot{b}^G$, note that $\dot{\bar{b}}^{\bar{G}}$ is club in $\bar{\lambda}$, as is $\bar{D}$. Note that $\Omega = \omega_1^{\bar{N}} = \omega_1^{\bar{N}[\bar{G}]}$. This is because $\sigma' : \bar{N} \prec N$ is elementary, so $\sigma'(\omega_1^{\bar{N}}) = \omega_1^N$, and σ' lifts to an elementary embedding $\sigma' : \bar{N}[\bar{G}] \prec N[G]$, as $\sigma'``\bar{G} \subseteq G$. Since G preserves ω_1, it follows that $\omega_1^{N[G]} = \omega_1^N$, which implies that $\omega_1^{\bar{N}[\bar{G}]} = \omega_1^{\bar{N}}$. It follows that $\bar{\lambda}$ has cofinality Ω in $\bar{N}[\bar{G}]$, since otherwise, $\omega_1^{\bar{N}}$ would be collapsed in $\bar{N}[\bar{G}]$. Hence, $\bar{D} \cap \dot{\bar{b}}^{\bar{G}}$ is club in $\bar{\lambda}$. But then, $\sigma``(\bar{D} \cap \dot{\bar{b}}^{\bar{G}}) = (\sigma')``(\bar{D} \cap \dot{\bar{b}}^{\bar{G}})$ (by (1)(a)) is unbounded in $\tilde{\lambda}$, and $(\sigma')``(\bar{D} \cap \dot{\bar{b}}^{\bar{G}}) \subseteq \dot{b}^G$. This shows that $\tilde{\lambda}$ is a limit point of $\dot{b}^G$, and thus the claim. $\square_{(2)}$

So, for every $\bar{G}'$ that's $\bar{\mathbb{P}}$-generic over $\bar{N}$, we can fix a condition $p_{\bar{G}'} \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{\sigma}_{\bar{G}'}$ such that $p_{\bar{G}'}$ forces that $\dot{\sigma} : \check{\bar{N}} \prec \check{N}$, $\dot{\sigma}_{\bar{G}'}(\bar{\theta}, \bar{f}, \bar{g}, \bar{\mathbb{P}}, \dot{\bar{b}}, \vec{\bar{\mathcal{C}}}) = \theta, f, g, \mathbb{P}, \dot{b}, \vec{\mathcal{C}}$ and $(\dot{\sigma})``\bar{G}' \subseteq \Gamma$ (where Γ is the canonical $\mathbb{P}$-name for the generic filter). Let us also fix a $C_{\bar{G}'} \in \mathcal{C}_{\tilde{\lambda}}$ such that $p_{\bar{G}'}$ forces that $\dot{b} \cap \check{\tilde{\lambda}} = \check{C}_{\bar{G}'}$ (by (2)).

Since $\mathbb{P}$ forces that $\dot{b}$ is not in V, it is straightforward to construct a system of filters $\langle \bar{G}_s \mid s : \omega \longrightarrow 2\rangle$ generic for $\bar{\mathbb{P}}$ over $\bar{N}$ such that if $s \neq t$, then $\dot{\bar{b}}^{\bar{G}_s} \neq \dot{\bar{b}}^{\bar{G}_t}$. Namely, fixing an enumeration $\langle D_n \mid n < \omega\rangle$ of all the dense subsets of $\bar{\mathbb{P}}$ that exist in $\bar{N}$, one can construct, by recursion on the length of $u \in {}^{<\omega}2$, a sequence $\langle q_u \mid u \in {}^{<\omega}2\rangle$ of conditions in $\bar{\mathbb{P}}$ such that $q_u \in D_{|u|}$, $u \subseteq v \implies q_v \leq_{\bar{\mathbb{P}}} q_u$, and such that for every $u \in {}^{<\omega}2$, there is an α such that $q_{u^\frown\langle 0\rangle} \Vdash_{\bar{\mathbb{P}}} \check{\alpha} \in \dot{\bar{b}}$ and $q_{u^\frown\langle 0\rangle} \Vdash_{\bar{\mathbb{P}}} \check{\alpha} \notin \dot{\bar{b}}$ or vice versa. Then, for every $s : \omega \longrightarrow 2$, the set $\{q_{s\restriction n} \mid n < \omega\}$ generates a $\bar{\mathbb{P}}$-generic filter $\bar{G}_s$ over $\bar{N}$, and the sequence $\langle \bar{G}_s \mid s : \omega \longrightarrow 2\rangle$ is as wished.

Since the cardinality of $\mathcal{C}_{\tilde{\lambda}}$ is less than 2^ω, we can find $s \neq t$ such that $C_{\bar{G}_s} = C_{\bar{G}_t}$. Set $\bar{G}_0 = \bar{G}_s$ and $\bar{G}_1 = \bar{G}_t$. Let $p_{\bar{G}_i} \in G_i$, G_i $\mathbb{P}$-generic over V, $\sigma'_i = (\dot{\sigma}_{\bar{G}_i})^{G_i}$, for $i < 2$. To summarize, we have:

(3) $\dot{b}^{G_0} \cap \tilde{\lambda} = \dot{b}^{G_1} \cap \tilde{\lambda}$, $\dot{\bar{b}}^{\bar{G}_0} \neq \dot{\bar{b}}^{\bar{G}_1}$ *and* $\sigma'_0\restriction\bar{D} = \sigma\restriction\bar{D} = \sigma'_1\restriction\bar{D}$.

But on the other hand, it follows that $\dot{\bar{b}}^{\bar{G}_0} = \dot{\bar{b}}^{\bar{G}_1}$, a contradiction. Namely, let $\bar{\gamma}$ be a limit point of $\dot{\bar{b}}^{\bar{G}_0} \cap \bar{D}$. Then $\dot{\bar{b}}^{\bar{G}_0} \cap \bar{\gamma} \in \bar{\mathcal{C}}_{\bar{\gamma}}$, i.e., for some $\bar{\rho} < (2^\omega)^{\bar{N}}$, $\dot{\bar{b}}^{\bar{G}_0} \cap \bar{\gamma} = \bar{C}^{\bar{\gamma}}_{\bar{\rho}}$. Since $\sigma'_0 : \bar{N}[\bar{G}_0] \prec N[G_0]$ is elementary, it follows that $\dot{b}^{G_0} \cap \sigma'_0(\bar{\gamma}) = C^{\sigma'_0(\bar{\gamma})}_{\sigma'_0(\bar{\rho})}$. By (1)(*b*), $\rho := \sigma'_0(\bar{\rho}) = \sigma(\bar{\rho}) = \sigma'_1(\bar{\rho})$. Moreover, by (1)(*a*), since $\bar{\gamma} \in \bar{D}$, $\gamma := \sigma'_0(\bar{\gamma}) = \sigma(\bar{\gamma}) = \sigma'_1(\bar{\gamma})$. So, since $\dot{b}^{G_0} \cap \tilde{\lambda} = \dot{b}^{G_1} \cap \tilde{\lambda}$, it follows that

$$\dot{b}^{G_1} \cap \gamma = \dot{b}^{G_0} \cap \gamma = C^{\sigma'_0(\bar{\gamma})}_{\rho} = C^{\sigma'_1(\bar{\gamma})}_{\rho} = C^{\gamma}_{\rho}$$

But $\dot{b}^{G_1} \cap \sigma'_1(\bar{\gamma}) = C^{\sigma'_1(\bar{\gamma})}_{\rho}$ means, by elementarity of σ'_1, that $\dot{\bar{b}}^{\bar{G}_1} \cap \bar{\gamma} = \bar{C}^{\bar{\gamma}}_{\bar{\rho}}$. So $\dot{\bar{b}}^{\bar{G}_0} \cap \bar{\gamma} = \dot{\bar{b}}^{\bar{G}_1} \cap \bar{\gamma}$. This is true for every limit point $\bar{\gamma}$ of $\dot{\bar{b}}^{\bar{G}_0} \cap \bar{D}$, and these are unbounded in $\bar{\lambda}$, so it follows that $\dot{\bar{b}}^{\bar{G}_0} = \dot{\bar{b}}^{\bar{G}_1}$, the desired contradiction. $\square$

Note that the assumption that $\mathrm{cf}(\lambda) = \omega_1$ in the previous lemma is necessary, because if $\mathrm{cf}(\lambda) \geq \omega_2$, then one can change the cofinality of λ to be equal to ω_2, by forcing with $\mathrm{Col}(\omega_2, \lambda)$, then force CH by adding a Cohen subset of ω_1, and then, subsequently, one can change the cofinality of λ to be ω, using Namba forcing (which is subcomplete, by CH, see [Jen14, P. 132, Lemma 6.2]). Changing the cofinality of λ to ω of course adds threads, because any cofinal subset of λ of order type ω, having no limit points less than λ, will then vacuously be a thread. The case of interest is that the coherent sequence in the lemma is a $\square(\lambda, {<}2^\omega)$-sequence, which for this reason can only happen if $\mathrm{cf}(\lambda) > \omega$. Finally, it is not hard to see that if $\mathrm{cf}(\lambda) = \omega_1$, then $\square(\lambda)$ holds - see, for example, [Vel86, p. 48].

Theorem 3.23. $\underset{\sim}{\mathsf{RA}}_{\sigma\text{-}closed}(H_{\omega_2})/\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})$ *imply the failure of* $\square(\omega_2, \omega)$.

Proof. Suppose $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \omega_2,\ \alpha \text{ limit}\rangle$ were a $\square(\omega_2, \omega)$-sequence. Let $\kappa = \omega_2$. Let G be generic for $\mathrm{Col}(\omega_1, \omega_2)$ over V. In $\mathrm{V}[G]$, the cofinality of κ is ω_1, and by Lemma 3.19 (with $\lambda = \omega_1$), $\vec{\mathcal{C}}$ is still a $\square(\kappa, \omega)$-sequence in $\mathrm{V}[G]$. Let $M = \langle H_{\omega_2}, \in, \vec{\mathcal{C}}\rangle$, where $\vec{\mathcal{C}}$ is coded as a subset of ω_2 in some canonical way. By $\underset{\sim}{\mathsf{RA}}_{\sigma\text{-closed}}(H_{\omega_2})/\ \underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})$, there is a forcing $\mathbb{Q} \in \mathrm{V}[G]$ that is countably closed/subcomplete in $\mathrm{V}[G]$, such that if H is $\mathbb{Q}$-generic over $\mathrm{V}[G]$, then in $\mathrm{V}[G][H]$, there is a structure $N = \langle H_{\omega_2}, \in, \vec{\mathcal{D}}\rangle$ such that $M \prec N$. But then, $\vec{\mathcal{D}}{\restriction}\kappa = \vec{\mathcal{C}}$, and so, every $T \in \mathcal{D}_\kappa$ is a thread through $\vec{\mathcal{C}}$. However, by Lemma 3.22, there can be no such thread in $\mathrm{V}[G][H]$, since $\mathrm{cf}^{\mathrm{V}[G]}(\kappa) = \omega_1$ and $\mathbb{Q}$ is subcomplete in $\mathrm{V}[G]$ (recall that every σ-closed forcing is subcomplete). □

Recall that by Corollary 3.16, $\underset{\sim}{\mathsf{RA}}_{\sigma\text{-closed}}(H_{\omega_2})/\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})$ implies $\square^*_{\omega_1}$, which, in turn, implies that $\square(\omega_2, \omega_1)$ holds, by the remarks after Definition 3.18. Thus, the previous theorem is optimal.

4 Climbing up the hierarchy

I will start by describing the relationship between higher resurrection axioms and the bounded forcing axioms.

Definition 4.1. Let Γ be a forcing class, and let κ be a cardinal. Then the *bounded forcing axiom for* Γ *at* κ, $\mathsf{BFA}(\Gamma, {\leq}\kappa)$, says that whenever $M = \langle |M|, \in, \vec{R}\rangle$ is a transitive model of size at most κ, $|\vec{R}| \leq \omega_1$, $\varphi(x)$ is a Σ_1-formula and $\mathbb{P}$ is a forcing in Γ that forces that $\varphi(M)$ holds, then there are in V a transitive model $\bar{M}$ with $\varphi(\bar{M})$ and an elementary embedding $j : \bar{M} \prec M$.

For more on the motivation for this way of defining the bounded forcing axioms, I refer the reader to [Fuc16a]. I will use the following weak resurrection axioms from time to time.

Definition 4.2. Let $\kappa \geq \omega_2$ be a cardinal, and let Γ be a forcing class. The *weak resurrection axiom for* Γ *at* H_κ, $\mathsf{wRA}_\Gamma(H_\kappa)$, says that whenever G is generic over V for some forcing $\mathbb{P} \in \Gamma$, then there is a forcing notion $\mathbb{Q}$ in $\mathrm{V}[G]$ and a λ such that whenever H is $\mathbb{Q}$-generic over $\mathrm{V}[G]$, then in $\mathrm{V}[G][H]$, λ is a cardinal and there is an elementary embedding

$$j : \langle H^{\mathrm{V}}_\kappa, \in\rangle \prec \langle H^{\mathrm{V}[G][H]}_\lambda, \in\rangle$$

with $j{\restriction}\omega_2 = \mathrm{id}$.

The principle $\underset{\sim}{\mathsf{wRA}}_\Gamma(H_\kappa)$ says that for every $A \subseteq H_\kappa$ and every G as above, there is a $\mathbb{Q}$ as above such that for every H as above, in $\mathrm{V}[G][H]$, there are a B and a j such that

$$j : \langle H^{\mathrm{V}}_\kappa, \in, A\rangle \prec \langle H^{\mathrm{V}[G][H]}_\lambda, \in, B\rangle,$$

with $j{\restriction}\omega_2 = \mathrm{id}$ and such that if κ is regular, then λ is regular in $\mathrm{V}[G][H]$.

If Γ is the class of subcomplete forcings, then $\mathsf{wRA}_{\mathsf{SC}}(H_\kappa)$, $\mathsf{w}\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_\kappa)$ stands for $\mathsf{wRA}_\Gamma(H_\kappa)$, $\mathsf{w}\underset{\sim}{\mathsf{RA}}_\Gamma(H_\kappa)$, respectively.

The reader will notice the requirement that $j{\restriction}\omega_2 = \mathrm{id}$ in the previous definition. In the regular principles $\mathsf{RA}_\Gamma(H_\kappa)/\underset{\sim}{\mathsf{RA}}_\Gamma(H_\kappa)$, such a requirement is not necessary, because ω_1 cannot be collapsed by any forcing in Γ, since the forcings notions in Γ will always be assumed to preserve stationary subsets of ω_1, see Fact 3.7. As a result, the critical point of an elementary embedding given by an application of $\mathsf{RA}_\Gamma(H_\kappa)$ or $\underset{\sim}{\mathsf{RA}}_\Gamma(H_\kappa)$ will always be greater than ω_1. However, in the weak form of the principle, the second forcing applied might conceivably collapse ω_1, since there is no requirement that $\mathbb{Q}$ belongs to $\Gamma^{\mathrm{V}[G]}$. Allowing for this to happen would result in a principle that does not generalize $\mathsf{w}\underset{\sim}{\mathsf{RA}}_\Gamma(H_{\omega_2})$, as introduced in Definition 3.13, and it would be different in spirit to the principles considered in [HJ14]. Thus, since it doesn't follow automatically in the context of weak resurrection axioms, I have to require explicitly that $j{\restriction}\omega_2 = \mathrm{id}$. I could have equivalently required that $\mathbb{Q}$ be ω_1-preserving in $\mathrm{V}[G]$. Note that these equivalent requirements are implicit in the definition of the principle in the case $\kappa = \omega_2$ (Definition 3.13), where it is asked that $j = \mathrm{id}$. Note also that requiring $j{\restriction}\omega_2$ is equivalent to the requiring $j{\restriction}H_{\omega_2} = \mathrm{id}$.

The following lemma was also observed in [HJ14, Theorem 4] for their version of the resurrection axioms, not involving elementary embeddings.

Lemma 4.3. *Let Γ be a forcing class and $\kappa > \omega_1$ a cardinal. Then*

$$\mathsf{wRA}_\Gamma(H_\kappa) \implies \mathsf{BFA}(\Gamma, {<}\kappa).$$

Proof. Let $M = \langle |M|, \in, R_0, R_1, \ldots, R_i, \ldots\rangle_{i<\omega_1}$ be a transitive model of size less than κ, let $\mathbb{P} \in \Gamma$ be a forcing, let G be generic for $\mathbb{P}$ over V, let $\varphi(x)$ be a Σ_1-formula, and suppose that $\mathrm{V}[G] \models \varphi(M)$. Let $\mathbb{Q} \in \Gamma^{\mathrm{V}[G]}$ and H be $\mathbb{Q}$-generic over $\mathrm{V}[G]$ such that there is in $\mathrm{V}[G][H]$ an elementary embedding $j : \langle H_\kappa, \in\rangle \prec \langle H_\lambda^{\mathrm{V}[G][H]}, \in\rangle$, where λ is a cardinal. Note that, since $\mathrm{V}[G] \models \varphi(M)$ and φ is Σ_1, it follows that $\mathrm{V}[G][H] \models \varphi(M)$. Further, $M \in H_\kappa \subseteq H_\lambda^{\mathrm{V}[G][H]}$, so that since $H_\lambda^{\mathrm{V}[G][H]} \prec_{\Sigma_1} \mathrm{V}[G][H]$, it follows that

$$\langle H_\lambda^{\mathrm{V}[G][H]}, \in\rangle \models \varphi(M)$$

Moreover, $j' := j{\restriction}M \in H_\lambda^{\mathrm{V}[G][H]}$, since $j' \subseteq M \times j(M) \in H_\lambda^{\mathrm{V}[G][H]}$. Since $j{\restriction}\omega_2 = \mathrm{id}$, we have that $j(M) = \langle j(M), \in, j(R_0), j(R_1), \ldots, j(R_i), \ldots\rangle_{i<\omega_1}$, and so, $j(M)$ is a model of the same language as M and $j' : M \prec j(M)$ is elementary. Hence, the statement "there is a transitive $\bar{M}$ such that $\varphi(\bar{M})$ holds and there is an elementary embedding $k : \bar{M} \prec j(M)$" is true in $H_\lambda^{\mathrm{V}[G][H]}$, as witnessed by M and j'. This is a statement about $j(M)$. So, by elementarity of j, the same statement is true in H_κ^{V} about M. Let $\bar{M}$ and k witness this. Then $\varphi(\bar{M})$ holds and $k : \bar{M} \prec M$, as wished. □

As a result, it follows that UR_Γ implies $\mathsf{FA}(\Gamma)$. Tsaprounis observed in [Tsa15, Corollary 2.6] that if Γ is a forcing class that is (Γ-necessarily) stationary set preserving, then UR_Γ implies the stronger forcing axiom $\mathsf{FA}^{++}(\Gamma)$, which says that given a poset $\mathbb{P} \in \Gamma$, a collection $\mathcal{A}$ of ω_1 many maximal antichains in $\mathbb{P}$ and ω_1 many names for stationary subsets of ω_1, there is an $\mathcal{A}$-generic filter in $\mathbb{P}$ which interprets each of these names as a stationary set. The main goal of the remainder of this section is now to show that the results from the previous section on the effects of $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})/\ \underset{\sim}{\mathsf{RA}}_{\sigma\text{-closed}}(H_{\omega_2})$ on stationary reflection and the failure of (Todorčević's) square carry over to higher cardinalities.

Lemma 4.4. *Assume $\underset{\sim}{\mathsf{RA}}_{\sigma\text{-closed}}(H_\kappa)$, where $\kappa > \omega_1$ is a regular cardinal. Then any sequence $\vec{S} = \langle S_i \mid i < \omega_1\rangle$ of stationary subsets of κ consisting of ordinals of countable cofinality reflects.*

Proof. Let $\vec{S}$ be given. Let $M = \langle H_\kappa, \tilde{S}\rangle$, where $\tilde{S} = \bigcup_{i<\omega_1}\{i\} \times S_i$. Let G be V-generic for $\mathrm{Col}(\omega_1, \kappa)$. Let $\mathbb{Q}$ be σ-closed and H be $\mathbb{Q}$-generic over $\mathrm{V}[G]$ such that in $\mathrm{V}[G][H]$, there is a model $N = \langle H_\lambda^{\mathrm{V}[G][H]}, \tilde{T}\rangle$ with $\mathrm{cf}(\lambda) > \omega_1$ and an elementary embedding $j : M \prec N$, by $\underset{\smile}{\mathsf{RA}}_{\sigma\text{-closed}}(H_\kappa)$. As pointed after Observation 2.3, the minimum requirement needed for arguments such as the present one seems to be that $\mathrm{cf}(\kappa) > \omega_1 \implies \mathrm{cf}^{\mathrm{V}[G][H]}(\lambda) > \omega_1$. Definition 2.2 actually gives us that λ is regular in $\mathrm{V}[G][H]$. Clearly, $j(\omega_1) = \omega_1$, since $\omega_1^M = \omega_1^N$.

In $\mathrm{V}[G][H]$, each S_i is stationary in κ, by Fact 3.9. Let $\theta = \sup j``\kappa$. Since $\mathrm{cf}^{\mathrm{V}[G][H]}(\lambda) > \omega_1$, yet $\mathrm{cf}^{\mathrm{V}[G][H]}(\kappa) = \omega_1$, it follows that $\theta < \lambda$. Fixing $i < \omega_1$, I claim that T_i reflects to θ in N (and, equivalently, in $\mathrm{V}[G][H]$). To see this, argue in $\mathrm{V}[G][H]$. Let $E \subseteq \theta$ be club. $j``\kappa$ is stationary in $\theta$, because $j``\kappa$ is closed under limits of cofinality ω (note that V and $\mathrm{V}[G][H]$ have the same ω-sequences of ordinals). So, $j``\kappa \cap E$ is stationary in $\theta$, and also closed under limits of cofinality $\omega$. So $\bar{E} = j^{-1}``E$ is unbounded in κ and closed under limits of cofinality ω. So, if we let $\bar{E}'$ be the union of $\bar{E}$ and its limit points below κ, $\bar{E}'$ is club in κ, and the limit points missing in $\bar{E}$ had uncountable cofinality. Let $\xi \in S_i \cap \bar{E}'$. Since S_i consists of ordinals of cofinality ω, it follows that $\xi \in \bar{E}$, so $j(\xi) \in T_i \cap E$. This shows that T_i reflects to θ.

So in N, the statement "there is a γ of cofinality ω_1 such that each T_i reflects to γ" is true, as witnessed by θ. Hence, by elementarity, M believes that there is a $\bar{\kappa}$ of cofinality ω_1 such that for every $i < \omega_1$, S_i reflects to $\bar{\kappa}$. Since H_κ contains every subset of $\bar{\kappa}$, M is right about that. □

Note that the assumption of the previous lemma could be weakened to $\mathrm{cf}(\kappa) > \omega_1$, but this is not interesting, because in that case, already $\underset{\smile}{\mathsf{RA}}_{\sigma\text{-closed}}(H_{\mathrm{cf}(\kappa)})$ implies the claimed stationary reflection principle. Namely, $\underset{\smile}{\mathsf{RA}}_{\sigma\text{-closed}}(H_{\mathrm{cf}(\kappa)})$ implies by Lemma 4.4 that any ω_1-sequence of stationary subsets of $\mathrm{cf}(\kappa)$, each consisting of ordinals of countable cofinality, reflects. But given an ω_1-sequence of such stationary subsets of κ, fixing a normal function $f : \mathrm{cf}(\kappa) \longrightarrow \kappa$ and letting $C = \{f(\gamma) \mid \gamma < \mathrm{cf}(\kappa) \text{ is a limit ordinal}\}$, one can reflect an ω_1-sequence $\langle T_\alpha \mid \alpha < \omega_1\rangle$ where each T_α is a stationary subset of κ consisting of ordinals of countable cofinality to a sequence $\langle S_\alpha \mid \alpha < \omega_1\rangle$, where $S_\alpha = f^{-1}``(T_\alpha \cap C)$. Each S_α is then a stationary subset of $\mathrm{cf}(\kappa)$, consisting of ordinals of countable cofinality, and hence $\vec{S}$ has a reflection point. The image of this point under f is then a simultaneous reflection point for $\vec{T}$.

In the next lemma, I show that the principle SFP_κ, which implies the simultaneous reflection of Lemma 4.4 (as pointed out after Definition 3.12), already follows from the assumption $\mathsf{w}\underset{\smile}{\mathsf{RA}}_{\mathsf{SC}}(H_\kappa)$.

Lemma 4.5. *Suppose that $\kappa > \omega_1$ is regular and $\mathsf{w}\underset{\smile}{\mathsf{RA}}_{\mathsf{SC}}(H_\kappa)$ holds. Then SFP_κ holds.*

Proof. Even though the proof works for the case $\kappa = \omega_2$ as well, the reader may think of the case $\kappa > \omega_2$ here, since the case $\kappa = \omega_2$ has been dealt with in Lemma 3.14. Let $\langle A_i \mid i < \omega_1\rangle$ be a sequence of stationary subsets of κ consisting of points of cofinality ω. Let $\langle D_i \mid i < \omega_1\rangle$ be a partition of ω_1 into stationary subsets. As in the proof of Lemma 3.14, let G be V-generic for the subcomplete forcing $\mathbb{P}$ to add a normal, cofinal function $f : \omega_1 \longrightarrow \kappa$ such that for every $i < \omega_1$, $f``D_i \subseteq A_i$, followed by the collapse of κ to ω_1. Let $M = \langle H_\kappa, \in, \langle A_i \mid i < \omega_1\rangle, \langle D_i \mid i < \omega_1\rangle\rangle$. By $\mathsf{w}\underset{\smile}{\mathsf{RA}}_{\mathsf{SC}}(H_\kappa)$, let $\mathbb{Q} \in \mathrm{V}[G]$ be a poset, let H be $\mathrm{V}[G]$-generic for $\mathbb{Q}$, and let

$$j : M \prec N = \langle H_\lambda^{\mathrm{V}[G][H]}, \in, \langle B_i \mid i < \omega_1\rangle, \langle \tilde{D}_i \mid i < \omega_1\rangle\rangle$$

be an elementary embedding in $\mathrm{V}[G][H]$ such that $j(\omega_1) = \omega_1$. Clearly, $D_i = \tilde{D}_i$, for $i < \omega_1$. Let $f' = j \circ f$. Since f is continuous, and since j is continuous at ordinals of cofinality ω (in H_κ), it follows that f' is continuous. Moreover, if $\xi \in D_i$, then $f(\xi) \in A_i$, and so, $f'(\xi) \in B_i$, by elementarity of j. Since λ is regular in $\mathrm{V}[G][H]$ (and it would be enough to know that $\mathrm{cf}(\lambda) > \omega_1$ in $\mathrm{V}[G][H]$), f' is in $H_\lambda^{\mathrm{V}[G][H]}$, and so, the statement that there exists an ordinal λ' and a normal function $h : \omega_1 \longrightarrow \lambda'$ such that for every $i < \omega_1$, $h``D_i \subseteq B_i$ is true in N, and hence, the corresponding statement is true in M, with B_i replaced by A_i. $\square$

Finally, I will replicate the results on the failure of square principles at higher cardinalities.

Lemma 4.6. *Countably closed forcing cannot add a thread to a coherent sequence of length λ all of whose elements have size less than 2^ω, if λ is an ordinal of uncountable cofinality.*

Proof. Let $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \lambda \rangle$ be a $\square(\lambda, {<}2^\omega)$-sequence, and suppose $\mathbb{P}$ is σ-closed, yet $\mathbb{P}$ adds a thread to $\vec{\mathcal{C}}$. Let $p \in \mathbb{P}$ force that $\dot{D}$ is a thread. By recursion on the length of $s \in {}^{<\omega}2$, one can define $p_s \in \mathbb{P}$, $\alpha_s, \delta_s < \lambda$, such that

1. $p_\emptyset \leq p$,
2. $p_{s^\frown 0} \Vdash \check{\delta}_s \in \dot{D}$,
3. $p_{s^\frown 1} \Vdash \check{\delta}_s \notin \dot{D}$,
4. $p_{s^\frown 0}, p_{s^\frown 1} \leq p_s$,
5. $p_s \Vdash \check{\alpha}_s \in \dot{D}$,
6. if $|s| < |t|$, then $\alpha_s < \alpha_t$, and if $|s| + 1 < |t|$, then $\delta_s < \alpha_t$.

The recursive construction proceeds as follows. At stage n, I assume that p_s and α_s have been defined for all s with $|s| < n$, and δ_s has been defined for all s with $|s| + 1 < n$. I will then define p_s and α_s for all s with $|s| = n$ and δ_s for all s with $|s| = n - 1$ (if $n > 0$).

Thus, at stage 0, I have to define $p_\emptyset$ and $\alpha_\emptyset$. To do this, let $p_\emptyset$ and $\alpha_\emptyset$ be such that $p_\emptyset \leq p$ and $p_\emptyset \Vdash \check{\alpha}_\emptyset \in \dot{D}$.

At stage $n > 0$ of the construction, making the assumptions listed above, let $|t| = n$, and let $s = t{\restriction}(n-1)$. Let $\gamma_{n-1} = \max\{\sup_{|u|<n} \alpha_u, \sup_{|u|+1<n} \delta_u\}$. First, it is clear that there is a δ such that for some extensions $p'_{s^\frown 0}$ and $p'_{s^\frown 1}$ of p_s, we have that $p'_{s^\frown 0} \Vdash \check{\delta} \in \dot{D}$ and $p'_{s^\frown 1} \Vdash \check{\delta} \notin \dot{D}$, since we are working below a condition which forces that $\dot{D} \notin \check{V}$. Let δ_s be such a δ. Since we are working below a condition that forces that $\dot{D}$ is unbounded in λ, it is now clear that there are ordinals $\alpha_{s^\frown 0}$ and $\alpha_{s^\frown 1}$, both greater than γ_{n-1}, such that for appropriate strengthenings $p_{s^\frown 0}$ and $p_{s^\frown 1}$ of $p'_{s^\frown 0}$ and $p'_{s^\frown 1}$, respectively, we have that $p_{s^\frown i} \Vdash \check{\alpha}_{s^\frown i} \in \dot{D}$, for $i = 0, 1$. This concludes the construction at stage n.

For each $f : \omega \longrightarrow 2$, let $p_f \in \mathbb{P}$ be a lower bound for the decreasing sequence $\langle p_{f{\restriction}n} \mid n < \omega \rangle$. Let $\alpha_\omega = \sup_{s \in {}^{<\omega}2} \alpha_s$. Then $\alpha_\omega < \lambda$, since $\mathrm{cf}(\lambda) \geq \omega_1$. Moreover, p_f forces that α_ω is a limit point of $\dot{D}$, because by 6., $\alpha_\omega = \sup_{n<\omega} \alpha_{f{\restriction}n}$ and by 5., each $\alpha_{f{\restriction}n}$ is forced by p_f to be in $\dot{D}$. Thus, p_f forces that $\dot{D} \cap \check{\alpha}_\omega \in \check{\mathcal{C}}_{\alpha_\omega}$. Hence, since there are 2^ω many functions from ω to 2, there must be $f \neq g$ such that p_f and p_g both force that $\dot{D} \cap \check{\alpha}_\omega = \check{C}$, for some $C \in \mathcal{C}_{\alpha_\omega}$. But, letting n be maximal such that $s := f{\restriction}n = g{\restriction}n$, it follows that $p_{f{\restriction}n+1}$ and $p_{g{\restriction}n+1}$ disagree about whether δ_s is in $\dot{D}$, while $\delta_s < \alpha_\omega$, by 6. This is a contradiction. $\square$

Theorem 4.7. *Let Γ be the class of countably closed forcings or the class of subcomplete forcings, let κ be a cardinal of cofinality greater than ω_1, and assume that $\underset{\sim}{\mathsf{RA}}_\Gamma(H_\kappa)$ holds. Then $\square(\kappa, \omega)$ fails.*

Proof. Assume, towards a contradiction, that there is a $\square(\kappa, \omega)$-sequence, let's call it $\vec{\mathcal{C}}$. Let g be $\mathrm{Col}(\omega_1, \kappa)$-generic. Now, apply $\underset{\sim}{\mathsf{RA}}_\Gamma(H_\kappa)$ to the structure $\langle H^{\mathrm{V}}_\kappa, \in, \vec{\mathcal{C}} \rangle$, where we can easily view $\vec{\mathcal{C}}$ as a subset of κ, let's say for concreteness, we identify it with the set of Gödel triples $\prec\alpha, \xi, n\succ$ such that $\xi \in C^n_\alpha$, where for every $\alpha < \kappa$, we fix an enumeration $\mathcal{C}_\alpha = \{C^n_\alpha \mid n < \omega\}$. So the principle gives us a forcing $\mathbb{Q}$ that's subcomplete/countably closed in $\mathrm{V}[g]$, such that, letting h be generic for $\mathbb{Q}$ over $\mathrm{V}[g]$, in $\mathrm{V}[g][h]$, there is a $\vec{\mathcal{D}}$, a cardinal λ and an elementary embedding

$$j : \langle H_\kappa, \in, \vec{\mathcal{C}}\rangle \prec \langle H_\lambda^{\mathrm{V}[g][h]}, \in, \vec{\mathcal{D}}\rangle$$

with $\mathrm{cf}^{\mathrm{V}[g][h]}(\lambda) > \omega_1$. Recall that if κ is regular, we are guaranteed that λ is regular in $\mathrm{V}[g][h]$, by Definition 2.2, and if κ is singular, then by Observation 2.4, we can assume that $j(\mathrm{cf}(\kappa)) = \mathrm{cf}^{\mathrm{V}[g][h]}(\lambda)$. Note also that ω_1 is the same in V, $\mathrm{V}[g][h]$, M and N, so that, as usual, $j(\omega_1) = \omega_1$.

(1) *$\vec{\mathcal{C}}$ is a $\square(\kappa, \omega)$-sequence in* $\mathrm{V}[g][h]$.

Proof of (1). By Lemma 4.6, $\vec{\mathcal{C}}$ is still a $\square(\kappa, \omega)$-sequence in $\mathrm{V}[g]$. But since $\mathrm{cf}^{\mathrm{V}[g]}(\kappa) = \omega_1$, Lemma 3.22 applies, so since $\mathbb{Q}$ is subcomplete in $\mathrm{V}[g]$, $\vec{\mathcal{C}}$ is still a $\square(\kappa, \omega)$-sequence in $\mathrm{V}[g][h]$. $\square_{(1)}$

Let $\theta = \sup j``\kappa$. So $\mathrm{cf}^{\mathrm{V}[g][h]}(\theta) = \omega_1$. Since $\mathrm{cf}^{\mathrm{V}[g][h]}(\lambda) > \omega_1$, it follows that $\theta < \lambda$.

(2) *$j``\kappa$ is stationary in θ (in* $\mathrm{V}[g][h]$*).*

Proof of (2). Arguing in $\mathrm{V}[g][h]$, let C be a club subset of θ. By recursion on n, define an increasing sequence $\vec{\alpha} = \langle \alpha_n \mid n < \omega\rangle$ in ${}^\omega\kappa$ such that, for every $n < \omega$, $[j(\alpha_n), j(\alpha_{n+1})) \cap C \neq \emptyset$. Since $\mathrm{cf}(\kappa) = \omega_1$, this sequence is bounded in κ. So $\alpha_\omega := \sup_{n<\omega} \alpha_n < \kappa$. Now in $\mathrm{V}[g]$, there is a bijection $f : \omega_1 \longrightarrow \kappa$. The pullback $\langle f^{-1}(\alpha_n) \mid n < \omega\rangle$ is then a sequence in $({}^\omega\omega_1)^{\mathrm{V}[g][h]}$, and hence is bounded in ω_1, say by β. But then, it can be coded by a real via a surjection $c : \omega \longrightarrow \beta$ with $c \in \mathrm{V}[g]$. That real is in $\mathrm{V}[g][h]$, and since subcomplete forcing doesn't add reals, it and the pullback it coded are in $\mathrm{V}[g]$. Since f is in $\mathrm{V}[g]$, it follows that $\vec{\alpha} \in \mathrm{V}[g]$. But since g is generic for a countably closed forcing, it follows that $\vec{\alpha} \in \mathrm{V}$, and hence that $\vec{\alpha} \in H_\kappa^{\mathrm{V}}$. It follows that H_κ sees that α_ω has cofinality ω, and as a result, $j(\alpha_\omega) = \sup_{n<\omega} j(\alpha_n)$, and this is a limit point of C, by construction. $\square_{(2)}$

Working in $\mathrm{V}[g][h]$, let $D \in \mathcal{D}_\theta$, and let D' be the set of limit points of D below θ. So by (2), $S = j``\kappa \cap D'$ is unbounded (stationary) in $\theta$. So, letting $\bar{S} = j^{-1}``S$, it follows that $\bar{S} = \{\alpha < \kappa \mid j(\alpha) \in D'\}$ is unbounded in κ. Now, for every $\alpha \in \bar{S}$, $D \cap j(\alpha) \in \mathcal{D}_{j(\alpha)} = j(\mathcal{C}_\alpha)$. For such α, let $n(\alpha)$ be such that

$$D \cap j(\alpha) = j(C_\alpha^{n(\alpha)})$$

and define

$$\bar{D} = \bigcup_{\alpha \in \bar{S}} C_\alpha^{n(\alpha)}.$$

(3) *$\bar{D}$ threads $\vec{\mathcal{C}}$.*

Proof of (3). First, note that if $\alpha < \beta$ with $\alpha, \beta \in \bar{S}$, then $C_\alpha^{n(\alpha)} = C_\beta^{n(\beta)} \cap \alpha$, because $D \cap j(\alpha) = j(C_\alpha^{n(\alpha)})$ and $D \cap j(\beta) = j(C_\beta^{n(\beta)})$, so that $j(C_\alpha^{n(\alpha)}) = j(C_\beta^{n(\beta)}) \cap j(\alpha)$. Applying j^{-1} yields that $C_\alpha^{n(\alpha)} = C_\beta^{n(\beta)} \cap \alpha$.

This implies that if $\alpha \in \bar{S}$, then $C_\alpha^{n(\alpha)} = \bar{D} \cap \alpha$. So, $\bar{D}$ is club in κ, and if β is a limit point of $\bar{D}$ below κ, then β is a limit point of $C_\alpha^{n(\alpha)}$, for some $\alpha \in \bar{S}$. Since $\vec{\mathcal{C}}$ is coherent, it follows that $\bar{D} \cap \beta = C_\alpha^{n(\alpha)} \cap \beta \in \mathcal{C}_\beta$, which means that $\bar{D}$ threads $\vec{\mathcal{C}}$. $\square_{(3)}$

This is a contradiction, since $\vec{\mathcal{C}}$ is a $\square(\kappa, \omega)$-sequence in $\mathrm{V}[g][h]$. $\square$

Observation 4.8. $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_3})/\underset{\sim}{\mathsf{RA}}_{\sigma\text{-}closed}(H_{\omega_3})$ *implies* $\mathsf{AD}^{L(\mathbb{R})}$.

Proof. This follows by assembling some consequences of these principles that were shown up to now. Recall that $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_3})/\underset{\sim}{\mathsf{RA}}_{\sigma\text{-closed}}(H_{\omega_3})$ imply $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})/\underset{\sim}{\mathsf{RA}}_{\sigma\text{-closed}}(H_{\omega_2})$, by the discussion after Observation 2.3. By Fact 3.1, $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})/\underset{\sim}{\mathsf{RA}}_{\sigma\text{-closed}}(H_{\omega_2})$ imply CH (even their lightface variants imply $\diamondsuit$). Moreover, $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_2})/\underset{\sim}{\mathsf{RA}}_{\sigma\text{-closed}}(H_{\omega_2})$ imply the failure of $\square(\omega_2, \omega)$ by Theorem 3.23, which certainly implies the failure of $\square(\omega_2)$. Finally, $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_3})/\underset{\sim}{\mathsf{RA}}_{\sigma\text{-closed}}(H_{\omega_3})$ imply the failure of $\square(\omega_3, \omega)$ (by Theorem 4.7), and in particular, the failure of $\square_{\omega_2}$ (see the discussion following Definition 3.18). The claim now follows from Steel's observation that the methods of proof of [Sch07], combined with Woodin's core model induction, show the following fact: if $2^\omega \le \omega_2$, $\square(\omega_2)$ fails and $\square_{\omega_2}$ fails, then $L(\mathbb{R})$ determinacy holds (see [Sch07, P. 90]). $\square$

5 Virtual resurrection

In analogy to the hierarchies of the weak forcing axioms, introduced in [Fuc16a], I now want to analyze a similar weakening of the resurrection axioms for higher cardinals. The resulting axioms will be much weaker, in particular, they will not have the striking effects on stationary reflection and the failure of square discussed in the previous section. On the upside, it will be possible to determine their consistency strengths precisely. The idea is to ask only that the elementary resurrection embeddings are generic embeddings, or virtual embeddings, meaning that they are only required to exist in a further forcing extension.

Definition 5.1. Let $\kappa \ge \omega_2$ be a cardinal, and let Γ be a forcing class. The *virtual resurrection axiom for* Γ *at* H_κ, $\mathsf{vRA}_\Gamma(H_\kappa)$, says that whenever G is generic over V for some forcing $\mathbb{P} \in \Gamma$, then there are a $\mathbb{Q} \in \Gamma^{\mathrm{V}[G]}$ and a λ such that whenever H is $\mathbb{Q}$-generic over $\mathrm{V}[G]$, then λ is a cardinal in $\mathrm{V}[G][H]$, and there is some further forcing $\mathbb{R} \in \mathrm{V}[G][H]$ such that if I is generic for $\mathbb{R}$ over $\mathrm{V}[G][H]$, then in $\mathrm{V}[G][H][I]$, there is an elementary embedding

$$j : \langle H_\kappa^{\mathrm{V}}, \in\rangle \prec \langle H_\lambda^{\mathrm{V}[G][H]}, \in\rangle$$

I will call such an embedding *virtual*.

The *boldface virtual resurrection axiom for* Γ *at* H_κ, $\mathsf{v}\underset{\sim}{\mathsf{RA}}_\Gamma(H_\kappa)$, says that for every $A \subseteq \kappa$ and every G as above, there are a $\mathbb{Q}$ and a λ as above such that for every H as above, there are a $B \in \mathrm{V}[G][H]$ and an $\mathbb{R}$ as above such that for every I as above, there is a j in $\mathrm{V}[G][H][I]$ such that

$$j : \langle H_\kappa^{\mathrm{V}}, \in, A\rangle \prec \langle H_\lambda^{\mathrm{V}[G][H]}, \in, B\rangle$$

and such that, if κ is regular in V, then λ is regular in $\mathrm{V}[G][H]$.

Finally, the *virtual unbounded resurrection axiom* vUR_Γ says that $\mathsf{vRA}_\Gamma(H_\kappa)$ holds for every cardinal $\kappa \ge \omega_2$.

I will frequently say that in some transitive model N of ZFC^-, containing structures M and M', there is a virtual embedding $j : M \prec M'$. This is just a shorthand for saying that N thinks that there is a poset $\mathbb{P}$ such that $\mathbb{P}$ forces the existence of such an elementary embedding.

Note that there is no requirement on the forcing notion $\mathbb{R}$ adding the embedding j - any forcing can be used. A little more can be said about it, though. First, if we are in the situation that M and N are models of the same first order language, and the universe of M is countable, then fixing an enumeration $f : \omega \longrightarrow M$, there is a canonical "tree searching for an elementary embedding from M to N". It is the tree consisting of functions $g : \{f(0), f(1), \ldots, f(n-1)\} \longrightarrow N$ such that $n < \omega$ and for any formula $\varphi(\vec{x})$ and any list of parameters $\vec{a}$ from $f``n$, we have that $M \models \varphi(\vec{a})$ iff $N \models \varphi(g(\vec{a}))$. So these are the functions that might be extended to an elementary embedding from M to N. The tree ordering is inclusion. It is now clear that T is ill-founded iff there is an elementary embedding from M to N. Now, let us drop the assumption that M is countable for a moment, let's say that M has size θ, and let's assume that there is some forcing $\mathbb{P}$ that adds an elementary embedding from M to N. Let G be generic over V for $\mathbb{P}$. If we let H be $\mathrm{Col}(\omega,\theta)$-generic over $\mathrm{V}[G]$, then we can form the tree searching for an elementary embedding from M to N (with respect to some enumeration of M by natural numbers) in $\mathrm{V}[H]$. Since there is such an embedding in $\mathrm{V}[H][G] = \mathrm{V}[G][H]$ (by the product lemma), it follows that this tree is ill-founded in $\mathrm{V}[H][G]$, and hence in $\mathrm{V}[H]$. A branch through this tree in $\mathrm{V}[H]$ gives rise to such an embedding in $\mathrm{V}[H]$. So, by the weak homogeneity of $\mathrm{Col}(\omega,\theta)$, it is forced by the weakest condition $1_{\mathrm{Col}(\omega,\theta)}$ that there is an elementary embedding from M to N, since this is a statement about elements of the ground model. Let's make a note of this, for future reference.

Observation 5.2. *Let M and N be models of the same first order language. If there is a forcing notion that adds an elementary embedding from M to N, then $\mathrm{Col}(\omega,\theta)$ adds such an elementary embedding, where θ is the cardinality of the universe of M.*

As before, the classes Γ I am interested in Γ-necessarily preserve ω_1, in which case it follows that $j\restriction H_{\omega_2}$ is the identity, where j is as in the previous definition. In particular, if $\kappa=\omega_2$, then j is the identity, and hence, no forcing is required to add the embedding. Let's also note this as an observation.

Observation 5.3. *Suppose that Γ is Γ-necessarily ω_1-preserving. Then $\mathsf{RA}_\Gamma(H_{\omega_2})$ is equivalent to $\mathsf{vRA}_\Gamma(H_{\omega_2})$, and $\underset{\sim}{\mathsf{RA}}_\Gamma(H_{\omega_2})$ is equivalent to $\mathsf{v}\underset{\sim}{\mathsf{RA}}_\Gamma(H_{\omega_2})$.*

As before, the requirement in Definition 5.1 that λ be regular if κ is, is a technical detail. The proof of Observation 2.4 goes through in the present context verbatim.

Observation 5.4. *Suppose κ is a singular cardinal and $\mathsf{v}\underset{\sim}{\mathsf{RA}}_\Gamma(H_\kappa)$ holds. Then for every $A\subseteq H_\kappa$ and every G generic for a forcing in Γ, there is a $\mathbb{Q}\in\Gamma^{\mathrm{V}[G]}$ such that if H is generic for $\mathbb{Q}$ over $\mathrm{V}[G]$, then in $\mathrm{V}[G][H]$, there are a B, a cardinal λ and a virtual elementary embedding j such that*

$$j:\langle H_\kappa^{\mathrm{V}},\in,A\rangle\prec\langle H_\lambda^{\mathrm{V}[G][H]},\in,B\rangle,$$

with $j(\mathrm{cf}^{\mathrm{V}}(\kappa))=\mathrm{cf}^{\mathrm{V}[G][H]}(\lambda)$.

The corresponding large cardinals are "virtual" strengthenings of the concept of (strongly) uplifting cardinals. Once they are strengthened, though, it becomes apparent that the correct terminology has to be phrased in terms of extendibility. The embeddings witnessing extendibility, however, are not required to exist in V but in a forcing extension. Hence the following definition. I use the notation $\kappa^{+\alpha}$ for the α-th cardinal successor of κ.

Definition 5.5. Let κ be an inaccessible cardinal and α an ordinal. Then κ is *virtually super α-extendible* if there are arbitrarily large inaccessible cardinals γ such that for some β, there is an elementary embedding j in $\mathrm{V}^{\mathrm{Col}(\omega,H_{\kappa^{+\alpha}})}$ such that

$$j:\langle H^{\mathrm{V}}_{\kappa^{+\alpha}},\in,\kappa\rangle\prec\langle H^{\mathrm{V}}_{\gamma^{+\beta}},\in,\gamma\rangle$$

where $j\restriction\kappa=\mathrm{id}$ (equivalently, $j\restriction H_\kappa=\mathrm{id}$). Here, κ and γ are used as predicates in these structures, and it follows that $\kappa=\mathrm{crit}(j)$ and $j(\kappa)=\gamma$ if $\alpha>0$.

κ is *strongly virtually super α-extendible* if for every $A\subseteq\kappa^{+\alpha}$, there are arbitrarily large inaccessible cardinals γ such that for some β and some $B\subseteq H_{\gamma^{+\beta}}$ (in V), there is an elementary embedding j in $\mathrm{V}^{\mathrm{Col}(\omega,H_{\kappa^{+\alpha}})}$ such that

$$j:\langle H^{\mathrm{V}}_{\kappa^{+\alpha}},\in,A,\kappa\rangle\prec\langle H^{\mathrm{V}}_{\gamma^{+\beta}},\in,B,\gamma\rangle$$

with $j\restriction\kappa = \mathrm{id}$, and such that, if $\kappa^{+\alpha}$ is regular, then $\gamma^{+\beta}$ is regular.

κ is virtually super $<\alpha$-extendible if it is virtually super $\bar{\alpha}$-extendible for every $\bar{\alpha} < \alpha$.

Note that κ is (strongly) uplifting iff κ is (strongly) virtually super 0-extendible. Note also that if $\alpha < \alpha'$ and κ is (strongly) virtually super α'-extendible, then κ is (strongly) virtually super α-extendible. In the future, I will omit the superscript V on $H_{\kappa^{+\alpha}}$, in the situation of the previous definition. The structures are understood to be in V, and only the elementary embedding j is added by forcing.

The concept of virtual extendibility was introduced in [BGS17], as follows.

Definition 5.6. An inaccessible cardinal κ is *virtually extendible* iff for every $\alpha > \kappa$, in some forcing extension of V, there is an elementary embedding $j : \mathrm{V}_\alpha^{\mathrm{V}} \prec \mathrm{V}_\beta^{V}$ such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.

Observation 5.7. *An inaccessible cardinal κ is virtually extendible iff it is virtually super α-extendible, for every ordinal α.*

Proof. From left to right, note first that if κ is virtually extendible, then there are arbitrarily large inaccessible cardinals, since if $j : \mathrm{V}_\alpha \prec \mathrm{V}_\beta$ is a virtual extendibility embedding, then $j(\kappa) > \alpha$ is inaccessible in V. So let α be given. To see that κ is virtually super α-extendible, let $\bar{\gamma}$ be some ordinal. We have to find an inaccessible $\gamma > \bar{\gamma}$ as in Definition 5.5. Let θ be an inaccessible cardinal greater than $\kappa^{+\alpha}$ and $\bar{\gamma}$, and let $j : \mathrm{V}_\theta \prec \mathrm{V}_{\theta'}$ be a virtual extendibility embedding. So j exists in some forcing extension of V, $\kappa = \mathrm{crit}(j)$, and $j(\kappa) > \theta$. Clearly then, $H_{\kappa^{+\alpha}} \in \mathrm{V}_\theta$. So if we let $j' = j\restriction H_{\kappa^{+\alpha}}$, $\beta = j(\alpha)$ and $\gamma = j(\kappa)$, then we get $j' : \langle H_{\kappa^{+\alpha}}, \in, \kappa\rangle \prec \langle H_{\gamma^{+\beta}}, \in, \gamma\rangle$, $j'\restriction\kappa = \mathrm{id}$ and $j'(\kappa) = \gamma > \bar{\gamma}$, and γ is inaccessible.

For the converse, let $\alpha > \kappa$ be given. Let $\alpha' > \alpha$ be inaccessible, $\gamma > \alpha'$ be inaccessible, and let $j : \langle H_{\kappa^{+\alpha'}}, \in, \kappa\rangle \prec \langle H_{\gamma^{+\beta'}}, \in, \gamma\rangle$ be a virtual super α'-extendibility embedding with $j\restriction\kappa = \mathrm{id}$, existing in some forcing extension. Then $j(\kappa) = \gamma > \alpha$ and $H_{\kappa^{+\alpha'}} = \mathrm{V}_{\alpha'}$. We have that $\mathrm{V}_\alpha \in \mathrm{V}_{\alpha'}$ and letting $\beta = j(\alpha)$, it follows that $j(\mathrm{V}_\alpha) = \mathrm{V}_\beta$. Thus, if we let $j' = j\restriction\mathrm{V}_\alpha$, it follows that $j' : \mathrm{V}_\alpha \prec \mathrm{V}_\beta$, $\kappa = \mathrm{crit}(j')$ and $j'(\kappa) = \gamma > \alpha$, that is, j' is a virtual extendibility embedding, as desired. □

For the following lemma, note that for a cardinal λ, we have that $H_\lambda \in H_{\lambda^+}$ iff the cardinality of H_λ, which is $2^{<\lambda}$, is equal to λ. I will also frequently use the fact, which is not hard to see, that if κ is a regular uncountable cardinal, $\mathbb{P}$ is a forcing notion in H_κ and G is $\mathbb{P}$-generic over V (equivalently, over H_κ), then $H_\kappa^{\mathrm{V}}[G] = H_\kappa^{\mathrm{V}[G]}$.

Lemma 5.8. *Suppose that κ is virtually super $\alpha + 1$-extendible, where $\alpha < \kappa$. Suppose that $H_{\kappa^{+\alpha}} \in H_{\kappa^{+\alpha+1}}$ or that $\kappa^{+\alpha}$ is regular. Then the set of $\bar{\kappa} < \kappa$ that are strongly virtually super α-extendible is stationary in κ.*

Proof. Let $C \subseteq \kappa$ be club. Let $j : \langle H_{\kappa^{+\alpha+1}}, \in, \kappa\rangle \prec \langle H_{\gamma^{+\alpha+1}}, \in, \gamma\rangle$, for some arbitrary inaccessible $\gamma > \kappa^{+\alpha+1}$, where $j\restriction\kappa = \mathrm{id}$ and j exists in a set-forcing extension $\mathrm{V}[G]$ of V obtained by forcing with $\mathrm{Col}(\omega, \theta)$, where θ is the cardinality of $H_{\kappa^{+\alpha+1}}$ in V, by Observation 5.2. In $\mathrm{V}[G]$, j has cardinality ω, since the domain of j is countable there. Thus, $j \in H_{\gamma^{+\alpha+1}}^{\mathrm{V}[G]} = H_{\gamma^{+\alpha+1}}^{\mathrm{V}}[G]$. There is a name $\check{H}$ for $H_{\gamma^{+\alpha+1}}$, definable in $H_{\gamma^{+\alpha+1}}$, which is class-sized from the point of view of $H_{\gamma^{+\alpha+1}}$, namely $\check{H} = \{\langle\check{x}, 1\rangle \mid x \in H_{\gamma^{+\alpha+1}}^{\mathrm{V}}\}$. Since $H_{\gamma^{+\alpha+1}}$ is a ZFC^--model, the forcing theorem holds over it, for the language which allows the usage of $\check{H}$ as a predicate. Since $H_{\gamma^{+\alpha}}$ is definable in $H_{\gamma^{+\alpha+1}}$, one can refer to $H_{\gamma^{+\alpha}}$ in the forcing language over $H_{\gamma^{+\alpha+1}}$, by relativizing the definition of $H_{\gamma^{+\alpha}}$ to $\check{H}$.

Let $k = j \upharpoonright H_{\kappa^{+\alpha}}$. Now in $\mathrm{V}[G]$, for every subset $A \subseteq \kappa^{+\alpha^{\mathrm{V}}}$ in V, we have that $A \in H^{\mathrm{V}}_{\kappa^{+\alpha+1}}$, and there is a $B \in H_{j(\kappa)^{+\alpha+1}}$ such that $k : \langle H_{\kappa^{+\alpha}}, \in, A, \kappa\rangle \prec \langle H_{j(\kappa^{+\alpha})}, \in, B, \gamma\rangle$ is elementary, because this is true for $B = j(A)$. Moreover, $k \in H_{\gamma^{+\alpha+1}}[G]$, since the forcing is in $H_{\gamma^{+\alpha+1}}$. Furthermore, if $\kappa^{+\alpha}$ is regular, then so is $j(\kappa^{+\alpha})$. I want to check that these facts can be expressed in the forcing language over $H_{\gamma^{+\alpha+1}}$. This is clear if $H_{\kappa^{+\alpha}} \in H_{\kappa^{+\alpha+1}}$, because this implies by elementarity of j that $H_{\gamma^{+\alpha}} \in H_{\gamma^{+\alpha+1}}$. If not, then by assumption, we have that $\kappa^{+\alpha}$ is regular in V, so that $\langle H_{\kappa^{+\alpha}}, \in, A, \kappa\rangle$ is a ZFC^- model (in the language with a two extra predicate symbols that are interpreted by A and κ), and the corresponding fact is true of $\langle H_{j(\kappa)^{+\alpha}}, \in, B, \gamma\rangle$, which is definable in $H_{\gamma^{+\alpha+1}}$, by elementarity of j. It is well-known that in this situation it is sufficient to say that k is Σ_1-elementary, since this then implies that it is fully elementary. Thus, in both cases, the elementarity of k can be expressed in the forcing language of $H_{\gamma^{+\alpha+1}}$. So, there is a condition in $\mathrm{Col}(\omega, \theta)$ that forces that there is a k as described that works for A and B. By homogeneity, the trivial condition of $\mathrm{Col}(\omega, \theta)$ will already force this, since A and B and the two models between which k is an elementary embedding are in the ground model.

So in $H_{\gamma^{+\alpha+1}}$, the following statement holds: "There is a $\bar{\kappa} \in j(C)$ such that for every $A \subseteq \bar{\kappa}^{+\alpha}$, there is a $B \subseteq j(\kappa^{+\alpha})$ such that $\mathrm{Col}(\omega, H_{\bar{\kappa}^{+\alpha}})$ forces that there is a $j' : \langle H_{\bar{\kappa}^{+\alpha}}, \in, A, \bar{\kappa}\rangle \prec \langle H_{j(\kappa)^{+\alpha}}, \in, B, j(\kappa)\rangle$ with $j' \upharpoonright \bar{\kappa} = \mathrm{id}$ and such that if $\bar{\kappa}^{+\alpha}$ is regular, then so is $j(\kappa^{+\alpha})$".

This is witnessed by $\bar{\kappa} = \kappa$ and $j' = k$. By elementarity of j, the pulled back version of this statement is true from the point of view of $H_{\kappa^{+\alpha+1}}$: there is a $\bar{\kappa} \in C$ such that for every $A \subseteq H_{\bar{\kappa}^{+\alpha}}$, there is a $B \subseteq H_{\kappa^{+\alpha}}$ such that $\mathrm{Col}(\omega, H_{\bar{\kappa}^{+\alpha}})$ forces the existence of an elementary embedding $j' : \langle H_{\bar{\kappa}^{+\alpha}}, \in, A, \bar{\kappa}\rangle \prec \langle H_{\kappa^{+\alpha}}, \in, B, \kappa\rangle$ such that $j' \upharpoonright \bar{\kappa} = \mathrm{id}$ and such that if $\bar{\kappa}^{+\alpha}$ is regular, then so is $\kappa^{+\alpha}$.

But since κ also is strongly virtually super α-extendible (as follows from an argument given above), there are arbitrarily large inaccessible γ' for which there is a $B' \subseteq H_{\gamma'^{+\alpha}}$, such that if $\kappa^{+\alpha}$ is regular, then so is $\gamma'^{+\alpha}$ and such that there is a virtual embedding $j'' : \langle H_{\kappa^{+\alpha}}, \in, B, \kappa\rangle \prec \langle H_{\gamma'^{+\alpha}}, \in, B', \gamma'\rangle$ with $j'' \upharpoonright \kappa = \mathrm{id}$. The composition $j'' \circ j'$, which exists in a forcing extension of V by the product of the forcing to add j' with the forcing to add j'', then witnesses that $\bar{\kappa}$ is strongly virtually super α-extendible. □

Here are the weak versions of the virtual resurrection axioms.

Definition 5.9. Let $\kappa \geq \omega_2$ be a cardinal, and let Γ be a forcing class. The *virtual weak resurrection axiom for* Γ *at* H_κ, $\mathsf{vwRA}_\Gamma(H_\kappa)$, says that whenever G is generic over V for some forcing $\mathbb{P} \in \Gamma$, then there are a poset $\mathbb{Q} \in \mathrm{V}[G]$ and a λ such that whenever H is $\mathbb{Q}$-generic over $\mathrm{V}[G]$, then λ is a cardinal in $\mathrm{V}[G][H]$, and there is some further forcing $\mathbb{R} \in \mathrm{V}[G][H]$ such that if I is generic for $\mathbb{R}$ over $\mathrm{V}[G][H]$, then in $\mathrm{V}[G][H][I]$, there is an elementary embedding

$$j : \langle H^{\mathrm{V}}_\kappa, \in\rangle \prec \langle H^{\mathrm{V}[G][H]}_\lambda, \in\rangle$$

with $j \upharpoonright \omega_2 = \mathrm{id}$.

The *boldface virtual weak resurrection axiom for* Γ *at* H_κ, $\utilde{\mathsf{vwRA}}_\Gamma(H_\kappa)$, says that for every $A \subseteq \kappa$ and every G as above, there are a $\mathbb{Q}$ and a λ as above such that for every H as above, there are a $B \in \mathrm{V}[G][H]$ and an $\mathbb{R}$ as above such that for every I as above, there is a j in $\mathrm{V}[G][H][I]$ such that

$$j : \langle H^{\mathrm{V}}_\kappa, \in, A\rangle \prec \langle H^{\mathrm{V}[G][H]}_\lambda, \in, B\rangle$$

with $j \upharpoonright \omega_2 = \mathrm{id}$, and such that if κ is regular in V, then λ is regular in $\mathrm{V}[G][H]$.

Finally, the *virtual weak unbounded resurrection axiom* vwUR_Γ says that $\mathsf{vwRA}_\Gamma(H_\kappa)$ holds for every cardinal $\kappa \geq \omega_2$.

Concerning the requirement that $j\restriction\omega_2 = \mathrm{id}$, a remark similar to the one made after Definition 4.2 applies here. As a result of making this requirement, $\mathsf{wRA}_\Gamma(H_{\omega_2})$ and $\mathsf{vwRA}_\Gamma(H_{\omega_2})$ are equivalent, and so are their boldface counterparts, because the embedding j will have to be the identity, and thus, no forcing is required to add it. Note that both $\mathbb{P}$ and $\mathbb{Q}$ in the definition have to preserve ω_1, since otherwise, it would follow that $j(\omega_1) > \omega_1$. So if $\mathsf{vwRA}_\Gamma(H_\kappa)$ holds, then Γ must be ω_1-preserving, and the resurrecting forcing $\mathbb{Q}$ must be as well.

Lemma 5.10. *Let Γ be a forcing class that contains $\mathrm{Col}(\omega_1, \delta)$, for arbitrarily large δ. Then*

1. *$\mathsf{vwRA}_\Gamma(H_{\omega_{2+\alpha}})$ implies that ω_2 is virtually super α-extendible in L.*
2. *If $\mathrm{cf}(\omega_{2+\alpha}) > \omega$, then $\underset{\sim}{\mathsf{vwRA}}_\Gamma(H_{\omega_{2+\alpha}})$ implies that ω_2 is strongly virtually super α-extendible in L.* [1]

Proof. For 1., let $\bar{\gamma}$ be an arbitrary ordinal, and let G be generic for $\mathrm{Col}(\omega_1, \bar{\gamma}')$, for some $\bar{\gamma}' \geq \bar{\gamma}$ such that $\mathrm{Col}(\omega_1, \bar{\gamma}') \in \Gamma$. This will ensure that the virtual embedding we get from our assumption will shoot ω_2 above $\bar{\gamma}$, as will be explained below.

By $\mathsf{vwRA}_\Gamma(H_{\omega_{2+\alpha}})$, let H be generic for some $\mathbb{Q} \in \mathrm{V}[G]$, so that for some filter I in a further forcing notion $\mathbb{R} \in \mathrm{V}[G][H]$, generic over $\mathrm{V}[G][H]$, there are a β and a $j \in \mathrm{V}[G][H][I]$ such that

$$j : \langle H^{\mathrm{V}}_{\omega_{2+\alpha}}, \in\rangle \prec \langle H^{\mathrm{V}[G][H]}_{\omega_{2+\beta}}, \in\rangle$$

Since $j\restriction\omega_2$ is the identity, and since ω_2 is definable (as a subclass if $\alpha = 0$) in $H_{\omega_{2+\alpha}}$, and by the same definition, $\omega_2^{\mathrm{V}[G][H]}$ is definable (as a subclass if $\beta = 0$) in $H^{\mathrm{V}[G][H]}_{\omega_{2+\beta}}$, we get that, setting $\kappa = \omega_2$ and $\gamma = \omega_2^{\mathrm{V}[G][H]}$,

$$j : \langle H^{\mathrm{V}}_{\kappa^{+\alpha}}, \in, \kappa\rangle \prec \langle H^{\mathrm{V}[G][H]}_{\gamma^{+\beta}}, \in, \gamma\rangle$$

and $j\restriction\kappa = \mathrm{id}$. Note that $\gamma > \bar{\gamma}$, since $\bar{\gamma}$ was collapsed to ω_1 by G.

I claim that there are an ordinal β' and a $\tilde{j}$ in $\mathrm{V}[G][H][I]$ such that

$$\tilde{j} : \langle (H_{\kappa^{+\alpha}})^L, \in, \kappa\rangle \prec \langle (H_{\gamma^{+\beta'}})^L, \in, \gamma\rangle \text{ with } \tilde{j}\restriction\kappa = \mathrm{id}.$$

To see this, consider two cases. The first case is that $(\kappa^{+\alpha})^L < (\kappa^{+\alpha})^{\mathrm{V}}$. In this case, it follows that $j((H_{\kappa^{+\alpha}})^L) = (H_{\gamma^{+j(\alpha)}})^L$, and so, we can let $\beta' = j(\alpha)$ and $\tilde{j} = j\restriction(H_{\kappa^{+\alpha}})^L$. The second case is that $(\kappa^{+\alpha})^L = (\kappa^{+\alpha})^{\mathrm{V}}$. In that case, noting that $L^{(H_{\kappa^{+\alpha}})^{\mathrm{V}}} = (H_{\kappa^{+\alpha}})^L = L_{(\kappa^{+\alpha})^L}$, it follows that if we let $\tilde{j} = j\restriction L^{(H_{\kappa^{+\alpha}})^{\mathrm{V}}}$, then $\tilde{j} : \langle L^{(H_{\kappa^{+\alpha}})^{\mathrm{V}}}, \in, \kappa\rangle \prec \langle L^{H^{\mathrm{V}[G][H]}_{\gamma^{+\beta}}}, \in, \gamma\rangle$. But of course, $(\gamma^{+\beta})^{\mathrm{V}[G][H]} = (\gamma^{+\beta'})^L$, for some β', and then $L^{H^{\mathrm{V}[G][H]}_{\gamma^{+\beta}}} = L_{(\gamma^{+\beta'})^L} = (H_{\gamma^{+\beta'}})^L$, as desired.

Let J be generic for $\mathrm{Col}(\omega, (\kappa^{+\alpha})^L)$ over $\mathrm{V}[G][H][I]$. Then in $L[J]$, there is a tree T searching for such an elementary embedding (with respect to some enumeration of $H^L_{\kappa^{+\alpha}}$ by natural numbers, see the discussion after Definition 5.1 - here, the tree can easily be modified to search only for embeddings which are the identity below κ). This tree T is ill-founded in $\mathrm{V}[G][H][I][J]$, hence in $L[J]$, which shows that such an embedding exists in $L[J]$.

To see that κ satisfies the requirements of Definition 5.5 in L, it still has to be checked that κ and γ are are inaccessible cardinals in L. Clearly, $\kappa = \omega_2^{\mathrm{V}}$ is regular in L, so to see that κ is inaccessible in L, it suffices to show that it is a limit cardinal in L, since GCH holds in L. But if $\delta < \kappa$ were the largest cardinal of L below κ, then it would follow by elementarity that δ is the largest cardinal below γ in L, but κ is a cardinal in L and hence in $H^L_\gamma = L_\gamma$, a contradiction. So κ is inaccessible in L. Similarly, $\gamma = \omega_2^{\mathrm{V}[G][H]}$ is regular in L, and since $H^L_\kappa = L_\kappa$ believes

[1] I do not know whether the assumption that $\mathrm{cf}(\omega_{2+\alpha}) > \omega$ is necessary here.

that there are arbitrarily large cardinals, the same is true in $H^L_\gamma = L_\gamma$, by elementarity, and so, γ is a regular limit cardinal in L, hence inaccessible in L, again since GCH holds in L.

Let's now turn to 2. Let $\kappa = \omega_2$, and let $\kappa' = (\kappa^{+\alpha})^L$. Let $A \subseteq \kappa'$ be in L. As before, given $\bar\gamma$, let $\bar\gamma' \ge \bar\gamma$ be such that $\mathrm{Col}(\omega_1, \bar\gamma') \in \Gamma$, and let G be generic over V for this forcing notion. If $\kappa' < \omega_{2+\alpha}$, then $A \in H_{\omega_{2+\alpha}}$, and one can argue as in 1., the point being that the virtual embedding given by $\underset{\sim}{\mathsf{vwRA}}_\Gamma(H_{\omega_{2+\alpha}})$ can be applied to A, and it will move a κ', if it is regular in L, to a regular L-cardinal.

So let us assume that $\kappa' = \omega_{2+\alpha}$. Then $A \in L_\delta$, for some $\delta < ({\kappa'}^+)^L$, where we may assume that $L_\delta \models \mathsf{ZFC}^-$. There is a set $E \subseteq \kappa' \times \kappa'$ in L that codes L_δ, in the sense that $\langle \kappa', E\rangle$ is isomorphic to $\langle L_\delta, \in\rangle$. Namely, working in L, we may choose a bijection $f : \kappa' \longrightarrow L_\delta$ and set $\mu E \nu$ iff $f(\mu) \in f(\nu)$. Then $f : \langle \kappa', E\rangle \longrightarrow \langle L_\delta, \in\rangle$ is the Mostowski collapse of $\langle \kappa', E\rangle$. In this sense, E codes both L_δ and f. If $\alpha > 0$, then f may be chosen so that $f{\restriction}(\kappa+1) = \mathrm{id}$.

Let $\mathbb{Q} \in \mathrm{V}[G]$ be a poset, H generic for $\mathbb{Q}$ over $\mathrm{V}[G]$, $F \in \mathrm{V}[G][H]$, $\mathbb{R}$ a poset in $\mathrm{V}[G][H]$, I generic for $\mathbb{R}$, β an ordinal and $j \in \mathrm{V}[G][H][I]$ an elementary embedding

$$j : \langle H_{\omega_{2+\alpha}}, \in, E\rangle \prec \langle H^{\mathrm{V}[G][H]}_{\omega_{2+\beta}}, \in, F\rangle$$

such that $j{\restriction}\omega_2 = \mathrm{id}$ and such that if $\omega_{2+\alpha}$ is regular in V, then $\omega^{\mathrm{V}[G][H]}_{2+\beta}$ is regular in $\mathrm{V}[G][H]$. Thus, if $\alpha = 0$, then $j = \mathrm{id}$, and if $\alpha > 0$, then clearly, $j(\omega_2) = j(\kappa) = \omega_2^{\mathrm{V}[G][H]} > \bar\gamma$, and if we let $\gamma = \omega_2^{\mathrm{V}[G][H]}$ and $\gamma' = (\gamma^{+\beta})^{\mathrm{V}[G][H]}$, then it follows that

$$j : \langle H_{\kappa'}, \in, E, \kappa\rangle \prec \langle H_{\gamma'}, \in, F, \gamma\rangle$$

with $j{\restriction}\kappa = \mathrm{id}$, and we have that if κ' is regular, then γ' is regular in $\mathrm{V}[G][H]$.

By Observation 5.4, $(\omega_{2+\beta})^{\mathrm{V}[G][H]}$ can be assumed to have uncountable cofinality in $\mathrm{V}[G][H]$, since we assumed that $\mathrm{cf}(\omega_{2+\alpha}) > \omega$, and hence, $H^{\mathrm{V}[G][H]}_{\omega_{2+\beta}}$ is closed under countable sequences in $\mathrm{V}[G][H]$. By elementarity of j, it follows that F is extensional and well-founded in $H^{\mathrm{V}[G][H]}_{\omega_{2+\beta}}$, and hence it is well-founded in $\mathrm{V}[G][H]$. Let X be the transitive set coded by F, and let $g : \langle \omega^{\mathrm{V}[G][H]}_{2+\beta}, F\rangle \longrightarrow \langle X, \in\rangle$ be the Mostowski isomorphism. Since $\langle \omega^{\mathrm{V}[G][H]}_{2+\beta}, F\rangle$ is a ZFC^- model that believes that $\mathrm{V} = L$, the same is true of $\langle X, \in\rangle$, and it follows that $X = L_{\delta'}$, for some ordinal δ'. Clearly, the embedding j (which in the case $\alpha = 0$ is the identity) induces an elementary embedding

$$j' : \langle L_\delta, \in\rangle \prec \langle L_{\delta'}, \in\rangle$$

which is defined by $j' = g \circ j \circ f^{-1}$.

Since $A \in L_\delta$, one may now restrict j' to $L_{\kappa'}$, and this will yield an elementary embedding

$$i : \langle L_{\kappa'}, \in, A\rangle \prec \langle L_{j'(\kappa')}, \in, j'(A)\rangle$$

The point of this construction is that the target model of this embedding is in L, that is, that $j'(A) \in L$. It follows that $i{\restriction}\kappa$ is the identity. Namely, for any given $\zeta < \kappa$, saying that $\xi = f^{-1}(\zeta)$ is equivalent to saying that $\langle \zeta, <\rangle$ is isomorphic to $\langle u, E{\restriction}u\rangle$, where u, the set of E-predecessors of ξ, is closed under E (namely, $u = f^{-1}``\zeta$). I used here that κ is regular, to conclude that $u \in H_\kappa$. So if $\xi = f^{-1}(\zeta)$, then this u is definable from ζ in $\langle H_{\omega_2}, \in, E\rangle$. By elementarity, u satisfies the same definition from ζ in $\langle H^{\mathrm{V}[G][H]}_{\omega_2}, \in, F\rangle$, and it follows that $\langle u, E{\restriction}u\rangle = \langle u, F{\restriction}u\rangle$, and so, $\xi = g^{-1}(\zeta)$. Thus, $j'(\zeta) = g(f^{-1}(\zeta)) = g(\xi) = \zeta$.

If $\alpha > 0$, then a similar argument shows that $i(\kappa) = j(\kappa) = \gamma$. Here, I use that $f{\restriction}(\kappa+1) = \mathrm{id}$, which guarantees that the collection of E-predecessors of $\kappa = f^{-1}(\kappa)$ is κ, and is hence an element of $H_{\kappa'}$.

This type of reasoning can be carried further if κ' is regular. Namely, it follows in this case that $i \restriction \kappa' = j \restriction \kappa'$. To see this, let $\eta < \kappa'$. Then $i(\eta) = g(j(f^{-1}(\eta)))$, and $f^{-1}(\eta)$ is the unique $\xi < \kappa'$ such that $\langle \eta, < \rangle$ is isomorphic to $\langle u, E \restriction u \rangle$, where u, the set of all E-predecessors of ξ, is closed under E-predecessors. We know that $u \in H_{\kappa'}$ because κ' is regular. It follows that $j(f^{-1}(\eta))$ is the unique ξ' such that $\langle j(\eta), \in \rangle$ is isomorphic to $\langle v, F \restriction v \rangle$, where v, which is the set of all F-predecessors of ξ', is closed under F-predecessors. But that object is $g^{-1}(j(\eta))$. Thus, $i(\eta) = g(j(f^{-1}(\eta))) = j(\eta)$.

One can see similarly that if κ' is regular, then $j'(\kappa') = \gamma'$. Namely, let $f^{-1}(\kappa') = \xi$. Then, letting U be the class of all E-predecessors of ξ, it follows that $\langle U, E \restriction U \rangle$ is isomorphic to κ'. In $\langle H_{\kappa'}, \in, E \rangle$, U is a proper class definable from ξ, and it satisfies that it is linearly ordered by E, every element ζ of U has only set-many E-predecessors (since κ' is regular), and it is closed under E-predecessors. Hence, if U' is the class of F-predecessors of $j(\xi)$, as defined in $\langle H_{\gamma'}^{\mathrm{V}[G][H]}, \in, F \rangle$, then U' has the corresponding properties there. It follows that $g(j(\xi)) = \gamma'$, so that $j'(\kappa') = g(j(f^{-1}(\kappa'))) = g(j(\xi)) = \gamma'$.

To see that this embedding i satisfies the requirements of Definition 5.5 in L, the only non-obvious point is now that if κ' is regular in L, then so is $j'(\kappa')$. If κ' is also regular in V, then we know that γ' is regular in $\mathrm{V}[G][H]$ as well, and the argument of the previous paragraph shows that $j'(\kappa') = \gamma'$, so we are done in this case. But if κ' is regular in L yet singular in V, then it follows that $0^{\#}$ exists, or else, by Jensen's Covering Lemma, one could cover a cofinal subset of κ' that has order type less than κ' by a subset of κ' in L that has size less than κ'. Thus, L would see that κ' is singular. But if $0^{\#}$ exists, then it is easy to see that each Silver-indiscernible, and in particular ω_2, is even virtually extendible (and much more, see [BGS17, Theorems 3.5, 3.8]).

Now the argument can be finished as in the proof of part 1. A tree searching for an embedding $\tilde{j}$ from $\langle L_{\kappa'}, \in, A, \kappa \rangle$ to $\langle L_{\gamma'}, \in, j'(A), \gamma \rangle$ with $\tilde{j} \restriction \kappa = \mathrm{id}$ exists in $L[J]$, where J is $\mathrm{Col}(\omega, \kappa')$-generic over $\mathrm{V}[G][H][I]$, since $L_{\kappa'}$ is countable there, and since $j'(A) \in L$. This tree is ill-founded in $\mathrm{V}[G][H][I][J]$, and hence in $L[J]$. Thus such an embedding exists in $L[J]$. By the homogeneity of the collapse, it follows that such an embedding exists in $L^{\mathrm{Col}(\omega,\kappa')}$. □

Going in the other direction, I will want to force the resurrection axioms over a model with a sufficiently virtually super-extendible cardinal, and the existence of an appropriate Menas function will help carry this out. It was shown in [HJ14, Theorem 13] that such functions suitable for uplifting cardinals exist, and the following lemma generalizes this to the present context.

Lemma 5.11. *Let κ be a cardinal. There is a virtually super-extendible Menas function, i.e., a function $m : \kappa \longrightarrow \kappa$ such that for any $\alpha \geq 1$ such that κ is virtually super α-extendible, and for every ordinal ζ, there are a cardinal γ, an ordinal β and a virtual embedding*

$$j : \langle H_{\kappa^{+\alpha}}, \in, \kappa \rangle \prec \langle H_{\gamma^{+\beta}}, \in, \gamma \rangle$$

with $j \restriction \kappa = \mathrm{id}$ and

$$j(m)(\kappa) > \zeta$$

Proof. For a cardinal ξ and an ordinal α, define $T(\xi, \alpha)$ to be the class of cardinals γ such that there are an ordinal β and a virtual embedding $j : \langle H_{\xi^{+\alpha}}, \in, \xi \rangle \prec \langle H_{\gamma^{+\beta}}, \in, \gamma \rangle$ with $j \restriction \xi = \mathrm{id}$. For $\xi < \kappa$, define

$$a(\xi) = \min\{\alpha < \kappa \mid T(\xi, \alpha) \cap \kappa \text{ is bounded in } \kappa\}$$

if this exists, and let $a(\xi)$ be undefined otherwise. Define

$$m(\xi) = \begin{cases} \sup(T(\xi, a(\xi)) \cap \kappa) & \text{if } a(\xi) \text{ is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\alpha \geq 1$, assume that κ is virtually super α-extendible, and let ζ be given. In particular, κ is uplifting, and as a result, it follows that for $\bar{\kappa}, \bar{\alpha} < \kappa$, $T(\bar{\kappa}, \bar{\alpha}) \cap \kappa = (T(\bar{\kappa}, \bar{\alpha}))^{H_\kappa}$. The inclusion from right to left is obvious here, and for the converse, the point is that if $\bar{\gamma} \in T(\bar{\kappa}, \bar{\alpha}) \cap \kappa$, and this is witnessed by a virtual embedding $j : \langle H_{\bar{\kappa}^{+\bar{\alpha}}}, \in, \bar{\kappa} \rangle \prec \langle H_{\bar{\gamma}^{+\beta}}, \in, \bar{\gamma} \rangle$, for some β, then it could be that $\bar{\gamma}^{+\beta} > \kappa$. But using an inaccessible $\gamma > \bar{\gamma}^{+\beta}$ with $H_\kappa \prec H_\gamma$, the existence of such a β and such a virtual embedding j reflects down to H_κ, with the consequence that $\bar{\gamma} \in (T(\bar{\kappa}, \bar{\alpha}))^{H_\kappa}$. The point is that the existence of such a virtual embedding is first order expressible, by saying that there is a forcing notion that adds it. We have seen that actually, if there is such a forcing notion, then already $\mathrm{Col}(\omega, H_{\bar{\kappa}^{+\bar{\alpha}}})$ will do. But independently of this, one could choose γ large enough to have the forcing notion needed to add the virtual embedding available in H_γ, making the existential statement in question true in H_γ and hence in H_κ.

This means that the functions a, m can be alternatively defined by

$$a(\xi) = \min\{\mu \mid H_\kappa \models T(\xi, \mu) \text{ is bounded}\}$$

if this exists (and otherwise, $a(\xi)$ is undefined), and

$$m(\xi) = \begin{cases} \sup T(\xi, a(\xi))^{H_\kappa} & \text{if } a(\xi) \text{ is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

Of course, from the point of view of H_κ, a and m are proper class functions.

To see that m is as wished, assume that κ is virtually super θ-extendible, $\theta > 0$, and let an ordinal ζ be given.

For every $\bar{\alpha} < \theta$, fix a $\gamma_{\bar{\alpha}} \in T(\kappa, \bar{\alpha})$ with $\gamma_{\bar{\alpha}} > \zeta$, and fix a $\beta_{\bar{\alpha}}$ be such that there is a virtual elementary embedding

$$j_{\bar{\alpha}} : \langle H_{\kappa^{+\bar{\alpha}}}, \in, \kappa \rangle \prec \langle H_{\gamma_{\bar{\alpha}}^{+\beta_{\bar{\alpha}}}}, \in, \gamma_{\bar{\alpha}} \rangle$$

witnessing that $\gamma_{\bar{\alpha}} \in T(\kappa, \bar{\alpha})$. Let $\tilde{\gamma} = \sup_{\bar{\alpha}<\theta} \gamma_{\bar{\alpha}}^{+\beta_{\bar{\alpha}}}$.

Let $\delta_0 \in T(\kappa, \theta)$, $\delta_0 > \max\{\zeta, \kappa^{+\theta}, \tilde{\gamma}\}$. Let $k_0 : \langle H_{\kappa^{+\theta}}, \in, \kappa \rangle \prec \langle H_{\delta_0^{+\varepsilon_0}}, \in, \delta_0 \rangle$ be a virtual embedding, for some ε_0, witnessing that $\delta_0 \in T(\kappa, \theta)$. Let

$$\delta_1 = \min(T(\kappa, \theta) \setminus (\delta_0^{+\varepsilon_0} + 1)$$

and let

$$k_1 : \langle H_{\kappa^{+\theta}}, \in, \kappa \rangle \prec \langle H_{\delta_1^{+\varepsilon_1}}, \in, \delta_1 \rangle$$

witness that $\delta_1 \in T(\kappa, \theta)$, for some ε_1. Note that since κ is inaccessible, so is δ_1, and that $\delta_1 > \theta$.

Note that $k_1(m) = m^{H_{\delta_1}}$ is the function defined in H_{δ_1} by the same formula by which m is defined in H_κ. Since $k_0 \in H_{\delta_1}^{\mathrm{Col}(\omega, H_{\kappa^{+\theta}})}$ (and $H_{\kappa^{+\theta}} \in H_{\delta_1}$), it follows that $\delta_0 \in T(\kappa, \theta)^{H_{\delta_1}}$. Moreover, $(T(\kappa, \theta))^{H_{\delta_1}} \subseteq (\delta_0^{+\varepsilon_0} + 1)$, because if there were a $\gamma' \in (T(\kappa, \theta))^{H_{\delta_1}} \setminus (\delta_0^{+\varepsilon_0} + 1)$, then it would follow that $\gamma' \in (T(\kappa, \theta)) \cap (\delta_0^{+\varepsilon_0}, \delta_1)$, but there is no such γ', by definition of δ_1. Thus, $T(\kappa, \theta)^{H_{\delta_1}}$ is bounded in δ_1, and hence,

$$a^{H_{\delta_1}}(\kappa) \leq \theta$$

Thus, $m^{H_{\delta_1}}(\kappa)$ is defined by the first case. If $a^{H_{\delta_1}}(\kappa) = \theta$, then it follows that

$$k_1(m)(\kappa) = m^{H_{\delta_1}}(\kappa) = \sup T(\kappa, \theta)^{H_{\delta_1}} \geq \delta_0 > \zeta$$

so that k_1 is as wished. If $\bar{\alpha} := a^{H_{\delta_1}}(\kappa) < \theta$, then it follows similarly that

$$k_1(m)(\kappa) = m^{H_{\delta_1}}(\kappa) = \sup T(\kappa, \bar{\alpha})^{H_{\delta_1}} \geq \gamma_{\bar{\alpha}} > \zeta$$

since $\gamma_{\bar{\alpha}} \in T(\kappa, \bar{\alpha})^{H_{\delta_1}}$, as $j_{\bar{\alpha}} \in H_{\delta_1}^{\mathrm{Col}(\omega, H_{\kappa^{+\bar{\alpha}}})}$ (and $H_{\kappa^{+\bar{\alpha}}} \in H_{\delta_1}$), completing the proof. □

Lemma 5.12. *Let Γ be either the class of semiproper, proper, countably closed or subcomplete forcings, and suppose that κ is a virtually super θ-extendible cardinal. Then there is a κ-c.c. poset in Γ which forces that $\mathsf{vRA}_\Gamma(H_{\omega_{2+\theta}})$ holds.*

Proof. To stratify the argument, let's assume that $\theta > 0$, for in the case $\theta = 0$, $\mathsf{vRA}_\Gamma(H_{\omega_{2+\theta}})$ is $\mathsf{vRA}_\Gamma(H_{\omega_2})$, which is equivalent to $\mathsf{RA}_\Gamma(H_{\omega_2})$ by Observation 5.3, and κ is virtually super 0-extendible iff it is uplifting (see the remark after Definition 5.5), and the claim follows from Fact 3.3 for the case that Γ is the collection of all countably closed or all subcomplete forcing notions, and it was shown in [HJ14, Theorems 18 and 19] for the case that Γ is the class of proper or semi-proper (among others) forcing notions that there is a κ-c.c. poset that forces $\mathsf{RA}_\Gamma(H_{2^\omega}) + 2^\omega = \omega_2$, which, by Observation 3.6, is equivalent to $\mathsf{RA}_\Gamma(H_{\omega_2})$.

I will use the Menas function m from Lemma 5.11 in combination with lottery sums in place of a Laver function, building on an idea of Hamkins, see also [Apt05]. The forcing will be an iteration of length κ. As usual, it suffices to specify the iterands $\langle \dot{\mathbb{Q}}_\alpha \mid \alpha \le \kappa \rangle$, setting $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, and the limit process used to form $\mathbb{P}_\alpha$ if α is a limit ordinal. Of course, the limit process depends on the forcing class in question. For semiproper or subcomplete forcings, revised countable support will be employed, and in the other cases, it will be countable support. Such constructions have been carried out, for example, in [HJ14, Theorems 18 and 19].

Suppose $\mathbb{P}_\alpha$ has been defined. Inductively, we will have that $\mathbb{P}_\alpha \in \mathrm{V}_\kappa$. Let $\dot{\mathbb{Q}}_\alpha \in \mathrm{V}_\kappa$ be a $\mathbb{P}_\alpha$-name such that $\mathbb{1}_{\mathbb{P}_\alpha}$ forces that $\dot{\mathbb{Q}}_\alpha$ is the lottery sum of all forcings in Γ of rank at most $m(\alpha)$, followed by the collapse of the size of this lottery sum to ω_1. The lottery sum of a set of forcing notions is the result of taking the union of pairwise disjoint isomorphic copies of the forcing notions in the set, with a new condition that's weaker than all the conditions in the disjoint union. A generic for this lottery sum will be generic for one of these forcings - each one is possible. One can think of the generic as "picking" one of the forcings in the collection.

At limit stages, take the appropriate limit. The forcing classes in question are closed under lottery sums, as can easily seen for countably closed forcing, and it is well known for the classes of proper or semi-proper forcing notions, see [HJ14, Theorems 18 and 19]. It has also been checked for the class of subcomplete forcings, see [Min17, Lemma 2.2.8]. Each of the classes of forcing under consideration is closed under two step iterations, and each of them contains all countably closed forcings. Thus, composing the lottery sum described with the collapse to ω_1 does not take us out of the class. It is necessary to carry out the collapse in the case of iterating semi-proper forcing or subcomplete forcing; this is what the revised countable support iteration theorems for these classes require (see [FMS88, P. 13], [Jen14, P. 142, Theorem 3]). It follows that $\mathbb{P}_\kappa$ is κ-c.c., see [VAS14, Lemma 3.12 and Theorem 3.13].

Let g be generic for $\mathbb{P}_\kappa$ over V. I claim that $\mathsf{vRA}_\Gamma(H_{\omega_{2+\theta}})$ holds in $\mathrm{V}[g]$. To see this, in $\mathrm{V}[g]$, let $\mathbb{P} = \dot{\mathbb{P}}^g$ be in $\Gamma^{\mathrm{V}[g]}$, and let h be $\mathbb{P}$-generic over $\mathrm{V}[g]$.

Let $\gamma > \kappa^{+\theta}$ be an inaccessible cardinal large enough that $\mathbb{P} \in H_\gamma[g]$ and $\mathbb{P} \in \Gamma^{H_\gamma[g]}$, β an ordinal, and j a virtual embedding $j : \langle H_{\kappa^{+\theta}}, \in, \kappa \rangle \prec \langle H_{\gamma^{+\beta}}, \in, \gamma \rangle$ with $j{\restriction}\kappa = \mathrm{id}$ and $j(m)(\kappa) > \mathrm{rnk}(\dot{\mathbb{P}})$. So $j \in \mathrm{V}[J]$, where J may be chosen to be $\mathrm{Col}(\omega, H_{\kappa^{+\theta}})$-generic over $\mathrm{V}[g][h]$.

In $H_{\gamma^{+\beta}}$, $\mathbb{P}_\kappa$ is an initial segment of $j(\mathbb{P}_\kappa)$, and g is generic for that initial segment. Moreover, $\mathbb{P}$ has rank less than $j(m)(\kappa)$, and $\mathbb{P} \in \Gamma^{H_{\gamma^{+\beta}}[g]}$. That's why we may let h be generic for the stage κ forcing of $j(\mathbb{P}_\kappa)$, opting for $\mathbb{P}$. Forcing with the part of the lottery sum that opts for $\mathbb{P}$ is like forcing with $\mathbb{P}$. So we may view h as being generic for $\mathbb{P}$ over $V[g][J]$, and let $g_{\mathtt{tail}}$ be generic for the rest of the $j(\mathbb{P}_\kappa)$ iteration, let's call it $\mathbb{P}_{\mathtt{tail}}$, over $\mathrm{V}[g][h][J]$. Then, arguing in $\mathrm{V}[g][h][J][g_{\mathtt{tail}}] = \mathrm{V}[g][h][g_{\mathtt{tail}}][J]$, clearly, j lifts to an embedding

$$j' : \langle H_{\kappa^{+\theta}}[g], \in \rangle \prec \langle H_{\gamma^{+\beta}}[g][h][g_{\mathtt{tail}}], \in \rangle$$

since $j\restriction\kappa = \mathrm{id}$ and the models in question satisfy ZFC^- and contain the forcing notions used as elements (I focused on the case $\theta > 0$ here; in the case $\theta = 0$, $\mathbb{P}_\kappa$ is a proper class iteration in H_κ, but the argument goes through in this case as well, see [HJ14, Theorems 18 and 19] for details). Observe that $\kappa = \omega_2^{\mathrm{V}[g]}$ and $\gamma = \omega_2^{\mathrm{V}[g][h][g_{\mathtt{tail}}]}$. Thus,

$$j' : \langle H^{\mathrm{V}[g]}_{\omega_{2+\theta}}, \in\rangle \prec \langle H^{V[g][h][g_{\mathtt{tail}}]}_{\omega_{2+\beta}}, \in\rangle$$

where $g_{\mathtt{tail}}$ is $\mathbb{P}_{\mathtt{tail}}$-generic over $\mathrm{V}[g][h]$. Since $\mathbb{P}_{\mathtt{tail}}$ is the tail of the iteration $j(\mathbb{P}_\kappa)$ in $H_{\gamma^{+\theta}}[g]$, it follows from standard iteration facts on the classes in question that in $\mathrm{V}[g][h]$, $\mathbb{P}_{\mathtt{tail}}$ belongs to Γ (for the case of subcomplete forcing, which is maybe less familiar, see [Jen14, P. 115, Lemma 2.1, 2.4] and [Jen14, P. 142, Theorem 3, in particular the claim on p. 143] for the relevant facts). Hence $\mathbb{P}_{\mathtt{tail}}$ can serve as our resurrecting poset. Recall that $j' \in \mathrm{V}[g][h][g_{\mathtt{tail}}][J]$, so this embedding can be added by forcing over $\mathrm{V}[g][h][g_{\mathtt{tail}}]$, as desired. □

Thus, Lemmas 5.10 and 5.12 provide level-by-level equiconsistencies between virtual bounded resurrection axioms and partially virtually super-extendible cardinals.

Corollary 5.13. *If κ is virtually extendible, and Γ is the class of semiproper, proper, countably closed or subcomplete forcings, then there is a κ-c.c. forcing extension in which vUR_Γ holds.*

Proof. The proof of Lemma 5.12 shows that there is a forcing $\mathbb{P}_\kappa$ such that for every $\theta > 0$, if κ is virtually super θ-extendible, then in $\mathrm{V}^{\mathbb{P}_\kappa}$, $\mathsf{vRA}_\Gamma(H_{\omega_{2+\theta}})$ holds. This is because the Menas function from Lemma 5.11 works for every $\theta > 0$, so the forcing is the same for every θ. Hence, if κ is virtually extendible, then it is virtually super θ-extendible for every θ, by Observation 5.7, so that vUR_Γ holds in $\mathrm{V}^{\mathbb{P}_\kappa}$. □

Lemma 5.14. *If κ is strongly virtually super θ-extendible and Γ is the class of semiproper, proper, countably closed or subcomplete forcings, then there is a poset in Γ that forces $\underset{\sim}{\mathsf{vRA}}_\Gamma(H_{\omega_{2+\theta}})$.*

Proof. We may assume that $\theta > 0$, because the case $\theta = 0$ is already covered by Fact 3.3 for the case of countably closed or subcomplete forcing, and letting Γ be either the class of proper or of semi-proper forcing notions, it was shown in [HJ, Theorem 19] that $\underset{\sim}{\mathsf{RA}}_\Gamma(H_{2^\omega})$ can be forced, using a forcing in Γ, assuming a strongly uplifting cardinal, which is the same as a strongly virtually super 0-extendible cardinal. By Observation 3.6, $\underset{\sim}{\mathsf{RA}}_\Gamma(H_{2^\omega})$ is equivalent to $\underset{\sim}{\mathsf{RA}}_\Gamma(H_{\omega_2})$, and this is equivalent to $\underset{\sim}{\mathsf{vRA}}_\Gamma(H_{\omega_2})$, by Observation 5.3.

Basically, the proof of Lemma 5.12 works here as well, but a slightly improved Menas function is needed. Namely, there is a function $m : \kappa \longrightarrow \kappa$ such that for every set $A \subseteq \kappa^{+\theta}$ and every ordinal ζ, there are a cardinal γ, an ordinal β, a set B (in V) and a virtual embedding $j : \langle H_{\kappa^{+\theta}}, \in, A, \kappa\rangle \prec \langle H_{\gamma^{+\beta}}, \in, B, \gamma\rangle$ with $j\restriction\kappa = \mathrm{id}$, $j(m)(\kappa) > \zeta$ (that is, j exists in some forcing extension of V, and equivalently, it exists in any forcing extension of V by $\mathrm{Col}(\omega, H_{\kappa^{+\theta}})$), and such that if $\kappa^{+\theta}$ is regular, then so is $\gamma^{+\beta}$. The construction of such a function works much as the proof of Lemma 5.11. For a cardinal ξ, an ordinal α and a set $A \subseteq \xi$, define $T(\xi, \alpha, A)$ to be the set of cardinals γ such that there are an ordinal β, a set $B \subseteq \gamma^{+\beta}$ and a virtual embedding $j : \langle H_{\xi^{+\alpha}}, \in, A, \xi\rangle \prec \langle H_{\gamma^{+\beta}}, \in, B, \gamma\rangle$ with $j\restriction\xi = \mathrm{id}$, such that if $\xi^{+\alpha}$ is regular, then so is $\gamma^{+\beta}$. For $\xi < \kappa$, set

$$a(\xi) = \min\{\alpha < \kappa \mid \exists A \subseteq \xi^{+\alpha} \quad T(\xi, \alpha, A) \cap \kappa \text{ is bounded in } \kappa\}$$

if this exists, and leave $a(\xi)$ undefined otherwise. Then, define

$$m(\xi, A) = \begin{cases} \sup(T(\xi, \alpha, A) \cap \kappa) & \text{if } \alpha = a(\xi) \text{ is defined,} \\ & \text{and } T(\xi, \alpha, A) \cap \kappa \text{ is bounded in } \kappa, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, let

$$m(\xi) = \begin{cases} \sup\{m(\xi, A) \mid A \subseteq \xi^{+\alpha}\} & \text{if } \alpha = a(\xi) \text{ is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that for $\xi < \kappa$, $a(\xi)$ is less than κ if it is defined, and it follows from the inaccessibility of κ that $m(\xi) < \kappa$.

To see that m is as wished, assume that κ is strongly virtually super θ-extendible, and let a set $A \subseteq \kappa^{+\theta}$ and an ordinal ζ be given. Since κ is uplifting, it follows as before that for $\bar{\kappa}, \bar{\alpha} < \kappa$ and $\bar{A} \subseteq \bar{\kappa}^{+\bar{\alpha}}$, $T(\bar{\kappa}, \bar{\alpha}, \bar{A}) \cap \kappa = (T(\bar{\kappa}, \bar{\alpha}, \bar{A}))^{H_\kappa}$. Thus, the function m can be defined in H_κ in the obvious way.

For every $\bar{\alpha} < \theta$ and every $\bar{A} \subseteq \kappa^{+\bar{\alpha}}$, fix a $\gamma_{\bar{\alpha},\bar{A}} \in T(\kappa, \bar{\alpha}, \bar{A})$ with $\gamma_{\bar{\alpha},\bar{A}} > \zeta$, a $\beta_{\bar{\alpha},\bar{A}}$ and a $\bar{B}_{\bar{\alpha},\bar{A}} \subseteq \gamma_{\bar{\alpha},\bar{A}}^{+\beta_{\bar{\alpha},\bar{A}}}$ such that there is a virtual elementary embedding

$$j_{\bar{\alpha},\bar{A}} : \langle H_{\kappa^{+\bar{\alpha}}}, \in, \bar{A}\rangle \prec \langle H_{\gamma_{\bar{\alpha},\bar{A}}^{+\beta_{\bar{\alpha},\bar{A}}}}, \in, \bar{B}_{\bar{\alpha},\bar{A}}\rangle$$

witnessing that $\gamma_{\bar{\alpha},\bar{A}} \in T(\kappa, \bar{\alpha}, \bar{A})$. Let $\tilde{\gamma} = \sup\{\gamma_{\bar{\alpha},\bar{A}}^{+\beta_{\bar{\alpha},\bar{A}}} \mid \bar{\alpha} < \theta \text{ and } \bar{A} \subseteq \kappa^{+\bar{\alpha}}\}$.

Let $\delta_0 \in T(\kappa, \theta, A)$, $\delta_0 > \max\{\zeta, \kappa^{+\theta}, \tilde{\gamma}\}$. Let $k_0 : \langle H_{\kappa^{+\theta}}, \in, A, \kappa\rangle \prec \langle H_{\delta_0^{+\varepsilon_0}}, \in, B_0, \delta_0\rangle$ be a virtual embedding witnessing that $\delta_0 \in T(\kappa, \theta, A)$, and let δ_1 and ε_1 be such that

$$\delta_1 = \min(T(\kappa, \theta, A) \setminus (\delta_0^{+\varepsilon_0} + 1),\ k_1 : \langle H_{\kappa^{+\theta}}, \in, A, \kappa\rangle \prec \langle H_{\delta_1^{+\varepsilon_1}}, \in, B_1, \delta_1\rangle$$

with k_1 being a virtual elementary embedding, $k_1 \restriction \kappa = \mathrm{id}$. Then $\delta_0 \in T(\kappa, \theta, A)^{H_{\delta_1}}$ and $(T(\kappa, \theta, A))^{H_{\delta_1}} \subseteq (\delta_0^{+\varepsilon_0} + 1)$ as before. So $T(\kappa, \theta, A)^{H_{\delta_1}}$ is bounded in δ_1, and hence, $a^{H_{\delta_1}}(\kappa) \leq \theta$. If $a^{H_{\delta_1}}(\kappa) = \theta$, then it follows that

$$k_1(m)(\kappa) = m^{H_{\delta_1}}(\kappa) \geq \sup T(\kappa, \theta, A)^{H_{\delta_1}} \geq \delta_0 > \zeta$$

so that k_1 is as wished. If $\bar{\alpha} := a^{H_{\delta_1}}(\kappa) < \theta$, then there is some $\bar{A} \subseteq \kappa^{+\bar{\alpha}}$ such that $T(\kappa, \bar{\alpha}, \bar{A})^{H_{\delta_1}}$ is bounded in δ_1, and it follows that

$$k_1(m)(\kappa) = m^{H_{\delta_1}}(\kappa) \geq \sup T(\kappa, \bar{\alpha}, \bar{A})^{H_{\delta_1}} \geq \gamma_{\bar{\alpha},\bar{A}} > \zeta$$

since $\zeta < \gamma_{\bar{\alpha},\bar{A}} \in T(\kappa, \bar{\alpha}, \bar{A})^{H_{\delta_1}}$, because $j_{\bar{\alpha},\bar{A}} \in H_{\delta_1}^{\mathrm{Col}(\omega, H_{\kappa^{+\bar{\alpha}}})}$. So k_1 is as wished in this case as well.

Now the forcing $\mathbb{P}_\kappa$ can be defined as in the proof of Lemma 5.12, but with respect to this improved, strongly virtually super θ-extendible Menas function. If g is generic for that forcing $\mathbb{P}_\kappa$, then in $\mathrm{V}[g]$, suppose $\mathbb{P} = \dot{\mathbb{P}}^g \in \Gamma$, h is generic for $\mathbb{P}$, and $A \subseteq \omega_{2+\theta}$. Since $\mathbb{P}_\kappa$ is κ-c.c. and every cardinal below κ is explicitly collapsed to ω_1, it follows as before that $\omega_2^{\mathrm{V}[g]} = \kappa$ and $\omega_{2+\theta}^{\mathrm{V}[g]} = (\kappa^{+\theta})^{\mathrm{V}}$. Moreover, again since $\mathbb{P}_\kappa$ is κ-c.c., there is a $\mathbb{P}_\kappa$-name $\dot{A} \subseteq H_{\kappa^{+\theta}}$ such that $\dot{A}^g = A$. Clearly, $\dot{A}$ can be chosen to have size at most $\kappa^{+\theta}$, so that it can be coded by a subset of $\kappa^{+\theta}$, and hence, we can choose a $\dot{B}$ and a γ such that there is a virtual embedding

$$j : \langle H_{\kappa^{+\theta}}, \in, \dot{A}, \kappa\rangle \prec \langle H_{\gamma^{+\beta}}, \in, \dot{B}, \gamma\rangle$$

with $j \restriction \kappa = \mathrm{id}$ and $\mathrm{rnk}(\dot{\mathbb{P}}) < j(m)(\kappa)$, and such that if $\kappa^{+\theta}$ is regular, then so is $\gamma^{+\beta}$. For definiteness, such an embedding j exists in $\mathrm{V}[J]$, for any J that is $\mathrm{Col}(\omega, H_{\kappa^{+\theta}})$-generic over V. In particular, we can let J be $\mathrm{Col}(\omega, H_{\kappa^{+\theta}})$-generic over $\mathrm{V}[g][h]$

As before, in $\mathrm{V}[g][h][J][g_{\mathtt{tail}}]$, j lifts to

$$j' : \langle H_{\kappa^{+\theta}}[g], \in, A\rangle \prec \langle H_{\gamma^{+\beta}}[g][h][g_{\mathtt{tail}}], \in, B\rangle$$

where $g_{\mathtt{tail}}$ is generic for the rest of the iteration $\mathbb{P}_{\mathtt{tail}} = j(\mathbb{P}_\kappa)/g{*}h$ after stage κ, over $\mathrm{V}[g{*}h][J]$, $A = \dot{A}^g$ and $B = \dot{B}^{g*h*g_{\mathtt{tail}}}$. Thus, $\mathbb{P}_{\mathtt{tail}}$ can serve as our resurrecting forcing notion in $\mathrm{V}[g][h]$. It follows by the usual iteration theorems for Γ that $\mathbb{P}_{\mathtt{tail}}$ is in $\Gamma^{\mathrm{V}[g][h]}$. The embedding j' is added by J, which is generic over $\mathrm{V}[g][h][g_{\mathtt{tail}}]$, so it is virtual in $\mathrm{V}[g][h]$. Since $j(\mathbb{P}_\kappa)$ is small forcing, $\gamma^{+\beta}$ remains regular in $\mathrm{V}[g][h][g_{\mathtt{tail}}]$ (if it was regular in V), and we are done, since $H_{\kappa^{+\theta}}[g] = H^{\mathrm{V}[g]}_{\omega_{2+\theta}}$ (since $\mathbb{P}_\kappa \in H_{\kappa^{+\theta}}$ and $\mathbb{P}_\kappa$ is κ-c.c., it follows that $H_{\kappa^{+\theta}}[g] = H^{\mathrm{V}[g]}_{\kappa^{+\theta}}$, and we have already seen that $(\kappa^{+\theta})^{\mathrm{V}[g]} = (\omega_{2+\theta})^{\mathrm{V}[g]}$) and $H_{\gamma^{+\beta}}[g][h][g_{\mathtt{tail}}] = H^{V[g][h][g_{\mathtt{tail}}]}_{\omega_{2+\beta}}$, because $j(\mathbb{P}_\kappa)$ is $j(\kappa) = \gamma$-c.c. in $H_{\gamma^{+\beta}}$, by elementarity of j, and hence it is γ-c.c. in V, and it also explicitly collapses every cardinal below γ to ω_1. □

Here is a summary of the equiconsistencies.

Theorem 5.15. *Let Γ be the class of semiproper, proper, countably closed or subcomplete forcings.*

1. *If κ is virtually super θ-extendible, then in a κ-c.c. forcing extension by a forcing in Γ, $\mathsf{vRA}_\Gamma(H_{\omega_{2+\theta}})$ holds.*
2. *If κ is virtually super $<\theta$-extendible, then in a κ-c.c. forcing extension by a forcing in Γ, $\mathsf{vRA}_\Gamma(H_{\omega_{2+\bar{\theta}}})$ holds, for every $\bar{\theta} < \theta$.*
3. *If κ is strongly virtually super θ-extendible, then in a κ-c.c. forcing extension by a forcing in Γ, $\underset{\sim}{\mathsf{vRA}}_\Gamma(H_{\omega_{2+\theta}})$ holds.*
4. *If κ is strongly virtually super $<\theta$-extendible, then in a κ-c.c. forcing extension by a forcing in Γ, $\underset{\sim}{\mathsf{vRA}}_\Gamma(H_{\omega_{2+\bar{\theta}}})$ holds, for every $\bar{\theta} < \theta$.*
5. *If κ is virtually extendible, then vUR_Γ holds in a κ-c.c. forcing extension by a forcing in Γ.*
6. *If $\mathsf{vRA}_\Gamma(H_{\omega_{2+\theta}})$ holds, then ω_2 is virtually super θ-extendible in L.*
7. *If $\underset{\sim}{\mathsf{vRA}}_\Gamma(H_{\omega_{2+\theta}})$ holds, where $\mathrm{cf}(\omega_{2+\theta}) > \omega$, then ω_2 is strongly virtually super θ-extendible in L.*
8. *The consistency strength of vUR_Γ is a virtually extendible cardinal.*

Proof. 1. is Lemma 5.12, 2. follows from the proof of that lemma (using the argument given in Corollary 5.13), 3. is Lemma 5.14, 4. follows from the proof of Lemma 5.14 (using an adaptation of the argument given in Corollar 5.13), 5. is Corollary 5.13, 6. is part one of Lemma 5.10, 7. is part two of that lemma, and 8. results from putting 5. and 6. together. □

6 How the hierarchies fit together

I would now like to establish the connections between the higher virtual resurrection axioms and the weak bounded forcing axioms, defined in [Fuc16a, Definition 4.6] as follows.

Definition 6.1. Let Γ be a forcing class and let λ be an uncountable cardinal. The *weak bounded forcing axiom for Γ at λ*, $\mathsf{wBFA}(\Gamma, \leq\lambda)$, says that whenever $M = \langle |M|, \in, R_0, R_1, \ldots, R_i, \ldots\rangle_{i<\omega_1}$ is a transitive model of size at most λ for a language $\mathcal{L}$ with ω_1 many predicates $\langle \dot{R}_i \mid i < \omega_1\rangle$ and the binary relation symbol $\dot{\in}$, and if $\varphi(x)$ is a Σ_1-formula and $\mathbb{P}$ is a forcing in Γ such that $\mathbb{P}$ forces that $\varphi(\check{M})$ holds, then there is (in V) a transitive model $\bar{M} = \langle |\bar{M}|, \in, \ldots, \bar{R}_i, \ldots\rangle_{i<\omega_1}$ for $\mathcal{L}$ such that $\varphi(\bar{M})$ holds (in V), and such that in $\mathrm{V}^{\mathrm{Col}(\omega,|\bar{M}|)}$, there is an elementary embedding $j : \bar{M} \prec M$.

I just write $\mathsf{wBFA}(\Gamma)$ for $\mathsf{wBFA}(\Gamma, \leq\omega_1)$, and the *weak forcing axiom for* Γ, $\mathsf{wFA}(\Gamma)$, is the statement that $\mathsf{wBFA}(\Gamma, \leq\lambda)$ holds for all uncountable cardinals λ. Similarly, $\mathsf{wBFA}(\Gamma, <\lambda)$ says that $\mathsf{wBFA}(\Gamma, \leq\bar{\lambda})$ holds for all uncountable cardinals $\bar{\lambda} < \lambda$.

If Γ is the class of subcomplete forcings, then I write wSCFA, wBSCFA, $\mathsf{wBSCFA}(\leq\lambda)$ and $\mathsf{wBSCFA}(<\lambda)$ for $\mathsf{wFA}(\Gamma)$, $\mathsf{wBFA}(\Gamma)$, $\mathsf{wBFA}(\Gamma, \leq\lambda)$ and $\mathsf{wBFA}(\Gamma, <\lambda)$, respectively. Similarly, the corresponding axioms for the class of proper forcings are denoted wPFA, wBPFA, etc.

Thus, the one obtains $\mathsf{wBFA}(\Gamma, <\lambda)$ by weakening the requirement of the existence of an elementary embedding in the definition of $\mathsf{BFA}(\Gamma, <\lambda)$ (see Definition 4.1) to the existence of just a virtual elementary embedding. So the relationship between $\mathsf{wBFA}(\Gamma, <\lambda)$ and $\mathsf{BFA}(\Gamma, <\lambda)$ is similar to that between $\mathsf{vRA}(\Gamma, H_\lambda)$ and $\mathsf{RA}(\Gamma, H_\lambda)$. Thus, it would have made sense to refer to the weak bounded forcing axioms as *virtual* bounded forcing axioms, but the "virtual" forcing axiom for proper forcing has already been named the weak proper forcing axiom in [BGS17]. Alternatively, it would have made sense to refer to the virtual resurrection axioms as the *weak* resurrection axioms, but again, the notion of weak resurrection axiom was already used in [HJ14], as in Definitions 3.13 and 4.2. As a result, the modifier "weak" has different meanings in the case of resurrection axioms and bounded forcing axioms, which I hope will not cause too much confusion.

The large cardinal relevant for the weak forcing axioms is Schindler's concept of remarkability. Remarkable cardinals can be defined in the following way, as in [BGS17].

Definition 6.2. A regular cardinal κ is *remarkable* if for every regular $\lambda > \kappa$, there is a regular cardinal $\bar{\lambda} < \kappa$ such that in $\mathrm{V}^{\mathrm{Col}(\omega, H_{\bar{\lambda}})}$, there is an elementary embedding $j : H_{\bar{\lambda}}^{\mathrm{V}} \prec H_{\lambda}^{\mathrm{V}}$ with $j(\mathrm{crit}(j)) = \kappa$.

It was shown in [BGS17, Theorems 6.3 and 6.4] that wPFA is equiconsistent with a remarkable cardinal, and in [Fuc16a, Theorem 4.5] that wSCFA is equiconsistent with a remarkable cardinal. In order to measure the consistency strengths of the weak bounded forcing axioms, I introduced the following large cardinals in [Fuc16a].

Definition 6.3. Let κ be an inaccessible cardinal and let $\lambda \geq \kappa$ be a cardinal. κ is *remarkably $\leq\lambda$-reflecting* if the following holds: for any $X \subseteq H_\lambda$ and any formula $\varphi(x)$, if there is a regular cardinal $\theta > \lambda$ such that $\langle H_\theta, \in\rangle \models \varphi(X)$, then there are cardinals $\bar{\lambda} < \bar{\theta} < \kappa$, such that $\bar{\theta}$ is regular, and there is a set $\bar{X} \subseteq H_{\bar{\lambda}}$ in V and an ordinal $\bar{\kappa} \leq \bar{\lambda}$ such that $\langle H_{\bar{\theta}}, \in\rangle \models \varphi(\bar{X})$, and a virtual embedding $j : \langle H_{\bar{\lambda}}, \in, \bar{X}, \bar{\kappa}\rangle \prec \langle H_\lambda, \in, X, \kappa\rangle$ (meaning that j exists in $\mathrm{V}^{\mathrm{Col}(\omega, H_{\bar{\lambda}})}$) such that $j{\upharpoonright}\bar{\kappa} = \mathrm{id}$.

κ is *remarkably $<\lambda$-reflecting* iff it is remarkably $\leq\bar{\lambda}$-reflecting, for every cardinal $\bar{\lambda} < \lambda$ with $\kappa \leq \bar{\lambda}$.

The connection between the weak bounded forcing axioms and the remarkably reflecting cardinals is as follows, see [Fuc16a, Lemma 4.13, Theorem 4.14, Lemma 4.15 and the following remark].

Theorem 6.4. *Let λ be a cardinal, and let Γ be the class of subcomplete, of proper or of semi-proper forcings.*

1. *If $\lambda \geq \omega_2$ and $\mathsf{wBFA}(\Gamma, \leq\lambda)$ holds, then ω_2 is remarkably $\leq\lambda$-reflecting in L.*
2. *If $\lambda \geq \omega_2$ and $\mathsf{wBFA}(\Gamma, <\lambda)$ holds, then ω_2 is remarkably $<\lambda$-reflecting in L.*
3. *If κ is remarkably $\leq\lambda$-reflecting, where $\kappa \leq \lambda$, then there is a κ-c.c. forcing notion in Γ which forces that $\mathsf{wBFA}(\Gamma, \leq\lambda)$ holds.*

4. *If κ is remarkably $<\lambda$-reflecting, where $\lambda > \kappa$, then there is a κ-c.c. forcing notion in Γ which forces that* $\mathsf{wBFA}(\Gamma, <\lambda)$ *holds.*

For the bounded forcing axiom, that is $\mathsf{wBFA}(\Gamma, \leq\omega_1)$, the relevant large cardinal concept is that of a reflecting cardinal.

Definition 6.5 ([GS95, Def. 2.2]). A regular cardinal κ is *reflecting* if for every $a \in H_\kappa$, and every formula $\varphi(x)$, the following holds: if there is a regular cardinal $\theta \geq \kappa$ such that $H_\theta \models \varphi(a)$, then there is a cardinal $\bar{\theta} < \kappa$ such that $H_{\bar{\theta}} \models \varphi(a)$.

The following was shown in [Fuc16a, Theorem 3.6].

Theorem 6.6. BSCFA *is equiconsistent with the existence of a reflecting cardinal.*

The corresponding result for proper forcing is also true, as was shown in [GS95].

Lemma 6.7. *Let Γ be a forcing class and $\kappa > \omega_1$ be a cardinal. Then*

$$\mathsf{vwRA}_\Gamma(H_\kappa) \implies \mathsf{wBFA}(\Gamma, <\kappa)$$

Proof. Let $M = \langle |M|, \in, R_0, R_1, \ldots, R_i, \ldots \rangle_{i<\omega_1}$ be a transitive model of size less than κ, let $\mathbb{P} \in \Gamma$ be a forcing notion, let G be generic for $\mathbb{P}$ over V, let $\varphi(x)$ be a Σ_1-formula, and suppose that $\mathrm{V}[G] \models \varphi(M)$. Let $\mathbb{Q} \in \mathrm{V}[G]$ be a poset, and let H be $\mathbb{Q}$-generic over $\mathrm{V}[G]$ such that in some further forcing extension $\mathrm{V}[G][H][I]$, there is a cardinal λ and an elementary embedding $j : \langle H_\kappa, \in \rangle \prec \langle H_\lambda^{\mathrm{V}[G][H]}, \in \rangle$ with $j{\restriction}\omega_2 = \mathrm{id}$.

If κ is a limit cardinal, then let $\kappa' = (|M| + \omega_1)^+$, otherwise let $\kappa' = \kappa$. Thus, κ' is a successor cardinal, so that $H_{\kappa'}$ is a model of ZFC^-, and $M \in H_{\kappa'}$. Similarly, if $\kappa' = \kappa$, then let $\lambda' = \lambda$, and if $\kappa' < \kappa$, then let $\lambda' = j(\kappa')$. It follows that λ' is a successor cardinal in $\mathrm{V}[G][H]$, and hence, $H_{\lambda'}^{\mathrm{V}[G][H]}$ is also a ZFC^- model. Moreover, the restriction $\bar{j}$ of j to $H_{\kappa'}$ is an elementary embedding from $\langle H_{\kappa'}, \in \rangle$ to $\langle H_{\lambda'}^{\mathrm{V}[G][H]}, \in \rangle$.

Since $\mathrm{V}[G] \models \varphi(M)$ and φ is Σ_1, it follows that $\mathrm{V}[G][H] \models \varphi(M)$. Further, $M \in H_{\kappa'} \subseteq H_{\lambda'}^{\mathrm{V}[G][H]}$, so that by reflection,

$$\langle H_{\lambda'}^{\mathrm{V}[G][H]}, \in \rangle \models \varphi(M)$$

since λ' is an uncountable cardinal in $\mathrm{V}[G][H]$, so that $H_{\lambda'}^{\mathrm{V}[G][H]} \prec_{\Sigma_1} \mathrm{V}[G][H]$. Let $j' := j{\restriction}M$. Then j' is added by $\mathrm{Col}(\omega, M)$, which is an element of $H_{\lambda'}^{\mathrm{V}[G][H]}$, so $j' \in H_{\lambda'}^{\mathrm{V}[G][H][I]} = H_{\lambda'}^{\mathrm{V}[G][H]}[I]$. Let $N = j(M)$. Then in $H_{\lambda'}^{\mathrm{V}[G][H]}$, the statement "there is a transitive model $\bar{M}$ with $\varphi(\bar{M})$ such that $\mathrm{Col}(\omega, \bar{M})$ adds an elementary embedding from $\bar{M}$ to N" holds, as witnessed by M (and the embedding j' – it is important here again that $j{\restriction}\omega_2 = \mathrm{id}$, so that M and $j(M)$ are models of the same language; recall that M has up to ω_1 many predicates). So, pulling this back via $\bar{j}$, keeping in mind that $N = j(M) = \bar{j}(M)$, it follows that $H_{\kappa'}^{\mathrm{V}}$ believes that there is a transitive model $\bar{M}$ with $\varphi(\bar{M})$ such that $\mathrm{Col}(\omega, \bar{M})$ adds an elementary embedding from $\bar{M}$ to M, as wished. Since $H_{\kappa'}$ is a ZFC^- model, forcing with $\mathrm{Col}(\omega, \bar{M})$ over V will add such an elementary embedding, and since $\varphi(\bar{M})$ is Σ_1, it will hold in V as well. □

As a result, vwUR_Γ implies $\mathsf{wFA}(\Gamma)$.

Observation 6.8. *Let Γ be either the class of proper, semiproper or subcomplete forcings.*

1. $\mathsf{vRA}_\Gamma(H_{\omega_2})$ *is equivalent to* $\mathsf{RA}_\Gamma(H_{\omega_2})$*, and similarly,* $\underset{\sim}{\mathsf{vRA}}_\Gamma(H_{\omega_2})$ *is equivalent to* $\underset{\sim}{\mathsf{RA}}_\Gamma(H_{\omega_2})$.
2. $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_3})$ *has strictly higher consistency strength than* $\mathsf{vUR}_{\mathsf{SC}}$.

3. $\mathsf{vRA}_{\mathsf{SC}}(H_\kappa)$ *implies* $\mathsf{wBSCFA}(<\kappa)$*, but, assuming the consistency of Martin's Maximum* MM*, not vice versa (for* $\kappa \geq \omega_2$*).*

Proof. 1. holds by Observation 5.3. 2. follows from Observation 4.8, which shows that $\underset{\sim}{\mathsf{RA}}_{\mathsf{SC}}(H_{\omega_3})$ implies that $\mathsf{AD}^{L(\mathbb{R})}$ holds. The consistency strength of this statement is at least as high as the existence of infinitely many Woodin cardinals, while the consistency strength of $\mathsf{vUR}_{\mathsf{SC}}$ is the existence of a virtually extendible cardinal, by Theorem 5.15.8, and these virtual large cardinals are consistent with $\mathrm{V} = L$, in fact, if κ is virtually extendible, then κ is virtually extendible in L. To see that 3. holds, note that the implication follows from Lemma 4.3. Assuming the consistency of Martin's Maximum, any model of MM shows that the reverse implication does not hold. Namely, MM implies $2^\omega = \omega_2$ (since this already follows from the bounded proper forcing axiom, by [Moo05]), and hence MM implies the failure of $\mathsf{vRA}_{\mathsf{SC}}(H_\kappa)$, because $\mathsf{vRA}_{\mathsf{SC}}(H_\kappa)$ implies $\mathsf{vRA}_{\mathsf{SC}}(H_{\omega_2})$, which is equivalent to $\mathsf{RA}_{\mathsf{SC}}(H_{\omega_2})$, which implies CH, by Fact 3.1. Moreover, MM implies SCFA, since every subcomplete forcing preserves stationary subsets of ω_1 (by [Min17, Proposition 2.2.4 and Theorem 2.1.4]), and, of course, SCFA implies $\mathsf{wBSCFA}(<\kappa)$. $\mathsf{BMM}(<\kappa)$ would have sufficed here, instead of MM. □

In order to compare the consistency strengths of the virtual resurrection axioms and the weak bounded forcing axioms, we have to compare the remarkably $\leq\lambda$-reflecting cardinals and the virtually super α-extendible cardinals. Observe that these large cardinal properties go down to L, and that the assumption that $H_{\kappa^{+\alpha}} \in H_{\kappa^{+\alpha+1}}$ is always satisfied in L.

Lemma 6.9. *Let* α *be an ordinal. Suppose that* κ *is virtually super* $\alpha + 1$*-extendible, and that* $H_{\kappa^{+\alpha}} \in H_{\kappa^{+\alpha+1}}$*. Then* κ *is remarkably* $\leq\kappa^{+\alpha}$*-reflecting. Moreover, if* $\alpha < \kappa$*, then the set* $\{\bar{\kappa} < \kappa \mid \bar{\kappa} \text{ is remarkably } \leq\bar{\kappa}^{+\alpha} - \text{reflecting in } H_\kappa\}$ *is stationary in* κ*.*

Proof. Let $\lambda = \kappa^{+\alpha}$. To show that κ is remarkably $\leq\lambda$-reflecting, let $X \subseteq H_\lambda$, $\theta > \kappa^{+\alpha}$ be regular, $\varphi(x)$ a formula and $\langle H_\theta, \in\rangle \models \varphi(X)$. Let $j : \langle H_{\lambda^+}, \in, \kappa\rangle \prec \langle H_\nu, \in, \gamma\rangle$ be a virtual embedding with $j{\restriction}\kappa = \mathrm{id}$, $\gamma > \theta$ inaccessible and large enough that $H_\lambda, H_\theta \in H_\gamma$. Note that $\nu = \gamma^{+\beta+1}$, for some β, $\kappa = \mathrm{crit}(j)$, $j(\kappa) = \gamma$, and that $j \in \mathrm{V}[J]$, for some J which is generic for $\mathrm{Col}(\omega, H_{\lambda^+})$. Thus, since this forcing is also in H_ν, $\langle H_\nu, \in\rangle$ sees that there are a cardinals $\bar{\kappa} \leq \bar{\lambda} < \bar{\theta} < j(\kappa)$, where $\bar{\theta}$ is regular, a set $\bar{X} \subseteq H_{\bar{\lambda}}$ and a virtual embedding $j' : \langle H_{\bar{\lambda}}, \in, \bar{X}, \bar{\kappa}\rangle \prec j(\langle H_\lambda, \in, X, \kappa\rangle)$, with $j'{\restriction}\bar{\kappa} = \mathrm{id}$ such that $\langle H_{\bar{\theta}}, \in\rangle \models \varphi(\bar{X})$. This is witnessed by $\bar{\lambda} = \lambda$, $\bar{X} = X$, $j' = j{\restriction}H_\lambda$ and $\bar{\theta} = \theta$. The assumption that $H_\lambda \in H_{\lambda^+}$ was used here, since it allowed us to apply j to $\langle H_\lambda, \in, X\rangle$. By elementarity, $\langle H_{\lambda^+}, \in\rangle$ sees that there are cardinals $\bar{\kappa} \leq \bar{\lambda} < \bar{\theta} < \kappa$, where $\bar{\theta}$ is regular, a set $\bar{X} \subseteq H_{\bar{\lambda}}$ and a virtual embedding $j' : \langle H_{\bar{\lambda}}, \in, \bar{X}, \bar{\kappa}\rangle \prec \langle H_\lambda, \in, X, \kappa\rangle$ with $j'{\restriction}\bar{\kappa} = \mathrm{id}$, such that $\langle H_{\bar{\theta}}, \in\rangle \models \varphi(\bar{X})$.

Hence, κ is remarkably $\leq\kappa^{+\alpha}$-reflecting.

A simple reflection argument shows that $\{\bar{\kappa} < \kappa \mid \bar{\kappa} \text{ is remarkably } \leq\kappa^{+\alpha} - \text{reflecting in } H_\kappa\}$ is stationary in κ if $\alpha < \kappa$. For this argument, I only use that κ is strongly uplifting and remarkably $\leq\kappa^{+\alpha}$-reflecting. Namely, given a club set $C \subseteq \kappa$, let $\gamma > \kappa$ be inaccessible and $D \subseteq \gamma$ such that $\langle H_\kappa, \in, C\rangle \prec \langle H_\gamma, \in, D\rangle$. Then $\kappa \in D$ and so, $\langle H_\gamma, \in, D\rangle$ thinks that there is a $\bar{\kappa} \in D$ that's remarkably $\leq\bar{\kappa}^{+\alpha}$-reflecting - it can easily be checked that $\langle H_\gamma, \in\rangle$ believes that κ is remarkably $\leq\kappa^{+\alpha}$-reflecting, since γ is an inaccessible cardinal greater than κ and α. Hence, $\langle H_\kappa, \in, C\rangle$ thinks that there is a $\bar{\kappa} \in C$ that's remarkably $\leq\bar{\kappa}^{+\alpha}$-reflecting. □

I will clarify the meaning of the statement of the following lemma below.

Lemma 6.10. *Let* $\alpha < \omega_2$*. Then the transitive model consistency strength of "*$\mathsf{wBSCFA}(\leq\omega_{2+\alpha}) + \alpha < \omega_2$*" is strictly lower than that of "*$\mathsf{vRA}_{\mathsf{SC}}(H_{\omega_{2+\alpha+1}}) + \alpha < \omega_2$*."*

Note that it wouldn't make sense to say that the consistency strength of $\mathsf{wBSCFA}(\leq\omega_{2+\alpha})$ is strictly lower than that of $\mathsf{vRA}_{\mathsf{SC}}(H_{\omega_{2+\alpha+1}})$, because α may not be absolutely definable, and then these forcing principles cannot be formulated without using the parameter α. But intuitively, the consistency strength of $\mathsf{wBSCFA}(\leq\omega_{2+\alpha})$ is a cardinal κ that's remarkably $\leq\kappa^{+\alpha}$-reflecting (by Theorem 6.4), which is strictly lower than a cardinal κ that's virtually super $\alpha+1$-extendible (in the sense of Lemma 6.9), which is the strength of $\mathsf{vRA}_{\mathsf{SC}}(H_{\omega_{2+\alpha+1}})$, by Theorem 5.15. The present lemma tries to make this intuitive difference in consistency strength precise. What I mean by the statement in the lemma is that if there is a set-sized transitive model M of $\mathsf{ZFC}+\mathsf{vRA}_{\mathsf{SC}}(H_{\omega_{2+\alpha+1}})+\alpha<\omega_2$, then in $\mathrm{V}^{\mathrm{Col}(\omega,M)}$, there is a (set-sized) transitive model with the same ordinals as M, in which $\mathsf{wBSCFA}(\leq\omega_{2+\alpha})+\alpha<\omega_2$ holds, but the converse is not true.

Proof. By Observation 6.8.3, $\mathsf{vRA}_{\mathsf{SC}}(H_{\omega_{2+\alpha+1}})$ outright implies $\mathsf{wBSCFA}(\leq\omega_{2+\alpha})$, so clearly, the transitive model consistency strength of the former is at least that of the latter.

To see that the converse is not true, assume that there is a transitive model M of $\mathsf{ZFC}+\mathsf{wBSCFA}(\leq\omega_{2+\alpha})+\alpha<\omega_2$. We know that then, in L^M, ω_2^M is remarkably $\leq\alpha$-reflecting, by Theorem 6.4.1. Now, if there is a γ (in V) such that L_γ is a model of ZFC and L_γ believes that there is an L_δ such that in L_δ, there is a cardinal κ such that κ is remarkably $\leq\alpha$-reflecting and $\alpha<\kappa$, then we can let γ be the least such, and we can let δ, κ be as described, and work in $\mathrm{V}=L_\gamma$. Otherwise, we work in $\mathrm{V}=L$, letting $\delta=\mathrm{On}\cap M$ and $\kappa=\omega_2^M$. So we are now in a universe where κ is remarkably $\leq\alpha$-reflecting in L_δ, $\alpha<\kappa$, $\mathrm{V}=L$, and there is no $\bar{\delta}<\delta$ such that in $L_{\bar{\delta}}$, there is a $\leq\alpha$-reflecting $\bar{\kappa}$ with $\alpha<\bar{\kappa}$.

Now, let g be V-generic for the forcing in L_δ to force $\mathsf{wBSCFA}(\leq\omega_{2+\alpha})$, as given by Theorem 6.4.3. This forcing is κ-c.c. and explicitly collapses every cardinal less than κ to ω_1 in L_δ, so in $\mathrm{V}[g]$, it is the case that $\alpha<\kappa=\omega_2^{L_\delta[g]}$. But if G is $\mathrm{Col}(\omega,\delta)$-generic over $\mathrm{V}[g]$, then in $\mathrm{V}[g][G]$, there can be no transitive ZFC-model N with $\mathrm{On}\cap N=\delta$ such that $\mathsf{ZFC}+\mathsf{vRA}_{\mathsf{SC}}(H_{\omega_{2+\alpha+1}})+\alpha<\omega_2$ holds in N, because otherwise, letting $\bar{\delta}=\omega_2^N<\delta$, it follows that $\bar{\delta}$ is virtually super $\alpha+1$-extendible in $L^N=L_\delta$, by Lemma 5.10, which implies by Lemma 6.9 that the set of $\bar{\kappa}<\bar{\delta}$ such that in $L_{\bar{\delta}}$, $\bar{\kappa}$ is $\leq\alpha$-remarkably reflecting, is stationary in $\bar{\delta}$ (from the point of view of L_δ). In particular, there is such a $\bar{\kappa}$ with $\alpha<\bar{\kappa}<\bar{\delta}<\delta$. This contradicts the minimality of δ. □

The previous lemma holds also for the classes of proper or semi-proper forcing notions, because the large cardinal strengths of both the virtual resurrection axioms and the weak bounded forcing axioms for any of these classes are measured by the remarkably reflecting or partially virtually super extendible cardinals, respectively. One can use Lemma 5.8 in a similar way to show that for $\alpha<\omega_2$, the transitive model consistency strength of $\mathsf{vRA}_\Gamma(H_{\omega_{2+\alpha+1}})$ is strictly higher than that of $\mathsf{vRA}_\Gamma(H_{\omega_{2+\alpha}})$, where Γ is any of these standard classes of forcing, or even the class of countably closed forcing notions, because the large cardinal strengths of the principles for all of these classes correspond to the hierarchy of the partially virtually super extendible cardinals. Note that the weak bounded forcing axioms are meaningless for countably closed forcings, in the sense that they are provable from ZFC.

Finally, for any of these classes of forcing (excluding the class of countably closed forcing notions), let's compare the consistency strength of wFA_Γ, which is a remarkable cardinal (by [Fuc16a, Theorem 4.5, Theorem 4.14 and the following remark], see also [BGS17, Theorems 6.3, 6.4]) with that of $\underset{\sim}{\mathsf{RA}}_\Gamma(H_{\omega_2})$, which is a strongly uplifting cardinal, by Theorem 5.15, noting that κ is uplifting iff it is strongly virtually super 0-extendible, as I pointed out after Definition 5.5. Since these large cardinals are known, fortunately, consulting the literature is all that's needed here.

It was shown in [GW11, Theorems 4.8, 4.11] that the consistency strength of a remarkable cardinal lies strictly between a 1-iterable and a 2-iterable cardinal. 1-iterable cardinals are precisely the weakly Ramsey cardinals (see [Git11, p. 539]), and in that paper, it was also shown that weakly Ramsey cardinals are limits of completely ineffable cardinals ([Git11, Theorem 1.7(6)]). Completely ineffable cardinals are clearly ineffable and hence subtle. In [HJ, Thm. 7], it was shown that if κ is subtle, then the set of cardinals less than κ that are strongly uplifting in V_κ is stationary in κ. So, putting this together, one sees that the consistency strength of a remarkable cardinal is higher than that of a strongly uplifting cardinal, which gives us the following observation.

Observation 6.11. *If Γ is the class of proper, semi-proper or subcomplete forcings, then* wFA_Γ *has strictly higher consistency strength than* $\undertilde{\mathsf{RA}}_\Gamma(H_{\omega_2})$.

The following diagram gives an overview of the relationships between the (virtual, bounded) resurrection axioms and the (weak, bounded) forcing axioms for subcomplete forcing. Solid arrows stand for implications, dotted arrows indicate that some intermediate principles (between which implications hold) are skipped in the diagram, and solid back-and-forth arrows stand for equivalences. Directly underneath some principles, I noted some of their combinatorial consequences. Large cardinal properties in square brackets give what's known about the consistency strength of the principle. Note that it is not the case that principles that are displayed at the same height have comparable consistency strengths.

The equivalences between $\mathsf{wBSCFA}(\leq\kappa)$ and $\mathsf{BSCFA}(\leq\kappa)$ for $\kappa = \omega_1, \omega_2$ are a special case of [Fuc16a, Observation 4.7]. The equivalences between the resurrection axioms at H_{ω_2} and their virtual counterparts have been shown in Observation 5.3. The implications going from the resurrection axioms to the bounded forcing axioms follow from Lemma 4.3. The implications going from virtual resurrection axioms to weak bounded forcing axioms follow from Lemma 6.7. The implications of $\diamondsuit$ are given in Fact 3.1. The implication arrows from resurrection axioms to failures of weak Todorčević square principles are shown in Theorem 4.7. The implication arrows from bounded forcing axioms to failures of Todorčević square principles follows from the proof of [Fuc16a, Lemma 4.17]. The consistency strength information for the resurrection axioms can be found in Fact 3.3 and Observation 4.8, and the consistency strength calculation for the virtual resurrection axioms is given by Theorem 5.15. The consistency strength lower bound for the bounded forcing axiom at ω_3 follows from the proof of [Fuc16a, Lemma 4.17], and the consistency strength facts about the weak bounded forcing axioms are given by Theorems 6.6 and 6.4, where a cardinal κ is +1-reflecting iff it is remarkably $\leq\kappa$-reflecting, see [Fuc16a, Lemma 4.9].

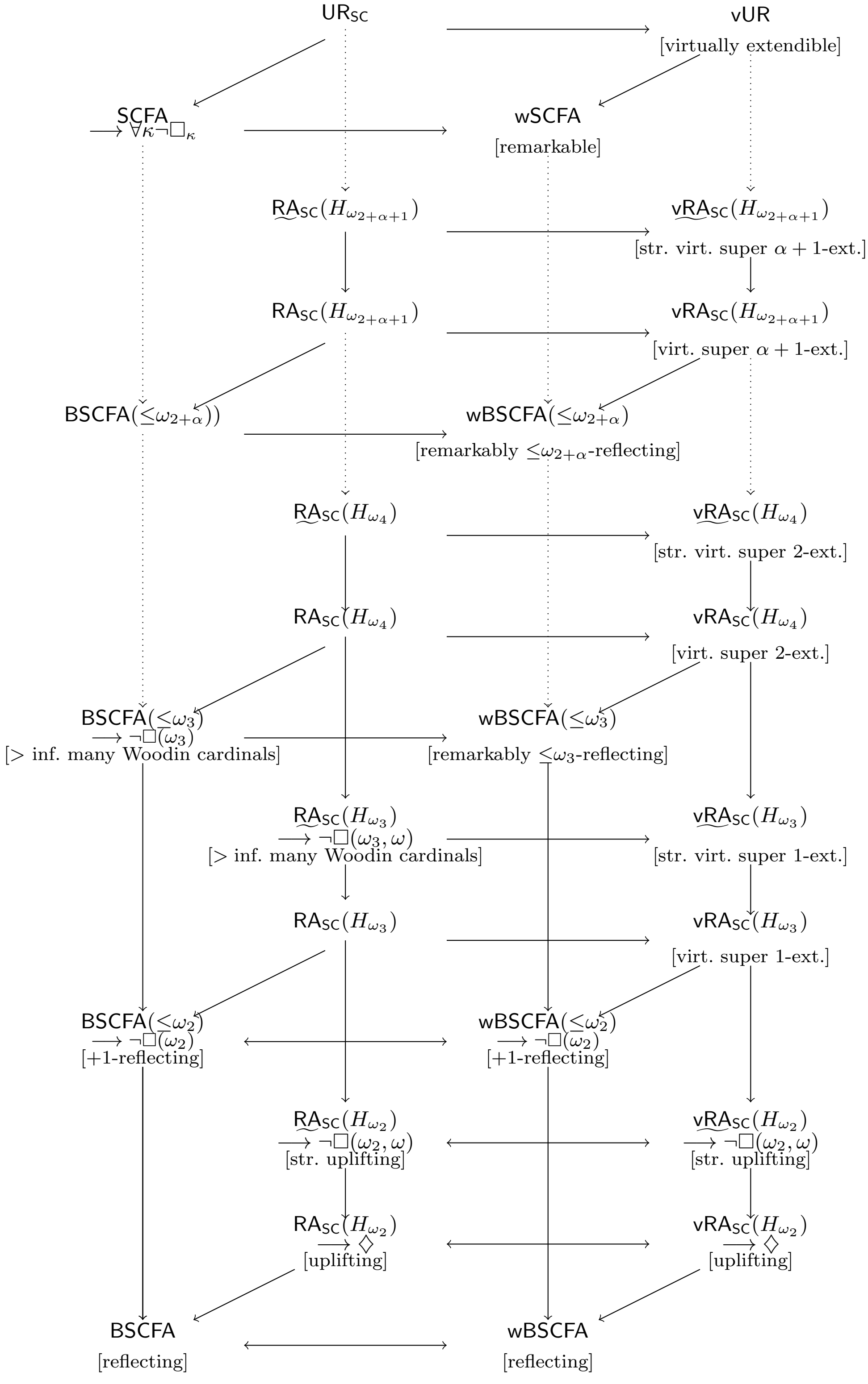

Figure 1: Overview of implications and consistency strengths.

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